1: 3-Resonance in a Hopf-Hopf bifurcation

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We investigate the bifurcating solutions at a Hopf-Hopf interaction point with an internal 1 : 3 resonance. It turns out, that the transitions from single to mixed modes can be described by Duffing or Mathieu scenarios.

1 Normal Form equations and investigation of the primary solution branches

If in some dynamical system, like the model of a fluid conveying tube, three parameters are varied, Hopf-Hopf mode interactions with internal low-order resonances may occur, which can lead to very interesting system dynamics. In this article we have a closer look at the 1 : 3-resonance, in which the resonance terms appear at third degree and therefore have the same order of influence as the leading terms in the non-resonant system. After shortly summarizing the results for a Hopf-Hopf interaction Manifold, applying nonlinear Normal Form and unfolding of the linearized system, we obtain the complex differential equations (1, 2)

\begin{equation}
\dot{z}_1 = (\lambda + i\omega + A_1|z_1|^2 + A_2|z_2|^2)z_1 + A_3z_1^2z_2, \quad (1)
\end{equation}

\begin{equation}
\dot{z}_2 = (\mu + 3i\omega + i\delta + A_4|z_1|^2 + A_5|z_2|^2)z_2 + A_6z_2^3, \quad (2)
\end{equation}

where \(\lambda, \mu\) and \(\delta\) are the unfolding parameters and the complex valued coefficients \(A_j = c_j + id_j\) are obtained from the nonlinear contributions of the original equations.

1.1 Bifurcation scenario for the non-resonant Hopf-Hopf bifurcation

If the frequencies \(\omega_j\) at the Hopf-Hopf interaction are not close to a low order resonance, also the terms \(A_3z_1^2r_1^2\) and \(A_6z_2^3\) can be eliminated by the Normal Form method. In polar coordinates \(z_j = r_j \exp(i\varphi_j)\) the Normal Form equations become

\begin{equation}
\dot{r}_1 = (\lambda + c_1r_1^2 + c_2r_2^2)r_1, \quad \dot{r}_2 = (\mu + c_4r_1^2 + c_5r_2^2)r_2, \quad (3)
\end{equation}

\begin{equation}
\dot{\varphi}_1 = (\omega_1 + d_1r_1^2 + d_2r_2^2), \quad \dot{\varphi}_2 = (\omega_2 + d_4r_1^2 + d_5r_2^2). \quad (4)
\end{equation}

The angles \(\varphi_j\) have completely disappeared from the equations for the radii and need not be taken into account during the further investigation, because the frequencies \(\omega_j\) are of order 1.

Besides the trivial solution with eigenvalues \(\lambda\) and \(\mu\) the equations (3) have the equilibria listed in Table 1. The mixed mode branch bifurcates from the primary solutions along the rays

\((\lambda, \mu) = -(c_1, c_4)r_1^2, \quad \text{and} \quad (\lambda, \mu) = -(c_2, c_5)r_2^2\)

by pitchfork bifurcations.

If the coefficients \(c_1\) and \(c_5\) have different signs and \(c_1c_5 - c_2c_4 > 0\), a tertiary branch of slow periodic solutions bifurcates from the mixed mode solutions. At the same parameter values a heteroclinic orbit connecting the single modes exists. In order to calculate the branching behaviour for the periodic solutions and distinguish it from the heteroclinic orbit, higher order terms in the bifurcation equations are needed.

Table 1: Nontrivial solution branches of the non-resonant Hopf-Hopf interaction

<table>
<thead>
<tr>
<th>Type</th>
<th>Branching equation</th>
<th>Eigenvalues (\sigma_j) resp. matrix</th>
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</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td>(\lambda + c_1r_1^2 = 0)</td>
<td>(\sigma_1 = 2c_1r_1^2), (\sigma_2 = \mu + c_4r_1^2)</td>
</tr>
<tr>
<td>Mode 2</td>
<td>(\mu + c_3r_2^2 = 0)</td>
<td>(\sigma_1 = c_4r_1^2), (\sigma_2 = 2c_5r_2^2)</td>
</tr>
<tr>
<td>Mixed mode</td>
<td>(\lambda + c_1r_1^2 + c_2r_2^2 = 0)</td>
<td>(\mu + c_4r_1^2 + c_5r_2^2 = 0)</td>
</tr>
</tbody>
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1.2 The slow solution branch in the resonant case

