

## ENTRIES OF INDEFINITE NEVANLINNA MATRICES

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ABSTRACT. In the early 1950's, M. G. Krein characterized the entire functions that are an entry of some Nevanlinna matrix, and the pairs of entire functions that are a row of some Nevanlinna matrix. In connection with Pontryagin space versions of Krein's theory of entire operators and de Branges' theory of Hilbert spaces of entire functions, an indefinite analog of the Nevanlinna matrices plays a role. In the paper, the above-mentioned characterizations are extended to the indefinite situation and the geometry of the associated reproducing kernel Pontryagin spaces is investigated.

### §1. INTRODUCTION

We call an entire  $(2 \times 2)$ -matrix-valued function  $W(z) = (w_{ij}(z))_{i,j=1}^2$  a *Nevanlinna matrix* if it is normalized by  $W(0) = I$ , its entries take real values on the real line,  $\det W(z) = 1$ ,  $z \in \mathbb{C}$ , and  $(\mathbb{C}^+$  denotes the open upper half-plane)

$$(1.1) \quad \operatorname{Im} \frac{w_{11}(z)t + w_{12}(z)}{w_{21}(z)t + w_{22}(z)} \geq 0, \quad z \in \mathbb{C}^+, \quad t \in \mathbb{R} \cup \{\infty\}.$$

It is well known that, equivalently, one could require that  $W(0) = I$ ,  $w_{ij}(\bar{z}) = \overline{w_{ij}(z)}$ ,  $\det W(z) = 1$ , and that the reproducing kernel ( $J$  denotes the signature matrix  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ )

$$(1.2) \quad H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \bar{w}}, \quad z, w \in \mathbb{C},$$

is positive definite.<sup>1 2</sup> This says that the class of Nevanlinna matrices coincides with the class of (normalized) entire  $(iJ)$ -inner  $(2 \times 2)$ -matrix-valued functions, see, e.g., [AD08]. Often in the literature the entries of a Nevanlinna matrix are required to be transcendental entire functions, see, e.g., [Akh61, Definition 2.4.3]. We do not include this condition.

The Nevanlinna matrices play a fundamental role in many places of functional and complex analysis. We mention the following, of course interrelated, instances.

M. G. Krein's theory of entire operators: there they appear as resolvent matrices in the description of all spectral functions, see, e.g., [GG97, Theorem 7.2].

L. de Branges' theory of Hilbert spaces of entire functions: there they are used to characterize whether a de Branges space is invariant under forming difference quotients, or to characterize the isometric inclusions of spaces, see, e.g., [dB68, Theorem 27, Theorems 33/34].

Direct and inverse spectral problems for two-dimensional canonical systems: there they appear as fundamental solution matrix, and are represented as multiplicative integrals

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<sup>1</sup>For  $z = \bar{w}$  the expression  $H_W(w, z)$  must be interpreted appropriately as a derivative.

<sup>2</sup>The kernel  $H_W(w, z)$  is said to be positive definite if for each  $L \in \mathbb{N}$ ,  $z_1, \dots, z_L \in \mathbb{C}$ , and  $a_1, \dots, a_L \in \mathbb{C}^2$ , the quadratic form  $\sum_{i,j=1}^L a_j^* H_W(z_j, z_i) a_i \cdot \xi_i \bar{\xi}_j$  is positive semidefinite.

in the sense of V. P. Potapov, see, e.g., [GK67, Chapter VI.1] and [AD08, Introduction, Theorem 4.13].

The theorems that hold true for Nevanlinna matrices in the above contexts have implications for a number of concrete problems. For example they can be used to describe the solution set of the problem of continuation of a positive definite function from an interval. Or, they can be used to describe spectral functions of certain differential operators. For these and other applications, see, e.g., [GG97, Theorems 3.1.6 and 3.2.6], [AD98], [KK68, §3, Fundamental Theorem], [KL14]. The — probably — most intensively studied subject where Nevanlinna matrices play a role is the power moment problem. Literature on this topic starts with a historical paper of T. J. Stieltjes dating back to 1894, and ranges from the classical work of H. Hamburger, R. Nevanlinna and M. Riesz in the 1920's to very recent contributions, see, e.g., [BP94, Ber95, Ped09].

A question that appears naturally is what entire functions may occur as an entry of *some* Nevanlinna matrix, or (especially when having in mind de Branges' theory) what pairs of entire functions may serve as a row of *some* Nevanlinna matrix. These questions were completely answered by M. G. Krein in the early 1950's, cf. [Kre52, §3, Theorems A,B,C]<sup>3</sup>.

In the theory of indefinite inner product spaces, a generalization of the notion of a Nevanlinna matrix occurs. Namely a certain class of entire matrix-valued functions arises, we speak of  $\mathcal{M}_{<\infty}$ , where the positivity requirement for the kernel  $H_W$  is weakened; see Definition 2.4 below for the details.

Matrices of the class  $\mathcal{M}_{<\infty}$  are of similar significance in Pontryagin space theory as the Nevanlinna matrices are in Hilbert space theory. In connection with entire operators see [KL78, Satz 6.9], in connection with de Branges spaces see [KW99a, Proposition 10.3, Theorem 12.2], and in connection with canonical systems see [KW10, 1.3]. Also various applications are found, e.g., to indefinite power moment problems, see [KL79, KL80], to the problem of continuation of a Hermitian indefinite function from an interval, see [GL74, KL85, KW98b], or to the spectral theory of certain differential operators with singular potentials, see [Wor12, LW].

Our aim in the present paper is to establish analogs of Krein's Theorems A,B,C for the class  $\mathcal{M}_{<\infty}$ , and to describe in detail the geometry of the reproducing kernel Pontryagin spaces generated by matrices  $W \in \mathcal{M}_{<\infty}$  with one prescribed row or entry. Our main results are Theorems 3.1, 3.4, and 3.5, which are the full indefinite analogs of Krein's Theorems A,B,C, and Theorems 4.2 and 4.3, where we investigate geometric structures.

The proof of Theorems 3.1, 3.4, and 3.5 is neither too difficult nor too laborious. They can be shown with some Pontryagin space arguments and some standard complex analysis using our previous (involved, but readily available) work [Wor11]. Nevertheless, we regard these results themselves as valuable; they beautifully round up the picture of the indefinite theory. Our method of proof uses the interplay of the Nevanlinna function theory and the theory of de Branges–Pontryagin spaces. This approach seems to be new also in the positive definite case. In particular, when specialized to the positive definite case, our results provide (probably) new proofs of Krein's Theorems A,B,C.

Matters are getting much more involved in Theorems 4.2 and 4.3. In order to analyze the geometric structure of the corresponding reproducing kernel spaces, we show one structure result about indefinite canonical systems and employ several facts from this theory.

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<sup>3</sup>Up to our knowledge, the relevant paper of Krein has not been translated to English. For the reader's convenience, we shall formulate his results in detail, see Theorems 2.1, 2.2, and 2.3 below. A proof of Theorem B can be found in [Akh61, Chapter 3, p. 133, no. 12], see also the more detailed and slightly more general exposition in [BP95, Theorems 3.6, 5.1].

Concerning the presentation of the paper, one comment is in order. The core content is arranged in two sections. These are §3 and §4. In the first of them, we prove the indefinite analogs of Krein's theorems. In the latter, we carry out the above-mentioned analysis of the geometric structure. In view of the amount of notions and results required from previous work, it turned out impossible to provide a fully self-contained exposition within an acceptable page range. Hence, we decided for a clear division: All auxiliary notation and knowledge needed for the proof of Theorems 3.1, 3.4, and 3.5 in §3 is provided in the preliminary Subsection 2.2. All what is required in §4 from the theory of indefinite canonical systems is clearly and extensively referenced, but otherwise used without further notice. We shall comment on this in more detail on p. 773.

## §2. PRELIMINARIES

First, in Subsection 2.1, we state the classical theorems in detail. Then, in Subsection 2.2, we provide a selection of auxiliary notation and results.

**2.1. Krein's Theorems A, B, C.** In the formulations below we already include our normalization  $W(0) = I$ . Let us moreover remark that Krein used the term "special matrix", whereas nowadays it is more common to talk of Nevanlinna matrices<sup>4</sup>.

We start with a description of all pairs of entire functions that appear as the second row of some Nevanlinna matrix. Thereby, we say that an entire function is real if it takes real values along the real line.

**2.1. Theorem** (see [Kre52, §3, Theorem A]). *Let  $F$  and  $G$  be two real entire functions with  $(F(0), G(0)) = (0, 1)$ . Then a Nevanlinna matrix  $W$  such that  $(F, G) = (0, 1)W$  exists if and only if the following conditions (i)–(iv) are fulfilled.*

- (i)  $F$  and  $G$  have no common zeros, and all zeros of  $F$  and  $G$  are real and simple.
- (ii)  $\operatorname{Im} \frac{G(z)}{F(z)} \geq 0$  for all  $z \in \mathbb{C}^+$ .
- (iii) The nonzero zeros  $\alpha_n$  of  $F$  and  $\beta_n$  of  $G$  satisfy

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)|\alpha_n^2} < \infty, \quad \sum_n \frac{1}{|F(\beta_n)G'(\beta_n)|\beta_n^2} < \infty.$$

- (iv) The function  $\frac{1}{FG}$  has an expansion

$$\begin{aligned} \frac{1}{F(z)G(z)} &= \frac{c_{-1}}{z} + c_0 + \sum_n \frac{1}{F'(\alpha_n)G(\alpha_n)} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} \right] \\ &\quad + \sum_n \frac{1}{F(\beta_n)G'(\beta_n)} \left[ \frac{1}{z - \beta_n} + \frac{1}{\beta_n} \right] \end{aligned}$$

with some  $c_{-1}, c_0 \in \mathbb{R}$ .

Second, the next theorem gives a description of all entire functions that appear as an entry of some Nevanlinna matrix.

**2.2. Theorem** (see [Kre52, §3, Theorem B]). *Let  $F$  be a real entire function with  $F(0) = 0$  or  $F(0) = 1$ . Then a Nevanlinna matrix  $W$  such that  $F$  is an entry of  $W$  exists if and only if all zeros of  $F$  are real and simple, the nonzero zeros  $\alpha_n$  of  $F$  satisfy*

$$\sum_n \frac{1}{|F'(\alpha_n)|\alpha_n^2} < \infty,$$

<sup>4</sup>The term "Nevanlinna matrix" was probably used for the first time in [Akh61], and is motivated by the fact that such matrices appear in Nevanlinna's parameterization of the solutions of a power moment problem.

and the function  $\frac{1}{F}$  has an expansion

$$\frac{1}{F(z)} = \frac{c_{-1}}{z} + c_0 + \sum_n \frac{1}{F'(\alpha_n)} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} \right]$$

with some  $c_{-1}, c_0 \in \mathbb{R}$ .

Third, thinking of the left lower entry of some Nevanlinna matrix as being fixed, we present a description of all entire functions that form, together with this function, the second row of some (other) Nevanlinna matrix.

**2.3. Theorem** (see [Kre52, §3, Theorem C]). *Let  $F$  be a real entire functions with  $F(0) = 0$ , and assume that  $F$  is subject to the conditions of Theorem 2.2. In order that a real entire function  $G$ ,  $G(0) = 1$ , form, together with  $F$ , the second row of some Nevanlinna matrix, it is necessary and sufficient that the functions  $F$  and  $G$  satisfy the following conditions (i)–(iii).*

- (i)  $F$  and  $G$  have no common zeros, and all zeros of  $G$  are real and simple.
- (ii)  $\text{Im} \frac{G(z)}{F(z)} \geq 0$  for all  $z \in \mathbb{C}^+$ .
- (iii)  $\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)|\alpha_n^2} < \infty$ .

A similar statement holds true when we regard  $G$  as fixed and  $F$  as varying.

Actually, in [Kre52, §3, Theorem C] a different condition was stated, and it was remarked afterwards that this condition is in its turn equivalent to the one we state here. However, this equivalence is immediate. Hence, we repeat only one of the conditions (choosing the one that is more suitable in the present context).

**2.2. Some classes of analytic and meromorphic functions.** We recall some terminology and present some (mainly) well-known facts from complex analysis and Pontryagin space theory.

To start with, we give the definition of the main players in the present paper: matrices of class  $\mathcal{M}_{<\infty}$ .

**2.4. Definition.** Let  $W = (w_{ij})_{i,j=1}^2$  be a  $(2 \times 2)$ -matrix-valued function, and let  $\kappa \in \mathbb{N}_0$ . We write  $W \in \mathcal{M}_\kappa$  if the following conditions are satisfied.

- (M1) The entries  $w_{ij}$  of  $W$  are real entire functions.
- (M2)  $\det W(z) = 1$  for  $z \in \mathbb{C}$ , and  $W(0) = I$ .
- (M3) The reproducing kernel  $H_W$  defined in (1.2) has  $\kappa$  negative squares<sup>5</sup>.

