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Congruences of convex algebras

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ABSTRACT

We provide a full description of congruence relations of finitely generated convex, positively convex, and absolutely convex algebras. As a consequence of this result we obtain that finitely generated convex (positively convex, absolutely convex) algebras are finitely presentable. Convex algebras are important in the area of probabilistic systems. In particular positively convex algebras are, as they are the Eilenberg–Moore algebras of the subdistribution monad.

1. Introduction

In this paper we present a study of the equational classes CA, PCA, and ACA, of convex, positively convex, and absolutely convex algebras. We describe all congruence relations of such algebras. Knowing the congruences, we obtain that finitely generated convex (positively convex, absolutely convex) algebras are finitely presentable.

A convex algebra is an algebra with an infinite set of operations of arbitrary positive arities providing convex combinations of the arguments, which satisfy two axioms (axiom schemes): (1) the projection axiom stating that a convex combination with a single coefficient equal to 1 equals the identity map, and (2) the barycenter axiom stating that a convex combination of convex combinations equals the convex combination with suitably multiplied and summed coefficients. Positively convex and absolutely convex algebras are defined in a similar way from larger convex structures (sub-convex combinations for positively convex algebras and linear combinations with coefficients whose absolute values are sub-convex for absolutely convex algebras). The full definitions and details follow in Section 3.

Examples of convex algebras are provided by convex subsets of a vector space over the scalar field $\mathbb{R}$ (a subset of a vector space is convex, if it contains with each two points the whole line segment connecting them): If C is such, then C is the carrier of a convex algebra with the operations inherited from the vector

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space. However, these examples do not exhaust the class CA; the major obstacle being possible failure of cancellation laws in general convex algebras. Examples of positively convex algebras are provided by convex subsets of a vector space over $\mathbb{R}$ which contain the zero vector. Examples of absolutely convex algebras are provided by convex subsets of a vector space $\mathbb{R}$ which are symmetric around the zero vector.

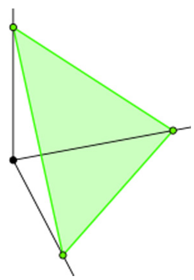
Among others, convex algebras appear in a categorical context. To explain this, e.g. for the PCA-situation, consider the category Vec_1^+ whose objects are regularly ordered normed vector spaces over the scalar field $\mathbb{R}$ and morphisms are positive and linear contractions between such spaces. The functor $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Sets}$ which acts on objects as

$$\Delta(V) := \big\{x \in V \,\big|\, \|x\| \leq 1,\ x \geq 0\big\},$$

and on morphisms as restriction to $\Delta(V)$, has a left adjoint. It turns out that the algebraic category PCA is the category of Eilenberg–Moore algebras of the monad induced by this adjunction, cf. [24]. Moreover, the monad in question is actually the discrete subprobability distribution monad, hence PCA is the category of Eilenberg–Moore algebras of the subprobability distribution monad [9,10]. In recent line of research, (positively) convex algebras are recognized as state transformers in the duality between predicates and states, capturing the essence of the semantics of program logics for probabilistic and quantum computation [15–17].

Our aim in this paper is to achieve full understanding of the structure of finitely generated algebras in CA (PCA and ACA). We manage this with Theorems 4.10 and 4.11 below, where we describe the congruences on any polytope in the euclidean space $\mathbb{R}^n$ considered as a convex algebra.

It is simple to check that, for each $n \in \mathbb{N}^+$, the free algebra F_n in CA with n generators is given by the standard $(n-1)$-simplex in $\mathbb{R}^n$ (a particular polytope). For $n = 3$, we can picture this algebra as[1]



(1.1)

Clearly, knowing all congruences of the free algebras F_n, $n \in \mathbb{N}^+$, is enough to understand all finitely generated algebras in CA. These results on CA can be transferred to PCA and ACA. Therefore, we also achieve full understanding of the structure of finitely generated positively convex or absolutely convex algebras.