Due to the presence of the term $A_6 z_1^3$ in (2) no pure slow mode can exist, there will always be a contribution of $O(|z_1|^3)$ in the second mode. Taking care of this term by applying a center manifold reduction to the fast equation (2), which is possible as long as $\mu + i \delta + A_4 |z_1|^2$ is sufficiently far from 0, we find that this term only contributes high order terms to the slow equation (1). Therefore we recover the slow oscillation from the non-resonant case and observe that outside the critical region it has the same branching behaviour. But its stability with respect to oscillations in the second mode is quite different: If $z_1(t)$ oscillates like $r_1 \exp(i \Omega t)$, the term $A_6 z_1^3$ acts like an external Duffing-like excitation in (2). With $\Omega = \omega + d_1 r_1^2$ we find that the primary resonance in this equation occurs when

$$3\omega + \delta + d_4 r_1^2 = 3(\omega + d_1 r_1^2) \quad \text{and} \quad \mu + c_4 r_1^2 = 0.$$  

Introducing for fixed values of $r_1$ the shifted “parameters”

$$\check{\delta} = \delta + (d_4 - 3d_1) r_1^2 \quad \text{and} \quad \check{\mu} = \mu + c_4 r_1^2,$$

and setting $z_1 = r_1 \exp(\check{\delta} t)$ and $z_2 = w_2 \exp(3 \check{\delta} t)$, we obtain the autonomous differential equation

$$\dot{w}_2 = (\check{\mu} + i \check{\delta} + A_4 |w_2|^2) w_2 + A_6 r_1^3,$$  

(5)

whose stationary values satisfy the Duffing-like equation

$$\left( (\check{\mu} + c_4 r_1^2)^2 + (\check{\delta} + d_4 r_1^2)^2 \right) r_2^2 = |A_6|^2 r_1^6.$$  

(6)

For given values of $r_1$ and $r_2$ the possible values of $\check{\mu}$ and $\check{\delta}$ lie on circles with radius $|A_6|^2 r_1^2 / r_2$ and center at $(-c_4 r_1^2, -d_4 r_1^2)$. The backbone of this family of solutions, which is obtained by setting $A_6 = 0$, corresponds to the mixed-mode solution of the non-resonant case.

These stationary solutions correspond to slow periodic oscillations of the full system (1,2) and approximate both the mode-1 solution and the mixed mode solution of the non-resonant case, as long as the contribution of the slow mode is dominant. We observe, that the pitchfork bifurcation from the slow mode to the mixed mode in the non-resonant case is governed by a Duffing scenario in the $1 : 3$-resonance.

1.3 The fast solution branch in the resonant case

If we set $z_1 = 0$, (1) is fulfilled and for the fast mode we obtain the same branch equation as in the non-resonant case. The solution oscillates with constant amplitude $r_2$ and frequency $\Omega_2 = 3\omega + \delta + d_5 r_2^2$.

The stability of $z_1 = 0$ is lost, when $\lambda + c_2 r_2^2 = 0$. Since in that case the imaginary linear part $\Omega_1 = \omega + d_2 r_2^2 \neq 0$, we would expect a Hopf bifurcation to occur. But the situation is somewhat more difficult, because the term $A_3 \vec{\pi}_2^2 z_2$ forces an oscillation close to the slow eigenfrequency, therefore we have to deal with a nonlinear Mathieu scenario. In order to investigate the bifurcation in this case we assume $|z_1| \ll |z_2|$ and that the influence of $z_1$ on the motion of $z_2$ can be neglected.

We introduce the new parameters

$$\tilde{\lambda} = \lambda + c_2 r_2^2 \quad \text{and} \quad \Delta = \Omega_1 - \Omega_2 / 3 = -\delta / 3 + (d_2 - d_5 / 3) r_2^2$$  

(7)

and set $z_1 = w_1 \exp(\Omega_2 t / 3)$. Then $w_1$ satisfies the autonomous equation

$$\dot{w}_1 = (\tilde{\lambda} + i \Delta + A_4 |w_1|^2) w_1 + A_3 \vec{\pi}_2^2 r_2$$  

(8)

The steady states are obtained from the equation

$$\left( \tilde{\lambda} + c_4 r_1^2 \right)^2 + (\Delta + d_4 r_1^2)^2 = |A_3|^2 r_1^6,$$  

(9)

which for fixed $r_1$ and $r_2$ describes a circle with radius $|A_3| r_1 r_2$ and center $(-c_4 r_1^2, -d_4 r_1^2)$ in the $(\tilde{\lambda}, \Delta)$-plane. The backbone curve of this solution set, given by $A_3 = 0$, coincides again with the mixed-mode solution of the non-resonant case. If $\Delta = 0$, a pair of slow periodic oscillations bifurcates from the fast solution branch. Otherwise first a quasiperiodic oscillation with frequencies $\Omega_2$ and $\Omega_1 + d_1 r_1^2$ is found. For larger amplitudes of $r_1$ the two oscillations can synchronize into the frequency ratio $1 : 3$.

Due to the parametric excitation term $A_3 \vec{\pi}_2^2 z_2$ the pitchfork bifurcation from the fast mode in the non-resonant case changes to a Mathieu-like transition in the resonant case.

References