Moreover, we set

$$\mathcal{M}_{<\infty} := \bigcup_{\kappa \in \mathbb{N}_0} \mathcal{M}_\kappa,$$

and write  $\text{ind}_- W = \kappa$  to express that  $W \in \mathcal{M}_\kappa$ .

As we already said in the Introduction, the class  $\mathcal{M}_0$  is nothing but the class of all Nevanlinna matrices.

**2.5. Remark.** For the present considerations it is more practical to use the indefinite version of the definition of Nevanlinna matrices via the reproducing kernel  $H_W$ , rather than the indefinite version of the initially stated classical definition of Nevanlinna matrices via the fractional linear transformations. As in the positive definite case, these two approaches are equivalent, see, e.g., [Kal02, Proposition 2.3, Theorem 6.1].

<sup>5</sup>By this we mean that  $\kappa$  is the maximal number of negative squares of the quadratic forms  $\sum_{i,j=1}^L a_j^* H_W(z_j, z_i) a_i \cdot \xi_i \bar{\xi}_j$  with  $L \in \mathbb{N}$ ,  $z_1, \dots, z_L \in \mathbb{C}$ , and  $a_1, \dots, a_L \in \mathbb{C}^2$ .

It is an important fact that the class  $\mathcal{M}_{<\infty}$  is closed with respect to taking products. In fact,

$$(2.1) \quad \text{ind}_-(W_1W_2) \leq \text{ind}_- W_1 + \text{ind}_- W_2, \quad W_1, W_2 \in \mathcal{M}_{<\infty},$$

cf. [KW11, Lemma 2.10].

**2.6. Polynomial matrices.** *Examples of matrices of class  $\mathcal{M}_{<\infty}$  are obtained from polynomials. Let  $p$  be a real polynomial without constant term. Then we have*

$$(2.2) \quad W_p := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{<\infty},$$

and ( $n := \deg p$ ,  $a_n$  is leading coefficient of  $p$ , and  $\lfloor x \rfloor$  denotes the largest integer not exceeding  $x$ )

$$\text{ind}_- \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 1 & \text{if } n \text{ odd, } a_n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

This fact is well known; an explicit reference is, e.g., [KW11, Proposition 2.8].

If  $W$  is a real  $(2 \times 2)$ -matrix-valued polynomial with  $W(0) = I$  and  $\det W = 1$ , then  $W$  belongs to  $\mathcal{M}_{<\infty}$  and can be factorized into a product of “rotations” of elementary factors of the form (2.2). This was shown in [ADL04, Theorem 3.1]; for a purely algebraic approach see [KW06, Theorem 3.1].

**2.7. Changing roles of rows and columns.** *Each of the classes  $\mathcal{M}_\kappa$ ,  $\kappa \in \mathbb{N}_0$ , is invariant under the transformations*

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{22} & -w_{21} \\ -w_{12} & w_{11} \end{pmatrix}, \quad \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{11} & -w_{21} \\ -w_{12} & w_{22} \end{pmatrix}, \\ \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \mapsto \begin{pmatrix} w_{22} & w_{12} \\ w_{21} & w_{11} \end{pmatrix},$$

see, e.g., [KWW06, Lemma 2.3]. Hence, a pair  $(F, G)$  is the second row of some matrix  $W \in \mathcal{M}_{<\infty}$ , if and only if  $(G, -F)$  is the first row, or if and only if  $(-F, G)^T$  is the second column, or if and only if  $(G, F)^T$  is the first column of some matrix  $W \in \mathcal{M}_{<\infty}$ .

Next, we discuss some facts from classical complex analysis.

**2.8. Entire functions of Cartwright class.** *An entire function is said to be of Cartwright class if it is of finite exponential type, i.e., satisfies*

$$\limsup_{|z| \rightarrow \infty} \frac{1}{|z|} \log |F(z)| < \infty,$$

and if

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1+x^2} dx < \infty.$$

Functions of Cartwright class are well-behaved concerning their growth and distribution of zeros, see, e.g., [Lev80, V. 4, Theorem 11]. In the present context the following is important. Assume that  $F$  is of Cartwright class, and that all but finitely many zeros of  $F$  are real and simple. Denote by  $(\alpha_n)$  the sequence of all nonzero real and simple zeros of  $F$ , let  $(\gamma_j)$  be the remaining nonzero zeros (with multiplicities  $d_j \in \mathbb{N}$ ), and let  $d_0 \in \mathbb{N}_0$  be the multiplicity of 0 as a zero of  $F$ . Moreover, denote by  $\alpha_n^+$  and  $\alpha_n^-$  the sequences of positive or negative, respectively, elements of  $(\alpha_n)$  arranged according to increasing modulus. Then

- (i) the limits<sup>6</sup>  $\lim_n \frac{n}{\alpha_n^+}$  and  $\lim_n \frac{n}{\alpha_n^-}$  exist in  $[0, \infty)$  and are equal;

<sup>6</sup>We tacitly assume that the limit of a finite sequence is equal to zero.

- (ii) the limit  $\lim_{R \rightarrow \infty} \sum_{|\alpha_n| \leq R} \frac{1}{\alpha_n}$  exists in  $\mathbb{R}$ ;
- (iii) the function  $F$  has the representation

$$(2.3) \quad F(z) = \frac{F^{(d_0)}(0)}{d_0!} z^{d_0} \cdot \prod_j \left(1 - \frac{z}{\gamma_j}\right)^{d_j} \cdot \lim_{R \rightarrow \infty} \prod_{|\alpha_n| \leq R} \left(1 - \frac{z}{\alpha_n}\right).$$

However, in general, these properties do not imply that  $F$  is of Cartwright class.

**2.9. Meromorphic functions of bounded type.** Let  $f$  be a function meromorphic in the half-plane  $\mathbb{C}^+$ . Then we say that  $f$  is of bounded type in this half plane if it can be written as the quotient of two functions analytic and bounded in this half-plane.

The set of all functions of bounded type in  $\mathbb{C}^+$  is a field. Two simple but important examples are the following.

- (i) Each rational function is of bounded type in  $\mathbb{C}^+$ . Indeed, the constant function 1 and the function  $(z - i)^{-1}$  (obviously) have this property.
- (ii) Each function that is analytic in  $\mathbb{C}^+$  and has nonnegative imaginary part throughout this half-plane is of bounded type. To see this, consider the fractional linear transformation  $L(z) := (z - i)(z + i)^{-1}$ . It is analytic in  $\mathbb{C} \setminus \{-i\}$  and maps the closed half-plane  $\overline{\mathbb{C}^+}$  onto the closed unit disk. If  $q$  is analytic with  $\text{Im } q \geq 0$  throughout  $\mathbb{C}^+$ , then  $L \circ q$  is analytic and bounded in  $\mathbb{C}^+$ . Now we obtain the representation

$$q = L^{-1} \circ (L \circ q) = i \frac{1 + (L \circ q)}{1 - (L \circ q)}$$

of  $q$  as the quotient of two bounded analytic functions<sup>7</sup>.

Functions of bounded type in the lower half-plane  $\mathbb{C}^-$  are defined in the same way, and enjoy similar properties. This follows by conformal invariance.

**2.10. Analytic functions of bounded type.** If  $F$  is analytic in  $\mathbb{C}^+$ , a more intrinsic characterization of the property to be of bounded type reads as follows, cf. [RR94, Theorem 3.20]. An analytic function  $F$  in  $\mathbb{C}^+$  is of bounded type if and only if the subharmonic function  $\log^+ |F(z)|$  has a harmonic majorant in  $\mathbb{C}^+$ .

Finally, we turn to entire functions. It is an important result due to M. G. Krein that an entire function is of Cartwright class if and only if its restrictions to  $\mathbb{C}^+$  and  $\mathbb{C}^-$  are both of bounded type in the respective half-planes. Moreover, if  $F$  is of bounded type in both half-planes, then the exponential type of  $F$  can be computed as

$$(2.4) \quad \max \left\{ \lim_{y \rightarrow +\infty} \frac{1}{y} \log^+ |F(iy)|, \lim_{y \rightarrow -\infty} \frac{1}{|y|} \log^+ |F(iy)| \right\}.$$

For these facts see [Kre47, Theorems 3 and 2] or, e.g., [RR94, Theorems 6.17 and 6.18].

Now we recall some notions and results from the indefinite world; among them generalized Nevanlinna functions and de Branges–Pontryagin spaces. These include the most important tools for our present work, and are probably the least commonly known among our required prerequisites.

**2.11. Generalized Nevanlinna functions.** A function  $q$  meromorphic in  $\mathbb{C} \setminus \mathbb{R}$  is called a generalized Nevanlinna function if  $q(\bar{z}) = \overline{q(z)}$  and the reproducing kernel  $(\rho(q))$  denotes the domain of holomorphy of  $q$

$$N_q(w, z) := \frac{q(z) - q(\bar{w})}{z - \bar{w}}, \quad z, w \in \rho(q),$$

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<sup>7</sup>Functions with nonnegative imaginary part throughout the half-plane are even outer, cf. [RR94, V. Examples and Addenda 1].

has a finite number of negative squares. The set of all generalized Nevanlinna functions is denoted by  $\mathcal{N}_{<\infty}$ . If  $q \in \mathcal{N}_{<\infty}$ , then we write  $\text{ind}_- q$  for the actual number of negative squares of the kernel  $N_q$ , and set

$$\mathcal{N}_\kappa := \{q \in \mathcal{N}_{<\infty} : \text{ind}_- q = \kappa\}, \quad \kappa \in \mathbb{N}_0.$$

It is a historical result<sup>8</sup> that a function  $q$  belongs to  $\mathcal{N}_0$  if and only if it is analytic throughout  $\mathbb{C} \setminus \mathbb{R}$ , satisfies  $q(\bar{z}) = \overline{q(z)}$ , and  $\text{Im } q(z) \geq 0$  for  $z \in \mathbb{C}^+$ .

Simple examples of generalized Nevanlinna functions are rational functions. Each rational function  $q$  with real coefficients belongs to the class  $\mathcal{N}_{<\infty}$ , and  $\text{ind}_- q$  cannot exceed the maximal degree of the numerator and denominator of  $q$  (this is a well-known fact; a short and explicit proof can be found, e.g., in [Wor97, Theorem 1]). If  $q$  is a polynomial, it is easy to compute  $\text{ind}_- q$  explicitly: we have ( $n := \deg q$ ,  $a_n$  leading coefficient of  $q$ )

$$\text{ind}_- \left[ \frac{n}{2} \right] + \begin{cases} 1 & \text{if } n \text{ odd, } a_n < 0, \\ 0 & \text{otherwise,} \end{cases}$$

see, e.g., [KL77, Lemma 3.3] (applied with  $\sigma = 0$ ).

We mention that the class  $\mathcal{N}_{<\infty}$  is closed with respect to sums, in fact,  $\text{ind}_-(q_1 + q_2) \leq \text{ind}_- q_1 + \text{ind}_- q_2$ .

It is a deep result shown in [KL77, Satz 3.1] that the functions of class  $\mathcal{N}_{<\infty}$  have an integral representation analogous to the Herglotz integral representation of functions with nonnegative imaginary part.

Finally, recall that all but finitely poles of a generalized Nevanlinna function are real, simple, and have negative residuum. This follows from the mentioned integral representation or, alternatively, from the multiplicative representation of a generalized Nevanlinna function stated in 2.13 below.

**2.12. Computing the negative index.** If  $q \in \mathcal{N}_{<\infty}$ , one can give a formula for  $\text{ind}_- q$  based on the structure of the poles of  $q$  and on the asymptotic growth of the measure in its integral representation towards certain critical points, cf. [KL77, Satz 3.4]. For our present needs the following estimate is sufficient. Let  $q \in \mathcal{N}_{<\infty}$  and assume that  $q$  is meromorphic in the whole plane. Denote by  $(\alpha_n)$  the sequence of all real and simple poles of  $q$  with negative residuum, let  $(\gamma_j)$  be the remaining poles (with multiplicities  $d_j \in \mathbb{N}$ ), and set

$$\delta_j := \begin{cases} 1 & \text{if } d_j \text{ odd, } \lim_{z \rightarrow \gamma_j} (z - \gamma_j)^{d_j} q(z) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(2.5) \quad \text{ind}_- q \geq \sum_{\gamma_j \in \mathbb{R}} \left( \left[ \frac{d_j}{2} \right] + \delta_j \right) + \sum_{\text{Im } \gamma_j > 0} d_j + \min \left\{ m \in \mathbb{N}_0 : \sum_n \frac{|\text{Res}(q; \alpha_n)|}{1 + \alpha_n^{2(m+1)}} < \infty \right\}.$$

Sometimes, especially when dealing with functions meromorphic in the whole plane, it is practical not to refer to [KL77, Satz 3.4] directly. A useful method for “counting negative squares” is to employ [KL77, Satz 1.13] to split into a sum with one summand well-behaved at  $\infty$  and one well-behaved off  $\infty$ . Then the negative index of the first summand can be computed by using [KL81, Theorem 3.5] and [Lan86], and for the second summand by using [KL77, Lemma 3.3].