Besides its obvious intrinsic interest, our motivation to investigate finitely generated algebras in CA (PCA or ACA) originates in a problem related to probabilistic systems. The probability subdistribution monad arising from the above mentioned adjunction, including the functor Δ, and its Eilenberg–Moore algebras play a crucial role in connection with the axiomatization of trace semantics for probabilistic systems given in [38]. There the question arose whether or not each finitely generated algebra in PCA is also finitely presentable. Using the newly established knowledge about congruence relations, we can answer this question affirmatively, cf. Corollary 5.5.

Historically, work on convex algebras (commonly called convex modules or convex spaces or abstract convex sets) can be traced back von Neumann and Morgenstern's book on the theory of games and economic behavior. In the 1960's and 1970's quite some mathematicians investigated convex algebras, motivated by problems in physics and chemistry. The theory of absolutely convex algebras (and their analogues allowing

[1] Picture source: http://en.wikipedia.org/wiki/Simplex.

infinitary operations) started in [28], where they were realized to be the Eilenberg–Moore algebras associated with the adjunction induced by the unit ball functor from the category of Banach spaces (with linear contractions) to **Sets**. A similar treatment of positively convex algebras was given shortly after in [24]. Later on these notions were extensively studied, mainly focusing on the categorical viewpoint and topological questions, see, e.g., [29,3,18,25–27] and the references therein. A far reaching generalization, namely the concept of convexity theories, has been developed in a series of papers involving several authors which started with [34], and went on (at least) till [35].

Previous work which is the closest to our approach is [30,19,20], where congruence relations in (infinitary or p-) absolutely convex modules (algebras) are studied. Some parts of our results read similarly and several geometric ideas employed there can also be used in the present setting. In order to prevent confusion concerning terminology, let us note explicitly that in the literature algebras in CA (PCA or ACA) are also called "finitely (positively/absolutely or totally) convex modules/spaces". The term "finitely" thereby refers to the fact that they carry only finitary operations. However, in the present paper we stick to the purely algebraic setting and do not touch upon the possibility of allowing infinitary operations. Hence, we omit the prefix "finitely" from the notation. Moreover, we also choose the term "algebra" over "module" or "space" since it has been used in recent work regarding positively convex algebras [9,10,38] and "absolutely" over "totally" as it is preferred in more recent work [25].

The structure of the paper is as follows. After the introduction, we recall some notions and facts from convex geometry in Section 2. In Section 3, we present the equational classes of convex, positively convex and absolutely convex algebras. After collecting some basic facts, we investigate the relationship between CA, PCA, and ACA. Interestingly, it turns out that CA and PCA are closely related, whereas ACA carries a significantly stronger structure. Section 4 is the core of the paper. There we formulate and prove Theorems 4.10 and 4.11 that describe the congruence relations on a polytope K in euclidean space. It turns out that a congruence on K is fully determined by two ingredients: (1) a family of linear subspaces, describing the congruence classes in the interior of K and in the interior of each of its lower dimensional facets; (2) a graph, describing how the interiors of K and each of its facets are related to the lower dimensional facets forming the respective boundary. Finally, in Section 5, we give the already mentioned application, and show that in each of the algebraic categories corresponding to CA, PCA, and ACA, the notions "finitely generated" and "finitely presentable" coincide.

Our basic reference concerning terminology and results of universal algebra is the (old but still excellent) book [12], or the more recent book [6]. For the (few) notions from category theory which are used in this paper, we refer the reader to [22]. Our standard references concerning convex geometry are [5,13].

2. Preliminaries from convex geometry

Basic universal algebra and euclidean topology notions and results will be recalled when they are needed. In this section we explicitly recall some definitions and results from convex geometry, since they are maybe less widely known. The proofs of these simple properties can be found in standard books on convexity, e.g. [5,13,37] or, for the convenience of the reader, in an older technical report version of this paper [39]. We start with recalling the definition of a convex set.

Definition 2.1. A subset C of a vector space V over the field $\mathbb{R}$ is convex if for all $x, y \in C$ and any scalar $\lambda \in [0, 1]$ it holds that $\lambda x + (1 - \lambda)y \in C$. Geometrically, this means that C contains, together with each two points, the whole line segment connecting them.

Linear functions map convex sets to convex sets. The following simple property shows that convexity is the same as being closed under arbitrary convex linear combinations.

Lemma 2.2. *Let C be a subset of a vector space V over $\mathbb{R}$. Then C is convex if and only if*

$$\sum_{i=1}^{n} p_i x_i \in C$$

for all $n \in \mathbb{N}^+$, all $x_i \in C$, and all $p_i \in [0,1]$ with $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} p_i = 1$. $\square$

The following subsets, associated with a finite nonempty subset Y of V, play an important role throughout the paper:

$$\begin{aligned}
\operatorname{span} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \;\Big|\; \lambda_y \in \mathbb{R} \Big\}, \\
\operatorname{dir} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \;\Big|\; \lambda_y \in \mathbb{R},\ \sum_{y \in Y} \lambda_y = 0 \Big\}, \\
\operatorname{aff} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \;\Big|\; \lambda_y \in \mathbb{R},\ \sum_{y \in Y} \lambda_y = 1 \Big\}, \\
\operatorname{co} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \;\Big|\; \lambda_y \in [0,1],\ \sum_{y \in Y} \lambda_y = 1 \Big\}, \\
\breve{\operatorname{co}} Y &:= \Big\{ \sum_{y \in Y} \lambda_y y \;\Big|\; \lambda_y \in (0,1],\ \sum_{y \in Y} \lambda_y = 1 \Big\}.
\end{aligned}$$

We refer to $\operatorname{co} Y$ as the (closed) convex hull of Y and $\breve{\operatorname{co}} Y$ as the open convex hull of Y or the interior of $\operatorname{co} Y$. We will see later that this choice of terminology is indeed justified, cf. Lemma 2.5. The linear span $\operatorname{span} Y$ is the smallest vector subspace that contains Y. Moreover, we refer to $\operatorname{aff} Y$ as the affine space generated by Y and $\operatorname{dir} Y$ as the directions of $\operatorname{aff} Y$. Note that $\operatorname{dir} Y$ is a vector subspace. Clearly, for each nonempty finite set Y,

$$\breve{\operatorname{co}} Y \subseteq \operatorname{co} Y \subseteq \operatorname{aff} Y \subseteq \operatorname{span} Y \quad \text{and} \quad \operatorname{dir} Y \subseteq \operatorname{span} Y.$$

If Y contains only one element, then $\breve{\operatorname{co}} Y = \operatorname{co} Y = \operatorname{aff} Y = Y$ and $\operatorname{dir} Y = \{0\}$. If $|Y| \geq 2$, then $\breve{\operatorname{co}} Y \subset \operatorname{co} Y \subset \operatorname{aff} Y$ and $\operatorname{dir} Y \neq \{0\}$.

First, some simple geometric properties of these sets.

Lemma 2.3. *Let Y be a nonempty finite subset of a vector space V over $\mathbb{R}$. Then the following hold:*

(*i*) *For each $z \in \operatorname{aff} Y$, we have $\operatorname{aff} Y = z + \operatorname{dir} Y$.*
(*ii*) *For each $z \in \operatorname{aff} Y$, we have $\operatorname{dir} Y = \{w - z \mid w \in \operatorname{aff} Y\}$.*
(*iii*) *Also, $\operatorname{dir} Y = \{w - z \mid z, w \in \operatorname{aff} Y\}$.*
(*iv*) *Finally, for every $y_0 \in Y$, $\operatorname{dir} Y = \operatorname{span}\{y - y_0 \mid y \in Y\}$.* $\square$

In the situation when $V = \mathbb{R}^n$ some important topological properties hold. These are expressed in the following two lemmas. Note that whenever we mention topological properties, we have in mind the Euclidean topology in $\mathbb{R}^n$. In the sequel, by $\operatorname{Clos}(X)$ we denote the topological closure of a set $X \subseteq \mathbb{R}^n$.

Lemma 2.4. *Let Y be a finite nonempty subset of $\mathbb{R}^n$. Then $\operatorname{co} Y$ is compact and convex. Moreover, $\operatorname{co} Y = \operatorname{Clos}(\breve{\operatorname{co}} Y)$.* $\square$

Lemma 2.5. *Let Y be a nonempty finite subset of $\mathbb{R}^n$. Then $\check{\mathrm{co}}Y$ is open considered as a subset of* $\operatorname{aff} Y$. □

Lemma 2.5 implies an alternative characterization of $\operatorname{dir} Y$.