Let us also explicitly mention the following corollary. Let  $q \in \mathcal{N}_\kappa$  be meromorphic in  $\mathbb{C}$ . Then all but at most  $2\kappa$  poles of  $q$  are real and simple and have negative residuum.

<sup>8</sup>It can be traced back as far as to some work of G. Herglotz from 1911.

Moreover, if  $(\alpha_n)$  denotes the sequence of all real and simple poles of  $q$  with negative residuum, then

$$(2.6) \quad \sum_n \frac{|\operatorname{Res}(q; \alpha_n)|}{1 + \alpha_n^{2(\kappa+1)}} < \infty.$$

Using the sources mentioned above, one can also show the following more refined statement. Let  $q \in \mathcal{N}_{<\infty}$  be meromorphic in  $\mathbb{C}$ , and denote the number on the right-hand side of (2.5) by  $\kappa_0(q)$ . Then (we denote by  $\mathbb{R}[z]$  the set of polynomials with real coefficients)

$$(2.7) \quad \{ \operatorname{ind}_-(q + p) : p \in \mathbb{R}[z], p(0) = 0 \} = [\kappa_0(q), \infty) \cap \mathbb{N}_0.$$

**2.13.** *Multiplicative representations of  $q \in \mathcal{N}_{<\infty}$ . The following important multiplicative representation of a generalized Nevanlinna function has been established independently in [DLLS00, Corollary] and [DHdS99, Theorem 3.3]. Let a function  $q \in \mathcal{N}_{<\infty}$  be given. Then there exist relatively prime polynomials  $p, \tilde{p}$  whose zeros are all located in the closed upper half-plane, and a function  $q_0 \in \mathcal{N}_0$ , such that (we denote  $f^\#(z) := \overline{f(\bar{z})}$ )*

$$(2.8) \quad q(z) = \frac{\tilde{p}(z)\tilde{p}^\#(z)}{p(z)p^\#(z)}q_0(z).$$

A converse is also true, cf. [DHdS99, Proposition 3.2]. Let  $p, \tilde{p}$ , and  $q_0$  be as above, then the function  $q$  in (2.8) belongs to  $\mathcal{N}_\kappa$  with  $\kappa := \max\{\deg p, \deg \tilde{p}\}$ .

As a consequence of this representation we see that each generalized Nevanlinna function is of bounded type (in both half-planes  $\mathbb{C}^+$  and  $\mathbb{C}^-$ ).

**2.14.** *Indefinite Hermite–Biehler functions. We say that an entire function  $E$  belongs to the indefinite Hermite–Biehler class  $\mathcal{HB}_{<\infty}$  if it is normalized by  $E(0) = 1$ , the functions  $E$  and  $E^\#$  have no common zeros, and the reproducing kernel*

$$K_E(w, z) := i \frac{E(z)\overline{E(w)} - E^\#(z)E(\bar{w})}{2\pi(z - \bar{w})}, \quad z, w \in \mathbb{C},$$

has a finite number of negative squares. Again we denote the actual number of negative squares of this kernel by  $\operatorname{ind}_- E$ , and write  $\mathcal{HB}_\kappa$  for all functions with  $\operatorname{ind}_- E = \kappa$ .

The class  $\mathcal{HB}_0$  is a classical object: it consists of all entire functions such that  $E(0) = 1$ ,  $E$  and  $E^\#$  have no common zeros, and

$$|E(\bar{z})| \leq |E(z)|, \quad z \in \mathbb{C}^+.$$

Thus is nothing but the class of all Hermite–Biehler functions (sometimes also called “de Branges functions”) as studied, e.g., in [dB68], [Lev80, Chapter VII].

Let  $E \in \mathcal{HB}_{<\infty}$ . Then the reproducing kernel  $K_E$  generates a Pontryagin space of entire functions, cf. [ADRdS97, Theorem 1.1.3]. We denote this space by  $\mathcal{P}(E)$ , and refer to it as the de Branges–Pontryagin space associated with  $E$ . For the theory of such spaces, see [KW99a]. If  $\operatorname{ind}_- E = 0$ , this notion coincides with the classical concept of de Branges’ Hilbert spaces of entire functions, cf. [dB68].

**2.15.** *Functions associated with a de Branges–Pontryagin spaces. A central concept for our present purposes is the notion of functions  $N$ -associated with a de Branges space. For  $E \in \mathcal{HB}_{<\infty}$  and  $N \in \mathbb{N}$ , we set*

$$\operatorname{Assoc}_N \mathcal{P}(E) := \mathcal{P}(E) + z\mathcal{P}(E) + \cdots + z^N \mathcal{P}(E).$$

The following statements are crucial tools for our present considerations.

First, the fact whether or not  $\operatorname{Assoc}_N \mathcal{P}(E)$  contains a real and zerofree element is related to the existence of matrices  $W \in \mathcal{M}_{<\infty}$  whose first row equals  $(A, B)$ , where

$$A := \frac{1}{2}(E + E^\#), \quad B := \frac{i}{2}(E - E^\#).$$

Namely, we have

$$1 \in \bigcup_{N \in \mathbb{N}} \text{Assoc}_N \mathcal{P}(E) \Leftrightarrow \exists W \in \mathcal{M}_{<\infty} : (1, 0)W = (A, B).$$

This was shown in [KW99a, Proposition 10.3] and [Wor11, Proposition 6.1], where the first reference covers the case where  $N = 1$  and the second the case of  $N > 1$ .

The second major result is [Wor11, Theorem 3.2]. To recall this, we need to introduce some notation. For  $E \in \mathcal{HB}_{<\infty}$  and  $\varphi \in \mathbb{R}$ , set

$$S_\varphi(z) := \frac{1}{2i} (e^{i\varphi} E(z) - e^{-i\varphi} E^\#(z)).$$

Denote by  $(\alpha_{\varphi,n})$  the sequence of all nonzero real and simple zeros of  $S_\varphi$  such that the residuum of  $S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}$  at this point is negative, set  $\sigma_{\varphi,n} := -\text{Res}(S_\varphi^{-1} S_{\varphi+\frac{\pi}{2}}; \alpha_{\varphi,n})$ , let  $(\gamma_{\varphi,j})$  be the remaining nonzero zeros (multiplicities denoted as  $d_{\varphi,j} \in \mathbb{N}$ ), and let  $d_{\varphi,0} \in \mathbb{N}_0$  be the multiplicity of 0 as a zero of  $S_\varphi$ . Moreover, denote by  $\alpha_{\varphi,n}^+$  and  $\alpha_{\varphi,n}^-$  the sequences of positive or negative, respectively, elements of  $(\alpha_{\varphi,n})$  arranged according to increasing modulus. Finally, set

$$F_\varphi(z) = z^{d_{\varphi,0}} \cdot \prod_j \left(1 - \frac{z}{\gamma_{\varphi,j}}\right)^{d_{\varphi,j}} \cdot \lim_{R \rightarrow \infty} \prod_{|\alpha_{\varphi,n}| \leq R} \left(1 - \frac{z}{\alpha_{\varphi,n}}\right),$$

provided the product converges.

Now [Wor11, Theorem 3.2] says the following. Assume that  $\dim \mathcal{P}(E) = \infty$  and let  $N \in \mathbb{N}$ . Then  $\text{Assoc}_N \mathcal{P}(E)$  contains a real and zerofree function if and only if for some  $\varphi \in \mathbb{R}$  the above data satisfies conditions (i) and (ii) of 2.8 and

$$\sum_n |\alpha_{\varphi,n}|^{-2N} \frac{1}{|F'_\varphi(\alpha_{\varphi,n})|^2 \sigma_{\varphi,n}} < \infty.$$

If  $\text{Assoc}_N \mathcal{P}(E)$  contains a real and zerofree function, then these conditions are fulfilled for all  $\varphi \in \mathbb{R}$ , and the function  $F_\varphi^{-1} S_\varphi$  is a unique (up to scalar multiples) real zerofree element of  $\bigcup_{M \in \mathbb{N}} \text{Assoc}_M \mathcal{P}(E)$ .

If  $\dim \mathcal{P}(E) < \infty$ , then  $\text{Assoc}_1 \mathcal{P}(E)$  always contains a real and zerofree function. In fact,  $\mathcal{P}(E)$  is of the form

$$\mathcal{P}(E) = \{U(z)p(z) : p \in \mathbb{C}[z], \deg p < \dim \mathcal{P}(E)\},$$

with  $U$  real and zerofree, see, e.g., [Wor11, Remark 3.3].

**2.16. Relations between  $\mathcal{M}_{<\infty}$ ,  $\mathcal{N}_{<\infty}$ ,  $\mathcal{HB}_{<\infty}$ .** There is a variety of (analytic and geometric) relations between the classes  $\mathcal{M}_{<\infty}$ ,  $\mathcal{N}_{<\infty}$ , and  $\mathcal{HB}_{<\infty}$ , as well as between the respective reproducing kernel space. In the present paper the following facts are used.

(i) Let  $W = (w_{ij})_{i,j=1}^2 \in \mathcal{M}_\kappa$ . Then

$$\frac{w_{11}}{w_{21}}, \frac{w_{12}}{w_{22}}, \frac{w_{12}}{w_{11}}, \frac{w_{22}}{w_{21}} \in \bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'},$$

see, e.g., [KWW06, Corollary 2.10] or [KL78].

(ii) Let  $F, G$  be real entire functions,  $F(0) = 1$ ,  $G(0) = 0$ , that have no common zeros, and set  $E := F - iG$ . Then ( $\kappa \in \mathbb{N}_0$ )

$$E \in \mathcal{HB}_\kappa \iff \frac{G}{F} \in \mathcal{N}_\kappa,$$

see, e.g., [KW99a, Remark 5.2].

- (iii) We explicitly point out the next fact, which follows by combining items (i) and (ii). If  $W \in \mathcal{M}_{<\infty}$ , then  $E_+ := w_{11} - iw_{12}, E_- := w_{22} + iw_{21} \in \mathcal{HB}_{<\infty}$ , and  $\text{ind}_- E_+, \text{ind}_- E_- \leq \text{ind}_- W$ .

§3. THE INDEFINITE ANALOGUES OF KREIN’S THEOREMS

First, we give an indefinite analogue of Theorem 2.2. There, we include an additional item (III), because it provides an easier accessible condition on  $F$  and because it makes the proof more transparent.

**3.1. Theorem.** *Let  $F$  be a real entire function with  $F(0) = 0$  or  $F(0) = 1$ . The following statements are equivalent.*

- (I) *There exists a matrix  $W \in \mathcal{M}_{<\infty}$  such that  $F$  is an entry of  $W$ .*
- (II) *Denote by  $(\alpha_n)$  the (finite or infinite) sequence of all nonzero real and simple zeros of  $F$ . Then*

$$(3.1) \quad \exists N \in \mathbb{N} : \sum_n \frac{1}{|F'(\alpha_n)| \cdot |\alpha_n|^{N+1}} < \infty,$$

and the function  $\frac{1}{F}$  has the expansion

$$(3.2) \quad \frac{1}{F(z)} = R(z) + \sum_n \frac{1}{F'(\alpha_n)} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \dots + \frac{z^{N-1}}{\alpha_n^N} \right],$$

with some rational function  $R$ .

- (III) (a) *All but finitely many zeros of  $F$  are real and simple.*  
*Denote by  $(\alpha_n)$  the sequence of all nonzero real and simple zeros of  $F$ , let  $(\gamma_j)$  be the remaining nonzero zeros (with multiplicities  $d_j \in \mathbb{N}$ ), and let  $d_0 \in \mathbb{N}_0$  be the multiplicity of 0 as a zero of  $F$ . Moreover, denote by  $\alpha_n^+$  and  $\alpha_n^-$  the sequences of positive or negative, respectively, elements of  $(\alpha_n)$  arranged according to increasing modulus. Then:*
  - (b) *The limits<sup>9</sup>  $\lim_n \frac{n}{\alpha_n^+}$  and  $\lim_n \frac{n}{\alpha_n^-}$  exist in  $[0, \infty)$  and are equal.*
  - (c) *The limit  $\lim_{R \rightarrow \infty} \sum_{|\alpha_n| \leq R} \frac{1}{\alpha_n}$  exists in  $\mathbb{R}$ .*
  - (d) *The function  $F$  has the representation*

$$F(z) = \frac{F^{(d_0)}(0)}{d_0!} z^{d_0} \cdot \prod_j \left( 1 - \frac{z}{\gamma_j} \right)^{d_j} \cdot \lim_{R \rightarrow \infty} \prod_{|\alpha_n| \leq R} \left( 1 - \frac{z}{\alpha_n} \right).$$

- (e) *We have*

$$(3.3) \quad \exists N \in \mathbb{N}, \sigma_n > 0 : \sum_n \frac{\sigma_n}{|\alpha_n|^{N+1}} < \infty, \quad \sum_n \frac{1}{\sigma_n |F'(\alpha_n)|^2 \cdot |\alpha_n|^{N+1}} < \infty.$$

The proof of this result requires some preparation. First, we provide a version of [Kre47, Theorem 4], see also [Lev80, V.6. Theorem 13].