Lemma 2.6. *Let Y be a nonempty finite subset of $\mathbb{R}^n$. Then, for $z \in \operatorname{aff} Y$*

$$\operatorname{dir} Y = \operatorname{span}\{y - z \mid y \in \check{\mathrm{co}}Y\}.$$

Also

$$\operatorname{dir} Y = \operatorname{span}\{y_2 - y_1 \mid y_1, y_2 \in \check{\mathrm{co}}Y\}. \qquad \square$$

Let $C \subseteq \mathbb{R}^n$ be convex. A point $e \in C$ is called an extremal point of C if

$$e = tx + (1-t)y \quad \text{ with } x, y \in C,\ t \in (0,1) \ \Rightarrow\ x = y = e.$$

Geometrically, this means that e does not lie in the interior of any line segment with endpoints in C. We denote the set of all extremal points of C by $\operatorname{ext} C$.

Compact convex sets can be recovered from their extremal points. The Kreĭn–Milman theorem states in a very general context that each compact convex set is the closed convex hull of its extremal points, see, e.g., [36, 3.23]. The version of this theorem for subsets C of $\mathbb{R}^n$, that we use here, can be found in [13, 2.4.5].

We mainly deal with a certain kind of geometric objects called polytopes. Polytopes are of central interest to us since, as already mentioned in the introduction, the free finitely generated algebras in CA, PCA, ACA are carried by polytopes. We next recall the definition of a polytope.

Definition 2.7. Let K be a subset of the euclidean space $\mathbb{R}^n$. The set K is a polytope if it is of the form $K = \operatorname{co} Y$ for some finite nonempty set $Y \subseteq \mathbb{R}^n$.

A basic example of a polytope is a simplex.

Example 2.8 *(A d-dimensional simplex).* Let $a \in \mathbb{R}^n$, and let $\{u_1, \dots, u_d\}$ be a linearly independent subset of $\mathbb{R}^n$. Then the polytope

$$K := \operatorname{co}\big(\{a\} \cup \{a + u_i \mid i = 1, \dots, d\}\big)$$

is called a d-dimensional simplex.

For instance, for $d = n = 3$ and

$$a := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.1}$$

we obtain the pyramid having the triangle with corner points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ as its base and the point $(0,0,1)$ as its apex.

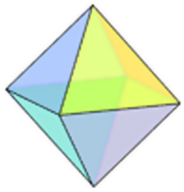
Another concrete example of a polytope is an octahedron.

Example 2.9 *(A d-dimensional octahedron).* Let $a \in \mathbb{R}^n$, and let $\{u_1, \dots, u_d\}$ be a linearly independent subset of $\mathbb{R}^n$. Then the polytope

$$K := \operatorname{co}\big(\{a + u_i \mid i = 1, \dots, d\} \cup \{a - u_i \mid i = 1, \dots, d\}\big)$$

is called a d-dimensional octahedron.

For instance, if $d = n = 3$ and a, u_1, u_2, u_3 are again as in (2.1), we obtain a regular octahedron with center at the origin.[2]



Polytopes can be defined in several equivalent ways. The definition used in [13] is presented in the next lemma. The fact that this definition is equivalent to the one above, i.e., the proof of the lemma is, in essence, a consequence of the Kreĭn–Milman theorem.

Lemma 2.10. *A subset $K \subseteq \mathbb{R}^n$ is a polytope if and only if K is compact, convex, and the set* $\operatorname{ext} K$ *of its extremal points is finite.* □

Note that if K is a polytope and $K = \operatorname{co} Y$ for some finite set Y, then $\operatorname{ext} K \subseteq Y$ and $K = \operatorname{co}(\operatorname{ext} K)$.

Remark 2.11. The mentioned concrete examples, the simplex from Example 2.8 and the octahedron from Example 2.9, are of particular interest in the present context. They are the free algebras with 3 generators in the equational classes PCA and ACA, respectively (similarly as the standard simplex (1.1) is in CA). This fact (of course for dimension n instead of 3), together with the results of Section 3 below, shows that describing all congruences of polytopes as convex algebras suffices to know all congruences of all finitely generated algebras in CA, PCA, and ACA.

3. The equational classes CA, PCA, and ACA

In this section we investigate the three convexity theories of convex, positively convex, and absolutely convex algebras and their induced equational classes. To start with, let us recall the definitions.

Definition 3.1. The variety of convex algebras is given by the signature (set of formal operations)

$$\mathcal{T}_{\mathrm{ca}} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \;\middle|\; n \in \mathbb{N}^+,\ p_1, \dots, p_n \geq 0,\ \sum_{i=1}^n p_i = 1 \right\},$$

where $(p_i)_{i=1}^n$ denotes an n-ary operation, and equations given by the following two axioms.

(1) The projection axiom:

$$(\delta_{ij})_{i=1}^n (x_1, \dots, x_n) = x_j, \quad n \in \mathbb{N}^+,\ j = 1, \dots, n,$$

where δ_{ij} denotes the Kronecker-delta

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

[2] Picture source: http://en.wikipedia.org/wiki/Octahedron.

(2) The barycenter axiom:

$$(p_i)_{i=1}^n\big((p_{1j})_{j=1}^m(x_1,\dots,x_m),\dots,(p_{nj})_{j=1}^m(x_1,\dots,x_m)\big)=\left(\sum_{i=1}^n p_ip_{ij}\right)_{j=1}^m(x_1,\dots,x_m),$$

whenever $n,m\in\mathbb{N}^+$, $(p_i)_{i=1}^n\in\mathcal{T}_{\mathrm{ca}}$, and $(p_{ij})_{j=1}^m\in\mathcal{T}_{\mathrm{ca}}$, $i=1,\dots,n$.

The operation appearing on the right-hand side of the barycenter axiom is well defined since

$$\sum_{j=1}^m\left(\sum_{i=1}^n p_ip_{ij}\right)=\sum_{i=1}^n p_i\left(\sum_{j=1}^m p_{ij}\right), \tag{3.1}$$

and hence $(\sum_{i=1}^n p_ip_{ij})_{j=1}^m\in\mathcal{T}_{\mathrm{ca}}$.

By CA we denote the equational class of convex algebras. A convex algebra $\mathbb{A}$ is then, as usual, a pair $\mathbb{A}=\langle A,(\tau_A(p_i)_{i=1}^n\mid(p_i)_{i=1}^n\in\mathcal{T}_{\mathrm{ca}})\rangle$ with $\tau_A(p_i)_{i=1}^n$ an n-ary operation on A, that satisfies the equations of CA.

Definition 3.2. The variety of positively convex algebra has the signature

$$\mathcal{T}_{\mathrm{pca}}:=\left\{(p_i)_{i=1}^n\in\mathbb{R}^n\;\middle|\;n\in\mathbb{N}^+,\ p_1,\dots,p_n\ge 0,\ \sum_{i=1}^n p_i\le 1\right\},$$

and equations given again by the projection axiom and the barycenter axiom, where in the latter $(p_i)_{i=1}^n$ and $(p_{ij})_{j=1}^m$ vary through $\mathcal{T}_{\mathrm{pca}}$.

We denote the equational class of all positively convex algebras as PCA. A positively convex algebra is then an algebra $\mathbb{A}=\langle A,(\tau_A(p_i)_{i=1}^n\mid(p_i)_{i=1}^n\in\mathcal{T}_{\mathrm{pca}})\rangle$ that satisfies the equations of PCA.

Definition 3.3. The variety of absolutely convex algebras has formal operations

$$\mathcal{T}_{\mathrm{aca}}:=\left\{(p_i)_{i=1}^n\in\mathbb{R}^n\;\middle|\;n\in\mathbb{N}^+,\ \sum_{i=1}^n |p_i|\le 1\right\},$$

and again equations given by the projection axiom and the barycenter axiom, where in the latter $(p_i)_{i=1}^n$ and $(p_{ij})_{j=1}^m$ vary through $\mathcal{T}_{\mathrm{aca}}$.