**3.2. Lemma.** *Let  $(\alpha_n)$  be a (finite or infinite) sequence of pairwise distinct and nonzero real numbers, let  $(\tau_n)$  be a corresponding sequence of real numbers, let  $N \in \mathbb{N}$ , and let  $R$  be a rational function. Assume that*

$$(3.4) \quad \sum_n \left| \frac{\tau_n}{\alpha_n^{N+1}} \right| < \infty,$$

<sup>9</sup>Again we tacitly assume that the limit of a finite sequence is equal to zero.

and consider the function<sup>10</sup>

$$(3.5) \quad \begin{aligned} f(z) &:= R(z) + \sum_n \tau_n \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \cdots + \frac{z^{N-1}}{\alpha_n^N} \right] \\ &= R(z) + z^N \sum_n \frac{\tau_n}{\alpha_n^N (z - \alpha_n)}. \end{aligned}$$

Then the meromorphic functions  $f|_{\mathbb{C}^+}$  and  $f|_{\mathbb{C}^-}$  are of bounded type in the respective half-planes.

Krein's theorem [Kre47, Theorem 4] is much stronger in the sense that also nonreal points  $\alpha_n$  satisfying the Blaschke condition are permitted. On the other hand, it is assumed a priori that  $f$  is the inverse of an entire function, i.e., has no zeros. In the above version this is not required.

*Proof of Lemma 3.2.* We rewrite

$$f(z) = R(z) + z^{N-1} \left( \sum_{\tau_n \alpha_n^{N-1} < 0} \frac{\tau_n}{\alpha_n^{N-1}} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} \right] - \sum_{\tau_n \alpha_n^{N-1} > 0} \frac{-\tau_n}{\alpha_n^{N-1}} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} \right] \right).$$

Each of the two sums is analytic in  $\mathbb{C}^+$ , and has nonnegative imaginary part throughout this half-plane. Thus, it is of bounded type, cf. 2.9, (ii).  $\square$

Second, we need a variant of an argument that is around (at least) since the basic paper [Kre47] was published, and repeatedly appeared more or less explicitly in the literature (see, e. g., [dB59, Lemma 2], [BP95, Lemma 6.3], or [Bak98, Theorem 3.1]). The formulation tailored to our present needs reads as follows.

**3.3. Lemma.** *Let  $F$  be a real entire function of Cartwright class, such that all but finitely many zeros of  $F$  are real and simple, and such that (3.1) is true. Then  $\frac{1}{F}$  has an expansion (3.2).*

*Proof.* The function  $F$  is of bounded type in  $\mathbb{C}^+$  and  $\mathbb{C}^-$ . Due to its representation (2.3), we have  $\lim_{|y| \rightarrow \infty} |F(iy)| = \infty$ .

Choose  $N \in \mathbb{N}$  in accordance with (3.1), and consider the function

$$H(z) := \sum_n \frac{1}{F'(\alpha_n)} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \cdots + \frac{z^{N-1}}{\alpha_n^N} \right].$$

By Lemma 3.2,  $H$  is of bounded type in  $\mathbb{C}^+$  and  $\mathbb{C}^-$ . By bounded convergence (remember (3.5)), we have  $\lim_{|y| \rightarrow \infty} |y|^{-(N+1)} |H(iy)| = 0$ .

Let  $R_0$  be the sum of all principal parts of the Laurent expansions of  $\frac{1}{F}$  at the poles different from the points  $\alpha_n$ . Then  $R_0$  is rational, hence of bounded type, and  $\lim_{|y| \rightarrow \infty} R_0(iy) = 0$ .

The function  $L := \frac{1}{F} - R_0 - H$  is entire, has bounded type in  $\mathbb{C}^+$  and  $\mathbb{C}^-$ , and satisfies  $\lim_{|y| \rightarrow \infty} |y|^{-(N+1)} |L(iy)| = 0$ . Referring to Krein's theorem [RR94, Theorem 6.18], cf. (2.4), we see that  $L$  is of minimal exponential type. The Phragmen–Lindelöf principle (e.g., in the form [Lev80, I.14. Corollary] adapted to an appropriate power) implies that  $L$  is a polynomial of degree at most  $N$ . We see that  $\frac{1}{F}$  admits an expansion (3.2).  $\square$

Now the equivalence of (II) and (III) in Theorem 3.1 is nearly obvious. The implication “(I)  $\Rightarrow$  (III)” is deduced with mainly standard methods, and “(III)  $\Rightarrow$  (I)” is established with help of [Wor11].

<sup>10</sup>Due to (3.4), the sum converges locally uniformly off the points  $\alpha_n$ . It represents a meromorphic function in  $\mathbb{C}$ , whose poles are all simple and located at the points  $\alpha_n$ .

*Proof of Theorem 3.1.* If  $F$  is a polynomial, each of statements (I), (II), and (III), is true. Thereby, (II) and (III) are trivial, for (I) remember 2.6. Hence, throughout the proof we may assume that  $F$  is transcendental.

*Step 1:* “(II)  $\Rightarrow$  (III)”. Only the poles of  $R$  may give rise to nonreal poles, or to poles with multiplicity greater than 1 of  $\frac{1}{F}$ . Thus (a) holds true. Lemma 3.2 together with Krein’s theorem recalled in 2.10 shows that  $F$  is of Cartwright class. Hence, we have (b), (c), and (d), cf. 2.8. Choose  $N \in \mathbb{N}$  according to (3.1) and set

$$\sigma_n := \frac{1}{|F'(\alpha_n)|}.$$

Then (3.3) is immediate from (3.1).

*Step 2:* “(III)  $\Rightarrow$  (II)”. Choose  $N, \sigma_n$  in accordance with (3.3). Since  $x + \frac{1}{x} \geq 1$  for all  $x > 0$ , writing relations (3.3) in the form

$$\sum_n \sigma_n |F'(\alpha_n)| \cdot \frac{1}{|F'(\alpha_n)| |\alpha_n|^{N+1}} < \infty, \quad \sum_n \frac{1}{\sigma_n |F'(\alpha_n)|} \cdot \frac{1}{|F'(\alpha_n)| |\alpha_n|^{N+1}} < \infty,$$

yields (3.1). It was shown in [LW02, Lemma 5.5]<sup>11</sup> that (a)–(d) in conjunction with (3.1) imply that  $F$  is of Cartwright class. Now Lemma 3.3 guarantees (3.2).

*Step 3:* “(I)  $\Rightarrow$  (III)”. We restrict the explicit proof to the case where  $F(0) = 1$ ; the case where  $F(0) = 0$  is treated in the same way. Moreover, by 2.7, we may restrict ourselves to considering the left upper entry of a matrix of class  $\mathcal{M}_{<\infty}$ .

Let  $W \in \mathcal{M}_{<\infty}$  be given, set  $\kappa := \text{ind}_- W$ , and consider the function  $F := w_{11}$ . By [LW13a, Proposition 2.7],<sup>12</sup> this function is of Cartwright class and, hence, satisfies (b)–(d), cf. 2.8.

The functions  $\frac{w_{12}}{F}$  and  $-\frac{w_{21}}{F}$  both belong to  $\bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'}$ , cf. 2.16. Since  $F$  and  $w_{12}$  (and  $F$  and  $w_{21}$ , respectively) have no common zeros, (a) follows, cf. 2.11. Moreover, by (2.6),

$$\sum_n \left| \frac{w_{12}(\alpha_n)}{F'(\alpha_n)} \right| \frac{1}{|\alpha_n|^{2(\kappa+1)}} < \infty, \quad \sum_n \left| \frac{w_{21}(\alpha_n)}{F'(\alpha_n)} \right| \frac{1}{|\alpha_n|^{2(\kappa+1)}} < \infty.$$

Set  $N := 2\kappa + 1$  and  $\sigma_n := \left| \frac{w_{12}(\alpha_n)}{F'(\alpha_n)} \right|$ . Then the first relation in (3.3) holds true. We have

$$(3.6) \quad 1 = \det W(\alpha_n) = -w_{12}(\alpha_n)w_{21}(\alpha_n),$$

whence

$$\left| \frac{w_{21}(\alpha_n)}{F'(\alpha_n)} \right| = \frac{1}{\sigma_n |F'(\alpha_n)|^2}.$$

Thus, the second relation in (3.3) is also fulfilled.

<sup>11</sup>Probably this fact has a longer history, but we are not aware of another explicit reference.

<sup>12</sup>For asymptotically well-behaved matrices  $W$ , this also follows by combining [KL78, Satz 4.2] with [KL78, §6.2].

*Step 4:* “(III)  $\Rightarrow$  (I)”. Again, we restrict explicit proof to the case where  $F(0) = 1$ . Choose  $N \in \mathbb{N}$  and  $\sigma_n > 0$  such that (3.3) is true. Set

$$q_0(z) := \sum_n \frac{\sigma_n}{|\alpha_n|^{N-1}} \left[ \frac{1}{\alpha_n - z} - \frac{1}{\alpha_n} \right] + \sum_{\substack{j: \gamma_j \in \mathbb{R} \\ d_j \text{ odd}}} \left[ \frac{1}{\gamma_j - z} - \frac{1}{\gamma_j} \right],$$

$$p(z) := \prod_{\gamma_j \in \mathbb{R}} (z - \gamma_j)^{\lfloor \frac{d_j}{2} \rfloor} \cdot \prod_{\text{Im } \gamma_j > 0} (z - \gamma_j)^{d_j},$$

$$q(z) := \frac{1}{p(z)p^\#(z)} \cdot q_0(z), \quad G(z) := F(z)q(z),$$

$$M := N + \sum_{\gamma_j \in \mathbb{R}} \left\lfloor \frac{d_j}{2} \right\rfloor + \sum_{\text{Im } \gamma_j > 0} d_j.$$

The function  $q_0$  is well defined by convergence of the first sum in (3.3), and is meromorphic in the whole plane. Moreover, it belongs to  $\mathcal{N}_0$ . It follows that  $q \in \mathcal{N}_{<\infty}$ , cf. 2.13. We have

$$\text{Res}(q; \alpha_n) = \frac{-\sigma_n}{|\alpha_n|^{N-1}} \cdot \frac{1}{|p(\alpha_n)|^2},$$

and, therefore,

$$(3.7) \quad \lim_{n \rightarrow \infty} \frac{|\text{Res}(q; \alpha_n)|}{\sigma_n} \cdot \frac{|\alpha_n|^{2M}}{|\alpha_n|^{N+1}} = 1.$$

The functions  $F$  and  $G$  have no common zeros and  $G(0) = 0$ . By what we recalled in 2.16,

$$E := F - iG \in \mathcal{HB}_{<\infty}.$$

Using (3.7), the convergence of the second sum in (3.3) implies

$$\sum_n \alpha_n^{-2M} \frac{1}{|F'(\alpha_n)|^2 |\text{Res}(q; \alpha_n)|} < \infty.$$

Remembering our present hypothesis (b) and (c), we see that [Wor11, Theorem 3.2] is applicable with the function  $E$  and the angle  $\varphi = \frac{\pi}{2}$ , cf. 2.15. By (d), it follows that  $1 \in \text{Assoc}_M \mathcal{P}(E)$ . Now [Wor11, Proposition 6.1], cf. 2.15, provides a matrix  $W \in \mathcal{M}_{<\infty}$  with  $(1, 0)W = (F, G)$ .  $\square$

Now we proceed to the indefinite versions of Theorems 2.1 and 2.3.

**3.4. Theorem.** *Let  $F$  and  $G$  be two real entire functions with  $(F(0), G(0)) = (0, 1)$ . Then a matrix  $W \in \mathcal{M}_{<\infty}$  such that  $(F, G) = (0, 1)W$  exists if and only if the following conditions  $(\alpha)$ – $(\delta)$  are satisfied.*

- ( $\alpha$ )  $F$  and  $G$  have no common zeros, and all but finitely many zeros of  $F$  and  $G$  are real and simple.
- ( $\beta$ ) The reproducing kernel

$$N_{\frac{G}{F}}(w, z) := \frac{1}{z - \bar{w}} \left( \frac{G(z)}{F(z)} - \frac{G(\bar{w})}{F(\bar{w})} \right)$$

has a finite number of negative squares.