We denote the equational class of all absolutely convex algebras as ACA. The term "absolutely convex" can be seen as a short form of "absolutely positively convex" which would be an appropriate term as the sum is still less than or equal to 1. An absolutely convex algebra is an algebra $\mathbb{A}=\langle A,(\tau_A(p_i)_{i=1}^n\mid(p_i)_{i=1}^n\in\mathcal{T}_{\mathrm{aca}})\rangle$ in ACA.

Note that, again because of (3.1), the operation appearing on the right-hand side of the barycenter axiom is always well defined, i.e., is in $\mathcal{T}_{\mathrm{pca}}$ or $\mathcal{T}_{\mathrm{aca}}$, respectively (to see this for $\mathcal{T}_{\mathrm{aca}}$, use the triangle inequality).

Slightly overloading the notation, we will also write CA, PCA, ACA for the categories of convex, positively convex, and absolutely convex algebras, respectively, with corresponding algebra homomorphisms. As already mentioned in the introduction, PCA is the category of Eilenberg–Moore algebras of the subdistribution monad.

It is obvious, since $\mathcal{T}_{\mathrm{ca}}\subseteq\mathcal{T}_{\mathrm{pca}}\subseteq\mathcal{T}_{\mathrm{aca}}$ and the axioms "coincide", that each absolutely convex algebra can be considered as a positively convex algebra, which in turn can be considered a convex algebra. To be precise, if $\mathbb{A} = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$ is a positively convex algebra, then $U_{\mathsf{CA}}(\mathbb{A}) = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{ca}})\rangle$ is a convex algebra. Similarly, if $\mathbb{A} = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{aca}})\rangle$ is an absolutely convex algebra, then $U_{\mathsf{PCA}}(\mathbb{A}) = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$ is a positively convex algebra, and in turn $U_{\mathsf{CA}}(U_{\mathsf{PCA}}(\mathbb{A}))$ is a convex algebra. More precisely, U_{PCA} and U_{CA} are the forgetful functors

$$\mathsf{ACA} \xrightarrow{U_{\mathsf{PCA}}} \mathsf{PCA} \xrightarrow{U_{\mathsf{CA}}} \mathsf{CA} \tag{3.2}$$

mapping morphisms to themselves. Due to this fact, many results immediately transfer from CA to PCA and ACA.

Another interesting fact, which we shall explain in the sequel, is that CA and PCA are closely related, whereas ACA is significantly different (Propositions 3.6 and 3.9 below).

When working with algebras in CA, PCA, or ACA, it is practical (and customary) to write operations as formal sums and/or to use vector notation:

$$\tau_A(p_i)_{i=1}^n(x_1,\dots,x_n) = \sum_{i=1}^n p_i x_i = (p_1,\dots,p_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

From now on, we will mainly use the formal-sum notation. In the next lemma we provide some simple but useful identities which follow from the projection and barycenter axioms. For the convenience of the reader, an explicit proof can be found in [39]. In the setting of ACA (with infinitary operations) these identities were shown in [28, Theorem 2.4] using a different proof.

Lemma 3.4. *For items (i)–(iii), let $\mathbb{A}$ be an algebra in any of the classes* CA, PCA, *or* ACA. *For items (iv) and (v), assume that $\mathbb{A}$ belongs to* PCA *or* ACA. *Let c stand for ca, pca, aca, if $\mathbb{A}$ is in* CA, PCA, ACA, *respectively.*

(i) *The operations are commutative, that is*

$$\sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_{\sigma(i)} x_{\sigma(i)}$$

whenever $n \in \mathbb{N}^+$, σ is a permutation of $\{1,\dots,n\}$, and $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{c}}$.

(ii) *The extended projection law*

$$\sum_{i=1}^n p_i x_i = \sum_{k=1}^m p_{i_k} x_{i_k}$$

holds whenever $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{c}}$ and $i_1,\dots,i_m$ satisfy

$$i_1 < \dots < i_m, \quad \{i_1,\dots,i_m\} \supseteq \big\{i \in \{1,\dots,n\} \bigm| p_i \neq 0\big\}.$$

(iii) *Whenever $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{c}}$ and $x \in A$, we have*

$$\sum_{i=1}^n p_i x = \left(\sum_{i=1}^n p_i\right) x.$$

(*iv*) *The elements*

$$\sum_{i=1}^{n} 0x_i, \quad \textit{for } n \in \mathbb{N}^+,\ x_1,\dots,x_n \in A$$

all coincide. We denote this element as 0_A.