- ( $\gamma$ ) The sequences  $(\alpha_n)$  and  $(\beta_n)$  of all nonzero real and simple zeros of  $F$  and  $G$ , respectively, satisfy

$$\exists N \in \mathbb{N} : \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{N+1}} < \infty, \quad \sum_n \frac{1}{|F(\beta_n)G'(\beta_n)| |\beta_n|^{N+1}} < \infty.$$

(δ) The function  $\frac{1}{FG}$  has the expansion

$$\frac{1}{F(z)G(z)} = R(z) + \sum_n \frac{1}{F'(\alpha_n)G(\alpha_n)} \left[ \frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \dots + \frac{z^{N-1}}{\alpha_n^N} \right] + \sum_n \frac{1}{F(\beta_n)G'(\beta_n)} \left[ \frac{1}{z - \beta_n} + \frac{1}{\beta_n} + \frac{z}{\beta_n^2} + \dots + \frac{z^{N-1}}{\beta_n^N} \right],$$

with some rational function  $R$ .

**3.5. Theorem.** Let  $F$  be an entire function with  $F(0) = 0$ , and assume that  $F$  is of Cartwright class. A real entire function  $G$ ,  $G(0) = 1$ , forms together with  $F$  the second row of some matrix  $W \in \mathcal{M}_{<\infty}$  if and only if conditions  $(\alpha)$  and  $(\beta)$  of Theorem 3.4 and the following condition  $(\gamma')$  are fulfilled.

$(\gamma')$  The sequence  $(\alpha_n)$  of all nonzero real and simple zeros of  $F$  satisfies

$$\exists N \in \mathbb{N} : \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{N+1}} < \infty.$$

A similar statement is valid when we regard  $G$  as fixed and  $F$  as varying.

In the positive definite case this result contains a slight improvement of Theorem 2.3; the a priori hypothesis on  $F$  is slightly weakened.

**3.6. Remark.** Following the presentation in [Kre52], we have formulated Theorem 3.4 and 3.5 for  $(F, G)$  being the second row (or the left lower element, respectively) of some matrix  $W \in \mathcal{M}_{<\infty}$ . From 2.7 it is obvious that the corresponding statements are true for the first row, or the first or the second column, instead of the second row.

The proof of Theorem 3.4 and 3.5 is carried out by using essentially the same arguments as in the proof of Theorem 3.1.

*Proof of Theorems 3.4 and 3.5.* For the “only if” part, assume that  $W \in \mathcal{M}_{<\infty}$  is given, and consider  $(F, G) := (0, 1)W$ . Condition  $(\alpha)$  follows, because  $\det W = 1$  and  $\frac{G}{F} \in \mathcal{N}_{<\infty}$ , cf. 2.11. Condition  $(\beta)$  holds by 2.16. By [Wor11, Proposition 6.1], cf. 2.15, there exists  $M \in \mathbb{N}$  with  $1 \in \text{Assoc}_M \mathcal{P}(E)$  where  $E := G + iF$  (apply 2.7 to switch the lower and upper row). Applying [Wor11, Theorem 3.2], cf. 2.15, with the function  $E$  and the angle  $\varphi$  equal to  $\frac{\pi}{2}$  gives

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{2M}} = \sum_n \alpha_n^{-2M} \frac{1}{|F'(\alpha_n)|^2 |\text{Res}(\frac{G}{F}; \alpha_n)|} < \infty.$$

This is  $(\gamma')$  with  $N := 2M - 1$ . Using the angle  $\varphi = 0$  gives

$$\sum_n \frac{1}{|F(\beta_n)G'(\beta_n)| |\beta_n|^{2M}} = \sum_n |\beta_n|^{-2M} \frac{1}{|G'(\beta_n)|^2 |\text{Res}(-\frac{F}{G}; \beta_n)|} < \infty,$$

and we see that even  $(\gamma)$  is true. By [LW13a, Proposition 2.7], the functions  $F$  and  $G$  are of Cartwright class. Hence, also  $FG$  is of Cartwright class. Due to  $(\gamma)$ , the hypothesis required to apply Lemma 3.3 is satisfied, and  $(\delta)$  follows.

For the “if” part, assume that  $F$  and  $G$  are given and satisfy the hypothesis of either Theorem 3.4 or of Theorem 3.5. In the latter case,  $F$  is of Cartwright class directly from the hypothesis. In the first case, we use Krein’s theorem (or Lemma 3.2) to conclude that the function  $FG$  is of bounded type in  $\mathbb{C}^+$  and  $\mathbb{C}^-$ . The function  $\frac{G}{F}$  belongs to  $\mathcal{N}_{<\infty}$ , and thus has the same property, cf. 2.13. It follows that  $F^2$  is of Cartwright class, and hence so is  $F$ . Again using  $(\beta)$ , we see that also  $G$  is of Cartwright class. From  $(\alpha)$  and  $(\beta)$ ,

$$E := G + iF \in \mathcal{HB}_{<\infty},$$

cf. 2.16, (ii). By  $(\gamma')$  (hence also by the stronger condition  $(\gamma)$ ), the hypothesis necessary to apply [Wor11, Theorem 3.2] with  $E$  and  $\varphi = \frac{\pi}{2}$  is fulfilled, cf. 2.15. Now [Wor11, Proposition 6.1], cf. 2.15, provides us with a matrix  $W \in \mathcal{M}_{<\infty}$  such that  $(G, -F) = (1, 0)W$ . It remains to apply 2.7 in order to pass to the lower row.  $\square$

#### §4. ON THE GEOMETRY OF REPRODUCING KERNEL SPACES

Consider a matrix  $W \in \mathcal{M}_{<\infty}$ . Then the reproducing kernel  $H_W(w, z)$  defined in (1.2) generates a reproducing kernel Pontryagin space  $\mathcal{K}(W)$  whose elements are 2-vector-valued entire functions. This space is obtained as the Pontryagin space completion of the linear space

$$\text{span} \left\{ H_W(w, \cdot) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : w \in \mathbb{C}, \alpha, \beta \in \mathbb{C} \right\}$$

that is endowed with an inner product  $[\cdot, \cdot]$  defined by linearity and the formula

$$\left[ H_W(w, \cdot) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, H_W(w', \cdot) \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \right] := \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}^* H_W(w, \cdot) \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$w, w' \in \mathbb{C}, \alpha, \beta, \alpha', \beta' \in \mathbb{C},$$

see, e.g., [ADRdS97, Theorem 1.1.3]. Basic theorems on spaces  $\mathcal{K}(W)$ , their relationship with de Branges–Pontryagin spaces on the one hand, and their relationship with the spectral theory of entire operators on the other, can be found in [KW99a, KW99b] for the former, and in [KW98a, KL78] for the latter. Standard literature dealing with the positive definite case is, e.g., [dB68] or [GG97, KL14].

**4.1.** *The subspaces  $\mathcal{K}_{\pm}(W)$ . Let  $W \in \mathcal{M}_{<\infty}$ . In the structure theory of the space  $\mathcal{K}(W)$  certain subspaces play a role. Namely (here “cls” stands for “closed linear span”)*

$$\mathcal{K}_+(W) := \text{cls} \left\{ H_W(w, \cdot) \begin{pmatrix} 1 \\ 0 \end{pmatrix} : w \in \mathbb{C} \right\},$$

$$\mathcal{K}_-(W) := \text{cls} \left\{ H_W(w, \cdot) \begin{pmatrix} 0 \\ 1 \end{pmatrix} : w \in \mathbb{C} \right\}.$$

*As we observed in 2.16, the functions  $E_+ := w_{11} - iw_{12}$  and  $E_- := w_{22} + iw_{21}$  belong to  $\mathcal{HB}_{<\infty}$ . Hence, they generate de Branges–Pontryagin spaces  $\mathcal{P}(E_+)$  and  $\mathcal{P}(E_-)$ . Denote by  $\pi_+$  the projection of a vector function onto its first component, i.e.,  $\pi_+ : \begin{pmatrix} F \\ G \end{pmatrix} \mapsto F$ . In [KW99a, Lemma 8.6] it was shown that  $\pi_+|_{\mathcal{K}_+(W)}$  maps  $\mathcal{K}_+(W)$  isometrically and surjectively onto  $\mathcal{P}(E_+)$ . Moreover, we have  $\ker(\pi_+|_{\mathcal{K}(W)}) = \mathcal{K}_+(W)^\perp$ . In particular,*

$$\ker(\pi_+|_{\mathcal{K}_+(W)}) = \mathcal{K}_+(W)^\circ.$$

*A similar statements hold true for the second row of  $W$ , i.e., with “+” everywhere replaced by “–”, and  $\pi_-$  being the projection onto the second component.*

The geometry of  $\mathcal{K}_+(W)$  and  $\mathcal{K}_-(W)$  as closed subspaces of the Pontryagin space  $\mathcal{K}(W)$  has consequences for the structure theory of  $W$ . For instance, two crucial result in this context are [KW99b, Theorem 5.7] and [Wor11, Lemma 6.3].

In the spirit of the present paper, the following question appears naturally.

*What values may the quantities*

$$\text{ind}_- \mathcal{K}(W), \quad \text{ind}_- \mathcal{K}_-(W), \quad \dim \mathcal{K}_-(W)^\circ$$

*take when  $W$  varies through all matrices having a prescribed second row (or one prescribed entry in this row)?*

In the two theorems below we give the answer. First, we regard the second row as prescribed.

**4.2. Theorem.** *Let  $F$  and  $G$  be entire functions subject to the conditions of either Theorem 3.4 or Theorem 3.5, so that there exist matrices of class  $\mathcal{M}_{<\infty}$  that have  $(F, G)$  as their second row. Denote by  $(\alpha_n)$  the sequence of all nonzero real and simple zeros of  $F$ . Then the following statements hold true.*

(i) *We have*

$$\{ \text{ind}_- \mathcal{K}(W) : W \in \mathcal{M}_{<\infty}, (0, 1)W = (F, G) \} = \left[ \text{ind}_- \frac{G}{F} + \dim \mathcal{K}_-(W)^\circ, \infty \right) \cap \mathbb{N}_0.$$

(ii) *For each  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W = (F, G)$ , we have*

$$\text{ind}_- \mathcal{K}_-(W) = \text{ind}_- \frac{G}{F}.$$

(iii) *For each  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W = (F, G)$ , we have*

$$\dim \mathcal{K}_-(W)^\circ = \min \left\{ M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{2(M+1)}} < \infty \right\}.$$

Second, we regard the left lower entry as prescribed. The corresponding result holds true if we fix the right lower entry; we do not give details.

**4.3. Theorem.** *Let  $F, F(0) = 0$ , be an entire function satisfying one (and hence each) condition of Theorem 3.1, so that there exist matrices of class  $\mathcal{M}_{<\infty}$  that have  $F$  as their left lower entry. Let the notation  $\alpha_n, \gamma_j, d_j$ , etc. be as in Theorem 3.1, and set*

$$\nu := \min \left\{ N \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)| \cdot |\alpha_n|^{N+1}} < \infty \right\},$$

$$\delta := \left( \left\lfloor \frac{d_0}{2} \right\rfloor + \begin{cases} 1 & \text{if } d_0 \text{ odd, } F^{(d_0)}(0) > 0, \\ 0 & \text{otherwise,} \end{cases} \right) + \sum_{\gamma_j \in \mathbb{R}} \left\lfloor \frac{d_j}{2} \right\rfloor + \sum_{\text{Im } \gamma_j > 0} d_j.$$

*Then the following statements are valid.*

(i) *If  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F$ , then*

$$(4.1) \quad \text{ind}_- \mathcal{K}(W) \geq \text{ind}_- \mathcal{K}_-(W) + \dim \mathcal{K}_-(W)^\circ \geq \delta + \nu - 1,$$

$$(4.2) \quad \text{ind}_- \mathcal{K}_-(W) \geq \delta.$$

(ii) *If  $\alpha, \beta, \gamma \in \mathbb{N}_0$  satisfy*

$$(4.3) \quad \alpha \geq \beta + \gamma \geq \delta + \nu - 1, \quad \beta \geq \delta,$$

*then there exists  $W \in \mathcal{M}_{<\infty}$  such that*

$$(0, 1)W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F,$$

$$\text{ind}_- \mathcal{K}(W) = \alpha, \quad \text{ind}_- \mathcal{K}_-(W) = \beta, \quad \dim \mathcal{K}_-(W)^\circ = \gamma.$$

We immediately point out one interesting consequence of this theorem: since in the first relation in (4.3) only the sum  $\beta + \gamma$  appears, one can trade off negative index against dimension of degeneracy.

**4.4. Corollary.** *Let  $F, F(0) = 0$ , be an entire function satisfying one (and hence each) condition of Theorem 3.1, so that there exist matrices of class  $\mathcal{M}_{<\infty}$  that have  $F$  as their left lower entry. Then there exist matrices  $W_1, W_2 \in \mathcal{M}_{<\infty}$  such that*

$$(0, 1)W_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F, \quad \text{ind}_- \mathcal{K}_-(W_1) = \delta.$$

$$(0, 1)W_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F, \quad \dim \mathcal{K}_-(W_2)^\circ = 0.$$

The choice of  $W_2$  can be made so that  $\mathcal{K}_-(W_2) = \mathcal{K}(W_2)$ .

The rest of this section is devoted to the proof of these results.

4.5. *Remark.* In the following discussion we extensively use the terminology and results of the theory of indefinite canonical systems and maximal chains of matrices. We refer the reader who wishes to dive into the details to [Wor11, §4], [LW13b, §2], or (the most exhaustive reference) [KW11, §2, §3]. There all definitions and a review of most relevant theorems can be found. Detailed references will be provided throughout the subsequent text.

To the reader who is not interested in details we suggest to skip Proposition 4.5, believe in Theorem 4.2, (i), and proceed directly to the proofs of Theorem 4.2, (ii) and (iii), and Theorem 4.3 and its corollary, which start on p. 114. To make the dependencies precise:

Theorem 4.2, (ii) and (iii) Theorem 4.3, Steps 1,2 Corollary 4.4	Proofs can be read without further prerequisites
Proposition 4.5, Theorem 4.2, (i)	Requires familiarity with indefinite canonical systems
Theorem 4.3, Step 3	No further prerequisites required, but uses Theorem 4.2, (i)

In this context we must say it very clearly that the theory of indefinite canonical systems has entered through the backdoor from the very beginning. The *proofs* in our previous work [Wor11] depend highly on this theory. Only, the theorems required from [Wor11] in the present paper can be *formulated* without using such notions (and this we did in 2.15).

Throughout the following we agree on a generic notation: if  $p$  is a polynomial with real coefficients and without constant term, we denote by  $P$  the matrix

$$P(z) := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}.$$

Let  $W \in \mathcal{M}_{<\infty}$ . From [KW99a, Corollary 9.8] we know that the totality of all matrices of class  $\mathcal{M}_{<\infty}$  that have the same second row as  $W$  is given by

$$(4.4) \quad \{\widetilde{W} \in \mathcal{M}_{<\infty} : (0, 1)\widetilde{W} = (0, 1)W\} = \{PW : p \in \mathbb{R}[z], p(0) = 0\}.$$

If  $\mathcal{K}_-(W)^\circ = \{0\}$ , then the spaces  $\mathcal{K}(PW)$  are accessible to explicit computation. On the other hand, if  $\mathcal{K}_-(W)$  degenerates, this is not anymore the case and things are getting more involved. In the following we describe what it means to pass from  $W$  to  $PW$  in terms of the associated general Hamiltonians. Thereby, we focus on the more difficult case where  $\mathcal{K}_-(W)^\circ \neq \{0\}$ . We also include the limit point situation (which means to work with Weyl coefficients instead of matrices of class  $\mathcal{M}_{<\infty}$ ).

**4.6. Proposition.** *Let  $\mathfrak{h}$  be a general Hamiltonian ([KW11, Definition 3.35]) given by the data<sup>13</sup>*

$$\sigma_0, \dots, \sigma_{n+1}, \quad H_0, \dots, H_n, \quad \ddot{o}_i, b_{i,j}, d_{i,j}, \quad i = 1, \dots, n, \quad E.$$

<sup>13</sup>We agree that  $b_{\ddot{o}_i+1} = 0$  unless indivisible intervals ([KW11, p. 259]) adjoin to both sides of  $\sigma_i$ .

Assume that  $n \geq 2$ , that  $(\sigma_0, \sigma_1)$  is indivisible of type 0 ([KW11, p. 259]), and that  $\sigma_1$  is not the left endpoint of an indivisible interval. Let  $\mathfrak{h}'$  be another general Hamiltonian, and assume that  $\mathfrak{h}$  and  $\mathfrak{h}'$  are either both regular or both singular ([KW11, Definition 3.35, (3.4)]). Then, with (i), (ii<sub>r</sub>), (ii<sub>s</sub>) as written out below, if  $\mathfrak{h}$  is regular, then (i)  $\Leftrightarrow$  (ii<sub>r</sub>), and if  $\mathfrak{h}$  is singular, then (i)  $\Leftrightarrow$  (ii<sub>s</sub>).

(i) The general Hamiltonians  $\mathfrak{h}$  and  $\mathfrak{h}'$  differ only in their data part at their first singularity. Precisely formulated, by this we mean that there exists a reparametrization ([KW11, Remark 3.38]) of  $\mathfrak{h}'$  that is given by the data

$$(4.5) \quad \sigma_0, \dots, \sigma_{n+1}, \quad H_0, \dots, H_n, \quad \delta'_i, b'_{i,j}, d'_{i,j}, \quad i = 1, \dots, n, \quad E,$$

where

$$(4.6) \quad \delta'_i = \delta_i, b'_{i,j} = b_{i,j}, d'_{i,j} = d_{i,j} \quad \text{for } i = 2, \dots, n.$$

(ii<sub>r</sub>) There exists  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , such that the monodromy matrices (these are the matrices  $\omega(\mathfrak{B}(\mathfrak{h}))$  in [KW11, Theorem 5.1, Proposition 4.29], cf. [KW11, top of p. 226])  $W$  and  $W'$  of  $\mathfrak{h}$  and  $\mathfrak{h}'$  are related as  $W' = PW$ .

(ii<sub>s</sub>) There exists  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , such that the Weyl coefficients ([KW11, Definition 5.2])  $q_{\mathfrak{h}}$  and  $q_{\mathfrak{h}'}$  of  $\mathfrak{h}$  and  $\mathfrak{h}'$  are related as  $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$ .

In the proof we use the following lemma.

**4.7. Lemma.** Let  $W \in \mathcal{M}_{<\infty}$  with  $\mathcal{K}_-(W)^\circ \neq \{0\}$ , and let  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ . Let  $\omega$  be a finite maximal chain going down from  $W$  ([KW11, Definition 3.7, 3.1]), and denote its domain by  $[\sigma_0, \sigma_{n+1}] \setminus \{\sigma_1, \dots, \sigma_n\}$ . Moreover, set

$$\omega_p(x) := \begin{cases} \omega(x) & \text{if } x \in [\sigma_0, \sigma_1), \\ P\omega(x) & \text{if } x \in (\sigma_1, \sigma_{n+1}] \setminus \{\sigma_2, \dots, \sigma_n\}. \end{cases}$$

Then  $\omega_p$  is a finite maximal chain going down from  $PW$ .

*Proof.* By [Wor11, Lemma 6.3] (applied with  $-JWJ$ ) the interval  $[\sigma_0, \sigma_1)$  in  $\omega$  is indivisible of type 0 and  $\sigma_1$  is not the left endpoint of an indivisible interval.

Since  $PW \in \mathcal{M}_{<\infty}$ , cf. 2.6 and (2.1), there exists a finite maximal chain going down from  $PW$  ([KW11, 3.9, p. 253]). Let  $\omega'$  be one such chain. By [LW13a, 5.16 (p. 310)], this chain starts with an indivisible interval of infinite length and type 0, and its first singularity is not the left endpoint of an indivisible interval. By [LW13a, Lemma 5.7] there exists an endsection of  $\omega'$  that is a reparametrization ([KW11, Definition 3.4]) of  $P\omega|_{(\sigma_1, \sigma_{n+1}] \setminus \{\sigma_2, \dots, \sigma_n\}}$ . The left endpoint of this endsection is a singularity of  $\omega'$  because (with the notation “ $\mathfrak{t}$ ” as in [KW11, Definition 2.1, (2.0)])

$$\lim_{x \searrow \sigma_1} \mathfrak{t}[P\omega(x)] = \mathfrak{t}[P] + \lim_{x \searrow \sigma_1} \mathfrak{t}[\omega(x)] = -\infty.$$

The intermediate Weyl coefficient ([KW03, p. 284, Proposition 5.1]) of  $\omega$  at its first singularity  $\sigma_1$  is equal to the constant  $\infty$ , and  $\omega'$  has the same property. We have (with the notation “ $\star$ ” as in [KW11, p. 246])

$$\lim_{x \searrow \sigma_1} [(P\omega(x)) \star \tau] = P \star \left( \lim_{x \searrow \sigma_1} [\omega(x)] \star \tau \right) = P \star \infty = \infty, \quad \tau \in \mathbb{R} \cup \{\infty\}.$$

Hence, the left endpoint of the mentioned endsection of  $\omega'$  must be the first singularity of  $\omega'$  (argue, e.g., as in [KW11, Proposition 3.10]). It follows that  $\omega'$  and  $\omega_p$  are related by a reparametrization, in particular,  $\omega_p$  is a finite maximal chain. □

**4.8. Corollary.** *Let  $\omega$  be a maximal chain ([KW11, Definition 3.1]) defined on  $[\sigma_0, \sigma_{n+1}) \setminus \{\sigma_1, \dots, \sigma_n\}$ , assume that  $[\sigma_0, \sigma_1)$  is indivisible of type 0 and that  $\sigma_1$  is not the left endpoint of an indivisible interval. Moreover, let  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , and set*

$$\omega_p(x) := \begin{cases} \omega(x) & \text{if } x \in [\sigma_0, \sigma_1), \\ P\omega(x) & \text{if } x \in (\sigma_1, \sigma_{n+1}) \setminus \{\sigma_2, \dots, \sigma_n\}. \end{cases}$$

*Then  $\omega_p$  is a maximal chain. The Weyl coefficients ([KW11, Definition 3.5])  $q_\omega$  and  $q_{\omega_p}$  of  $\omega$  and  $\omega_p$ , respectively, are related as*

$$q_{\omega_p} = q_\omega + p.$$

*Proof.* Apply Lemma 4.6 to each beginning section of  $\omega$  (and argue using [KW11, Remark 3.15]).  $\square$

*Proof of Proposition 4.5.* First assume (i), i.e., assume that  $\mathfrak{h}$  and  $\mathfrak{h}'$  differ only in their data part at their first singularity. Without loss of generality, we may assume that the parameterization of  $\mathfrak{h}'$  is chosen so that it is given by the data (4.5) with (4.6). Let  $\omega$  and  $\omega'$  be the (finite) maximal chains constructed from  $\mathfrak{h}$  and  $\mathfrak{h}'$  (as in [KW11, Definition 5.3]). Since the Hamiltonian functions of  $\mathfrak{h}$  and  $\mathfrak{h}'$  coincide, [KW11, Corollary 5.6] implies

$$(4.7) \quad \omega|_{[\sigma_0, \sigma_1)} = \omega'|_{[\sigma_0, \sigma_1)},$$

$$(4.8) \quad \omega(x)^{-1}\omega(y) = \omega'(x)^{-1}\omega'(y), \quad \sigma_1 < x \leq y, \quad x, y \in \text{dom } \omega.$$

We are going to define two more general Hamiltonians  $\tilde{\mathfrak{h}}$  and  $\tilde{\mathfrak{h}'}$ . Set  $s_1 := \min(E \cap (\sigma_1, \sigma_2))$ , choose  $\psi \in \mathbb{R}$  such that  $s_1$  is not the right endpoint of an indivisible interval of type  $\psi$ , and set

$$\tilde{H}_1(x) := \begin{cases} H_1(x) & \text{if } x \in (\sigma_1, s_1], \\ (\cos \psi, \sin \psi)^T (\cos \psi, \sin \psi) & \text{if } x \in (s_1, \infty). \end{cases}$$

Now we define  $\tilde{\mathfrak{h}}$  and  $\tilde{\mathfrak{h}'}$  as the sets of data

$$\begin{aligned} \tilde{\mathfrak{h}} &: \sigma_0, \sigma_1, \infty, \quad H_0, \tilde{H}_1, \quad \ddot{o}_1, b_{1,j}, d_{1,j}, \quad \{\sigma_0, s_1, \infty\}, \\ \tilde{\mathfrak{h}'} &: \sigma_0, \sigma_1, \infty, \quad H_0, \tilde{H}_1, \quad \ddot{o}'_1, b'_{1,j}, d'_{1,j}, \quad \{\sigma_0, s_1, \infty\}. \end{aligned}$$

Since  $\tilde{\mathfrak{h}}$  and  $\mathfrak{h}$ , and  $\tilde{\mathfrak{h}'}$  and  $\mathfrak{h}'$ , respectively, coincide to the left of  $s_1$ , we have

$$(4.9) \quad \tilde{\omega}|_{[\sigma_0, s_1] \setminus \{\sigma_1\}} = \omega|_{[\sigma_0, s_1] \setminus \{\sigma_1\}}, \quad \tilde{\omega}'|_{[\sigma_0, s_1] \setminus \{\sigma_1\}} = \omega'|_{[\sigma_0, s_1] \setminus \{\sigma_1\}}.$$

We apply [LW13b, Corollary 5.9] with  $\tilde{\mathfrak{h}}$  and  $\tilde{\mathfrak{h}'}$ . This provides us with a polynomial  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , such that the Weyl coefficients  $q_{\tilde{\mathfrak{h}}}$  and  $q_{\tilde{\mathfrak{h}'}}$  of  $\tilde{\mathfrak{h}}$  and  $\tilde{\mathfrak{h}'}$ , respectively, are related as

$$q_{\tilde{\mathfrak{h}'}} = q_{\tilde{\mathfrak{h}}} + p.$$

Consider the maximal chain  $\tilde{\omega}_p$  defined as in Corollary 4.7 starting from  $\tilde{\omega}$ . Then the Weyl coefficient  $q_{\tilde{\omega}_p}$  is equal to  $q_{\tilde{\omega}} + p$ , and it follows that

$$q_{\tilde{\omega}_p} = q_{\tilde{\omega}} + p = q_{\tilde{\mathfrak{h}}} + p = q_{\tilde{\mathfrak{h}'}} = q_{\tilde{\omega}'}$$

Hence,  $\tilde{\omega}_p$  and  $\tilde{\omega}'$  are reparametrizations of each other ([KW11, 3.6, p. 251]).

Let  $\alpha$  be an increasing bijection of  $[\sigma_0, \infty) \setminus \{\sigma_1\}$  onto itself, such that  $\tilde{\omega}_p = \tilde{\omega}' \circ \alpha$ . The interval  $[s_1, \infty)$  is maximal indivisible in the chain  $\tilde{\omega}'$  and in the chain  $\tilde{\omega}$ , hence also in  $\tilde{\omega}_p$ . It follows that  $\alpha(s_1) = s_1$ , and hence (remember (4.9))

$$\omega'(s_1) = \tilde{\omega}'(s_1) = \tilde{\omega}_p(s_1) = P\tilde{\omega}(s_1) = P\omega(s_1).$$

Using (4.8), we see that

$$\omega'(y) = P\omega(y), \quad y \geq s_1, \quad y \in \text{dom } \omega.$$

If  $\mathfrak{h}$  is regular, we may evaluate at  $y = \sigma_{n+1}$  and show that the monodromy matrices are related as  $W' = PW$ . If  $\mathfrak{h}$  is singular, we pass to the limit as  $y \nearrow \sigma_{n+1}$  and show that the Weyl coefficients are related as  $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$ .

Second, assume that  $\mathfrak{h}$  (and hence also  $\mathfrak{h}'$ ) is regular, and that the corresponding monodromy matrices are related as  $W' = PW$  with some  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ . Let  $\omega$  and  $\omega'$  be the finite maximal chains associated with  $\mathfrak{h}$  and  $\mathfrak{h}'$  (as in [KW11, Definition 5.3]). Moreover, let  $\omega_p$  be the finite maximal chain defined as in Lemma 4.6. Then  $(\sigma_{n+1}$  and  $\sigma'_{n'+1}$  denote the maximum of the domains of  $\omega$  and  $\omega'$ , respectively)

$$\omega_p(\sigma_{n+1}) = P\omega(\sigma_{n+1}) = PW = W' = \omega'(\sigma'_{n'+1}).$$

It follows that  $\omega_p$  and  $\omega'$  are reparametrizations of each other ([KW11, 3.9, p. 253]). Let  $\mathfrak{h}_p$  denote the regular general Hamiltonian constructed from  $\omega_p$  (as in [KW10, §2, §3] using the same splitting points  $E$  as in  $\mathfrak{h}$ ). Then  $\mathfrak{h}_p$  and  $\mathfrak{h}'$  are reparametrizations of each other ([KW10, 1.3, p. 514]).

From the definition of  $\omega_p$  we have

$$\begin{aligned} \omega_p(x) &= \omega(x), \quad x \in [\sigma_0, \sigma_1], \\ \omega_p(x)^{-1}\omega_p(y) &= \omega(x)^{-1}\omega(y), \quad \sigma_1 < x \leq y, \quad x, y \in \text{dom } \omega. \end{aligned}$$

Hence, the Hamiltonian functions of  $\mathfrak{h}_p$  and  $\mathfrak{h}$  all coincide and the data parts  $\ddot{o}_i, b_{i,j}, d_{i,j}$  for  $i > 1$  also coincide (apply [KW11, 3.6, 3.9] and notice that no proper reparametrization is possible).

Finally, consider the case where  $\mathfrak{h}$  (and hence also  $\mathfrak{h}'$ ) is singular and the corresponding Weyl coefficients are related as  $q_{\mathfrak{h}'} = q_{\mathfrak{h}} + p$  with some  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ . In this case the required form of  $\mathfrak{h}'$  follows with a word-for-word repetition of the above argument, only using Corollary 4.7 instead of Lemma 4.6 and the references for existence, uniqueness, etc., for maximal chains instead of finite maximal chains. We skip the details.  $\square$

Having this result on general Hamiltonians available, we are ready for the proof of Theorem 4.2. We start with the proof of item (i), which is actually the hard part where Proposition 4.5 is needed.

*Proof of Theorem 4.2, (i).* By our a priori hypothesis on  $F$  and  $G$ , we may choose a matrix  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W = (F, G)$ . As we have readily noticed, cf. (4.4), the task is to consider all matrices of the form  $PW$  with  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ .

Assume first that  $\mathcal{K}_-(W)^\circ = \{0\}$ . Then, by [KW99a, Proposition 10.3, Corollary 10.4], there exists a matrix  $W_1$  with  $(0, 1)W_1 = (0, 1)W$  and  $\mathcal{K}_-(W_1) = \mathcal{K}(W_1)$ . Since  $W_1$  is of the form  $P_1W$  with some  $p_1 \in \mathbb{R}[z]$ ,  $p_1(0) = 0$ , we have

$$\{PW_1 : p \in \mathbb{R}[z], p(0) = 0\} = \{PW : p \in \mathbb{R}[z], p(0) = 0\}.$$

The general Hamiltonian whose monodromy matrix equals  $W_1$  does not start with an indivisible interval of type 0 ([KW99b, Lemma 7.5], [Wor11, Lemma 6.3]). Hence, by [KW11, Proposition 3.17], we have

$$\text{ind}_-(PW_1) = \text{ind}_- P + \text{ind}_- W_1.$$

Recalling what we said in 2.6 and (2.1), we obtain

$$\{\text{ind}_- PW_1 : p \in \mathbb{R}[z], p(0) = 0\} = [\text{ind}_- W_1, \infty) \cap \mathbb{N}_0.$$

However,  $\text{ind}_- W_1 = \text{ind}_- \mathcal{K}_-(W_1) = \text{ind}_- \frac{G}{F}$ .

Assume now that  $\mathcal{K}_-(W)^\circ \neq \{0\}$ . Consider a general Hamiltonian  $\mathfrak{h}$  whose monodromy matrix equals  $W$ , and denote the data that  $\mathfrak{h}$  is composed of as in (4.12). Then,

by [Wor11, Lemma 6.3],  $(\sigma_0, \sigma_1)$  is indivisible of type 0 and  $\sigma_1$  is not the left endpoint of an indivisible interval. By Proposition 4.5, the totality of general Hamiltonians with monodromy matrices  $PW$ ,  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , is given (up to reparametrization) by all sets of data (4.5) subject to (4.6). By [KW11, Proposition 4.29], we have

$$\begin{aligned} \text{ind}_- PW &= \sum_{i=1}^n \left( \Delta_i + \left\lfloor \frac{\ddot{o}'_i}{2} \right\rfloor \right) + |\{1 \leq i \leq n : \ddot{o}'_i \text{ odd, } b'_{i,1} > 0\}| \\ &= \left( \Delta_1 + \left\lfloor \frac{\ddot{o}'_1}{2} \right\rfloor \right) + \begin{cases} 1 & \text{if } \ddot{o}'_1 \text{ odd, } b'_{1,1} > 0, \\ 0 & \text{otherwise,} \end{cases} \\ &\quad + \sum_{i=2}^n \left( \Delta_i + \left\lfloor \frac{\ddot{o}_i}{2} \right\rfloor \right) + |\{2 \leq i \leq n : \ddot{o}_i \text{ odd, } b_{i,1} > 0\}|. \end{aligned}$$

Since  $\ddot{o}'_1, b'_{1,j}, d'_{1,j}$  may be chosen arbitrarily, it follows that

$$\begin{aligned} &\{ \text{ind}_- PW : p \in \mathbb{R}[z], p(0) = 0 \} \\ &= \left[ \Delta_1 + \sum_{i=2}^n \left( \Delta_i + \left\lfloor \frac{\ddot{o}_i}{2} \right\rfloor \right) + |\{2 \leq i \leq n : \ddot{o}_i \text{ odd, } b_{i,1} > 0\}|, \infty \right) \cap \mathbb{N}_0. \end{aligned}$$

By [Wor11, Lemma 6.3], we have

$$\Delta_1 = \dim \mathcal{K}_-(W)^\circ.$$

Consider the matrix  $\text{rev } W$  ([KW11, Definition 2.6]). The general Hamiltonian  $\text{rev } \mathfrak{h}$  ([KW11, Definition 3.40]) ends with an indivisible interval of infinite length and type 0, and hence  $(\text{rev } W) \star \infty$  is the intermediate Weyl coefficient of  $\text{rev } \mathfrak{h}$  at the singularity  $-\sigma_1$ . Using [KW11, Theorem 5.1], we get

$$\text{ind}_- [(\text{rev } W) \star \infty] = \sum_{i=2}^n \left( \Delta_i + \left\lfloor \frac{\ddot{o}_i}{2} \right\rfloor \right) + |\{2 \leq i \leq n : \ddot{o}_i \text{ odd, } b_{i,1} > 0\}|.$$

However, a short computation shows that  $(\text{rev } W_1) \star \infty = -\frac{G}{F}$ , implying

$$\sum_{i=2}^n \left( \Delta_i + \left\lfloor \frac{\ddot{o}_i}{2} \right\rfloor \right) + |\{2 \leq i \leq n : \ddot{o}_i \text{ odd, } b_{i,1} > 0\}| = \text{ind}_- \frac{G}{F}. \quad \square$$

The proof of items (ii) and (iii) in Theorem 4.2 is again more elementary.

*Proof of Theorem 4.2, (ii) and (iii).* Item (ii) follows since  $\pi_-$  maps  $\mathcal{K}_-(W)$  surjectively and isometrically onto  $\mathcal{P}(G + iF)$ . In fact, using 2.16, (ii), we obtain

$$\text{ind}_- \mathcal{K}_-(W) = \text{ind}_- \mathcal{P}(G + iF) = \text{ind}_-(G + iF) = \text{ind}_- \frac{G}{F}.$$

We come to item (iii). By our a priori hypothesis on  $F$  and  $G$ , there exist matrices  $W \in \mathcal{M}_{<\infty}$  such that  $(0, 1)W = (F, G)$ . Hence,  $1 \in \bigcup_{N \in \mathbb{N}} \text{Assoc}_N \mathcal{P}(G + iF)$ , cf. 2.15. If  $1 \in \text{Assoc}_1 \mathcal{P}(G + iF)$ , set  $N_0 := 0$ . Otherwise, let  $N_0$  be a unique positive integer with

$$1 \in \text{Assoc}_{N_0+1} \mathcal{P}(G + iF) \setminus \text{Assoc}_{N_0} \mathcal{P}(G + iF).$$

Then, using [Wor11, Proposition 6.1] and [KW99a, Proposition 10.3], we have

$$\dim \mathcal{K}_-(W)^\circ = N_0, \quad W \in \mathcal{M}_{<\infty}, (0, 1)W = (F, G).$$

Applying [Wor11, Theorem 3.2], cf. 2.15, with  $E := G + iF$  and  $\varphi = 0$ , we see that

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{2(N_0+1)}} < \infty, \quad \text{but} \quad \sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{2N_0}} = \infty. \quad \square$$

For the proof of Theorem 4.3, we need a preparatory lemma, which contains a refinement of the argument used in the proof of Theorem 3.1, Steps 3, 4.<sup>14</sup> We denote by  $\mathfrak{d}_f$  the divisor of a meromorphic function  $f$ , i.e.,  $\mathfrak{d}_f(w)$  is the smallest integer  $n$  such that the  $n$ th coefficient in the Laurent expansion of  $q$  at  $w$  is nonzero (see, e.g., [Rem98, Chapter 3, §1.1]).

**4.9. Lemma.** *Let  $F$  be as in Theorem 4.3. Then the assignment*

$$q \mapsto Fq$$

*establishes a bijection between the sets*

$$(4.10) \quad \left\{ q \in \mathcal{N}_{<\infty} : q \text{ meromorphic in } \mathbb{C}, -\min\{\mathfrak{d}_q, 0\} = \mathfrak{d}_F, \lim_{z \rightarrow 0} [F(z)q(z)] = 1, \right. \\ \left. \exists M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 \cdot |\text{Res}(q; \alpha_n)| \cdot |\alpha_n|^{2(M+1)}} < \infty \right\}$$

and

$$(4.11) \quad \{G \text{ entire} : \exists W \in \mathcal{M}_{<\infty} \text{ with } (0, 1)W = (F, G)\}.$$

Thereby, for each  $q$  in the set (4.10) and each  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W = (F, Fq)$ , we have

$$(4.12) \quad \text{ind}_- \mathcal{K}_-(W) = \text{ind}_- q,$$

$$(4.13) \quad \dim \mathcal{K}_-(W)^\circ = \min \left\{ M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 \cdot |\text{Res}(q; \alpha_n)| \cdot |\alpha_n|^{2(M+1)}} < \infty \right\}.$$

*Proof.* Since  $F$  satisfies the assumptions of Theorem 3.1,  $F$  is of Cartwright class, all but finitely many zeros of  $F$  are real and simple, and (3.1) holds true.

Let  $q$  be an element of the set (4.10), and let  $G := Fq$ . Our aim is to apply Theorem 3.5. The function  $G$  is entire, has no common zeros with  $F$ , and satisfies  $G(0) = 1$ . The zeros of  $G$  coincide with the zeros of  $q$ , counting multiplicities. Since  $q \in \mathcal{N}_{<\infty}$ , all but finitely many zeros of  $G$  are real and simple, cf. 2.11. Clearly,  $\frac{G}{F} = q \in \mathcal{N}_{<\infty}$ . Moreover, we have

$$\text{Res}(q; \alpha_n) = \frac{G(\alpha_n)}{F'(\alpha_n)},$$

and hence (with some appropriate  $N' \in \mathbb{N}$ )

$$\sum_n \frac{1}{|F'(\alpha_n)G(\alpha_n)| |\alpha_n|^{N'}} < \infty.$$

Altogether, Theorem 3.5 applies, yielding a matrix  $W \in \mathcal{M}_{<\infty}$  such that  $(0, 1)W = (F, G)$ . Hence, the assignment  $q \mapsto Fq$  indeed maps (4.10) into (4.11). Clearly, it is injective.

Assume that  $G$  is entire and  $(F, G) = (0, 1)W$  for some  $W \in \mathcal{M}_{<\infty}$ . The function  $q := \frac{G}{F}$  is meromorphic in  $\mathbb{C}$  and satisfies  $-\min\{\mathfrak{d}_q, 0\} = \mathfrak{d}_F$  because  $F$  and  $G$  have no common zeros. Moreover,  $\lim_{z \rightarrow 0} [F(z)q(z)] = G(0) = 1$ . By statement  $(\beta)$  of Theorem 3.4, we

<sup>14</sup>At this point our presentation contains a slight redundancy. Accurately tracing back the logic of the proofs, one sees that a part of Steps 3, 4 in the proof of Theorem 3.1 could be skipped and substituted by the corresponding argument from Lemma 4.8. For the following reason we decided to arrange matters in this way, and accept a slight repetition. We find it interesting that a proof of the pure existence statement (I) Theorem 3.1, is not only technically simpler, but also can be carried out by using much coarser methods compared to what is required to get hands on negative indices and dimensions of degeneracy. The argument in the proof of Lemma 4.8 is not “just the same, only more complicated,” compared to the proof of Steps 3, 4 in Theorem 3.1 for instance, there we constructed the function  $q$  in a multiplicative way, using the trick (3.6) and referring only to the rough estimate (2.6).

have  $q \in \mathcal{N}_{<\infty}$ . Finally, statement  $(\gamma)$  of Theorem 3.4, shows that  $q$  belongs to the set (4.10). Thus  $q \mapsto Fq$  map (4.10) surjectively onto (4.11).

Formulas (4.12) and (4.13) are immediate from Theorem 4.2, statements (ii) and (iii).  $\square$

*Proof of Theorem 4.3.*

*Step 1:* Assume that  $W \in \mathcal{M}_{<\infty}$  with  $(0, 1)W \binom{1}{0} = F$  is given. Set

$$G := (0, 1)W \binom{0}{1}, \quad q := \frac{G}{F},$$

$$m_0 := \min \left\{ m \in \mathbb{N}_0 : \sum_n \frac{|\operatorname{Res}(q; \alpha_n)|}{|\alpha_n|^{2(m+1)}} < \infty \right\},$$

$$M_0 := \min \left\{ M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 |\operatorname{Res}(q; \alpha_n)| |\alpha_n|^{2(M+1)}} < \infty \right\}.$$

Then  $M_0 = \dim \mathcal{K}_-(W)^\circ$ , cf. (4.13). We use (2.5) to estimate  $\operatorname{ind}_- q$ . For this, we need to match notation: the “set of points  $\gamma_j$ ” in the present notation is a subset of the “set of points  $\gamma_j$ ” in 2.12, and the “set of points  $\alpha_n$ ” in 2.12 is a subset of the present “set of points  $\alpha_n$ ”. However, they differ only by finitely many points, namely, the real and simple poles of  $q$  with positive residuum and the pole at 0. We conclude that the minimum in (2.5) equals  $m_0$ . Moreover, the number  $\delta_0$  in (2.5) is nonzero if and only if  $d_0$  is odd and  $F^{(d_0)}(0)$  is positive. Thus we see that the sum in the first line of (2.5) is not less than  $\delta$ , and it follows that

$$\operatorname{ind}_- \mathcal{K}_-(W) + \dim \mathcal{K}_-(W)^\circ = \operatorname{ind}_- q + \dim \mathcal{K}_-(W)^\circ \geq \delta + m_0 + M_0.$$

By the Schwarz inequality in  $\ell^2$ , we have

$$\sum_n \frac{1}{|F'(\alpha_n)| |\alpha_n|^{m_0 + M_0 + 2}} \leq \left( \sum_n \frac{|\operatorname{Res}(q; \alpha_n)|}{|\alpha_n|^{2(m_0+1)}} \right)^{\frac{1}{2}} \left( \sum_n \frac{1}{|F'(\alpha_n)|^2 |\operatorname{Res}(q; \alpha_n)| |\alpha_n|^{2(M_0+1)}} \right)^{\frac{1}{2}} < \infty.$$

This shows that  $m_0 + M_0 + 1 \geq \nu$ , and the second inequality in (4.1) follows. The first inequality in (4.1) is obvious, and (4.2) follows from (2.5) by dropping the summands  $\delta_j$  (not  $\delta_0$ ) and the minimum.

*Step 2:* Let  $\beta, \gamma \in \mathbb{N}_0$  be given in accordance with (4.3), and set

$$\tau_n := \frac{|\alpha_n|^{\beta - \delta - \gamma}}{|F'(\alpha_n)|}.$$

Then

$$\sum_n \frac{\tau_n}{|\alpha_n|^{2([\beta - \delta] + 1)}} = \sum_n \frac{1}{|F'(\alpha_n)| |\alpha_n|^{\beta - \delta + \gamma + 2}} = \sum_n \frac{1}{|F'(\alpha_n)|^2 \tau_n |\alpha_n|^{2(\gamma + 1)}}.$$

By (4.3), we have  $\beta - \delta + \gamma + 2 \geq \nu + 1$ ; hence, the middle series converges.

Set

$$\sigma_n := \begin{cases} \tau_n & \text{if } \tau_n \leq \frac{1}{|F'(\alpha_n)|^2 |\alpha_n|^{2\gamma}}, \\ \frac{1}{|F'(\alpha_n)|^2 |\alpha_n|^{2\gamma}} & \text{otherwise.} \end{cases}$$

Then  $\sigma_n \leq \tau_n$ , whence

$$\sum_n \frac{\sigma_n}{|\alpha_n|^{2([\beta - \delta] + 1)}} < \infty.$$

Recalling that  $\beta - \delta \geq 0$  by (4.3), we have

$$m_0 := \min \left\{ m \in \mathbb{N}_0 : \sum_n \frac{\sigma_n}{|\alpha_n|^{2(m+1)}} < \infty \right\} \leq \beta - \delta.$$

Since  $\sigma_n |F'(\alpha_n)|^2 |\alpha_n|^{2\gamma} \leq 1$ , it follows that

$$\sum_n \frac{1}{|F'(\alpha_n)|^2 \sigma_n |\alpha_n|^{2\gamma}} = \infty.$$

On the other hand,

$$\frac{1}{|F'(\alpha_n)|^2 \sigma_n |\alpha_n|^{2(\gamma+1)}} = \begin{cases} \frac{1}{|F'(\alpha_n)|^2 \tau_n |\alpha_n|^{2(\gamma+1)}} & \text{if } \tau_n \leq \frac{1}{|F'(\alpha_n)|^2 |\alpha_n|^{2\gamma}}, \\ \frac{1}{|\alpha_n|^2} & \text{otherwise.} \end{cases}$$

By statement (III.b) of Theorem 3.1, the convergence exponent of the sequence  $(\alpha_n)_n$  cannot exceed 1. Hence,

$$\sum_n \frac{1}{|F'(\alpha_n)|^2 \sigma_n |\alpha_n|^{2(\gamma+1)}} \leq \sum_n \frac{1}{|F'(\alpha_n)|^2 \tau_n |\alpha_n|^{2(\gamma+1)}} + \sum_n \frac{1}{|\alpha_n|^2} < \infty,$$

and we conclude that

$$\min \left\{ M \in \mathbb{N}_0 : \sum_n \frac{1}{|F'(\alpha_n)|^2 \sigma_n |\alpha_n|^{2(M+1)}} < \infty \right\} = \gamma.$$

Set

$$q_0(z) := \left[ \frac{F^{(d_0)}(0)}{d_0!} \right]^{-1} \frac{1}{z^{d_0}} + \sum_{\gamma_j \in \mathbb{R}} \frac{1}{(\gamma_j - z)^{d_j}} + \sum_{\text{Im } \gamma_j > 0} \left( \frac{1}{(\gamma_j - z)^{d_j}} + \frac{1}{(\bar{\gamma}_j - z)^{d_j}} \right) + \sum_n \sigma_n \left( \frac{1}{\alpha_n - z} - \frac{1}{\alpha_n} - \dots - \frac{z^{2m_0}}{\alpha_n^{2m_0+1}} \right).$$

Since  $\beta \geq \delta + m_0$ , there exists a polynomial  $p \in \mathbb{R}[z]$ ,  $p(0) = 0$ , such that

$$\text{ind}_-(q_0 + p) = \beta,$$

cf. (2.7). Now Lemma 4.8 applied with  $q_0 + p$  provides us with a matrix  $W \in \mathcal{M}_{<\infty}$  such that

$$(4.14) \quad (0, 1)W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F, \quad \text{ind}_- \mathcal{K}_-(W) = \beta, \quad \dim \mathcal{K}_-(W)^\circ = \gamma.$$

*Step 3:* Suppose that, in addition to  $\beta$  and  $\gamma$  as in Step 2, we are given a number  $\alpha \in \mathbb{N}_0$  with  $\alpha \geq \beta + \gamma$ . Choose  $W \in \mathcal{M}_{<\infty}$  with (4.14). Then statement (i) of Theorem 4.2 applied with the functions  $F$  and  $G := (0, 1)W \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , yields a (possibly different) matrix  $\widetilde{W} \in \mathcal{M}_{<\infty}$  such that

$$(0, 1)\widetilde{W} = (0, 1)W, \quad \text{ind}_- \mathcal{K}_-(\widetilde{W}) = \alpha.$$

Since the quantities  $\text{ind}_- \mathcal{K}_-(W)$  and  $\dim \mathcal{K}_-(W)^\circ$  depend only on the second row of a matrix, we have

$$\text{ind}_- \mathcal{K}_-(\widetilde{W}) = \beta, \quad \dim \mathcal{K}_-(\widetilde{W})^\circ = \gamma.$$

Clearly,  $(0, 1)\widetilde{W} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F$ . □

Finally, we present the proof of Corollary 4.4.

*Proof of Corollary 4.4.* We apply statement (ii) of Theorem 4.3. Choosing  $\alpha := \delta + \nu - 1$ ,  $\beta := \delta$ ,  $\gamma := \nu - 1$ , we obtain  $W_1 \in \mathcal{M}_{<\infty}$  with

$$(0, 1)W_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F, \quad \text{ind}_- \mathcal{K}_-(W) = \delta.$$

Choosing  $\alpha := \delta + \nu - 1$ ,  $\beta := \delta + \nu - 1$ ,  $\gamma := 0$ , we see that  $W_2 \in \mathcal{M}_{<\infty}$  with

$$(0, 1)W_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F, \quad \dim \mathcal{K}_-(W_2)^\circ = 0.$$

It is easy to alter  $W_2$  so as to arrange that  $\mathcal{K}_-(W_2) = \mathcal{K}(W_2)$ . We apply [KW99b, Theorem 5.7, (iii)] with some parameter  $\tau$ , say  $\tau := 0$ . This provides us with a real polynomial  $p$  having the properties stated there. An application of [KW99b, Theorem 5.7, (ii)] with the matrix and parameter

$$\widetilde{W}_2 := PW_2, \quad \tau := 0,$$

yields  $\mathcal{K}_-(\widetilde{W}_2) = \mathcal{K}(\widetilde{W}_2)$ . Clearly, the left lower entry of  $\widetilde{W}_2$  equals  $F$ .  $\square$

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