(*v*) *The element* 0_A *plays the role of a zero element: Let* $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{c}}$ *and let* $i_1,\dots,i_m$ *satisfy*

$$i_1 < \cdots < i_m, \quad \{i_1,\dots,i_m\} \supseteq \big\{i \in \{1,\dots,n\} \,\big|\, x_i \neq 0_A\big\}.$$

Then

$$\sum_{i=1}^{n} p_i x_i = \begin{cases} \sum_{k=1}^{m} p_{i_k} x_{i_k}, & m \geq 1 \\ 0_A, & m = 0 \end{cases}$$

Note that items (*iv*) and (*v*) would not at all make sense within CA, since the operations appearing in them do not belong to $\mathcal{T}_{\mathrm{ca}}$.

Remark 3.5. Let $\mathbb{A} \in \mathsf{CA}$. A consequence of the extended projection law is that the barycenter axiom remains valid when the sequences $(p_{ij})_{j=1}^m$ appearing therein are no more of the same length m and no more bound to the same variables. More precisely, let

$$n, m \in \mathbb{N}^+, \quad K_i \subseteq \mathbb{N}^+,\ i = 1,\dots,n \text{ with } \bigcup_{i=1}^{n} K_i = \{1,\dots,m\},$$

$$(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{ca}}, \tag{3.3}$$

$$(p_{ij})_{j\in K_i} \quad \text{with } p_{ij} \geq 0,\ \sum_{j \in K_i} p_{ij} = 1,\ i = 1,\dots,n. \tag{3.4}$$

Set $m_i := |K_i|$, and write $K_i = \{\kappa_k^i \mid k = 1,\dots,m_i\}$ with $\kappa_k^i < \kappa_{k+1}^i$. Then

$$\sum_{i=1}^{n} p_i \left(\sum_{k=1}^{m_i} p_{i\kappa_k^i} x_{\kappa_k^i} \right) = \sum_{j=1}^{m} \left(\sum_{\substack{i=1 \\ j \in K_i}}^{n} p_i p_{ij} \right) x_j,$$

whenever $x_1,\dots,x_m \in A$.

The same holds when $\mathbb{A} \in \mathsf{PCA}$ and the conditions (3.3) and (3.4) are replaced by

$$(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}},$$

$$(p_{ij})_{j\in K_i} \quad \text{with } p_{ij} \geq 0,\ \sum_{j \in K_i} p_{ij} \leq 1,\ i = 1,\dots,n.$$

Furthermore, the same holds when $\mathbb{A} \in \mathsf{ACA}$ and (3.3) and (3.4) are replaced by

$$(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{aca}},$$

$$(p_{ij})_{j\in K_i} \quad \text{with } p_{ij} \in \mathbb{R},\ \sum_{j \in K_i} |p_{ij}| \leq 1,\ i = 1,\dots,n.$$

This remark also clarifies the note made in [27, p. 110] immediately after the definition of positively convex algebra (in which the above stronger form of the barycenter axiom is required).

Positively convex algebras vs. convex algebras

In this subsection we make precise the connection between CA and PCA.

Let $\mathbb{A} \in \mathsf{CA}$. An algebra $\mathbb{A}^{\mathrm{pca}}$ in PCA is an extension of $\mathbb{A}$ to PCA if $U_{\mathsf{CA}}(\mathbb{A}^{\mathrm{pca}}) = \mathbb{A}$. In order words, if $\mathbb{A} = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{ca}})\rangle$ and $\mathbb{A}^{\mathrm{pca}} = \langle A, (\bar\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$, then $\mathbb{A}^{\mathrm{pca}}$ is an extension of $\mathbb{A}$ to PCA if

$$\bar\tau_A\alpha = \tau_A\alpha, \quad \alpha \in \mathcal{T}_{\mathrm{ca}}. \tag{3.5}$$

We will soon be able to show that any algebra in CA with nonempty carrier has an extension to PCA, i.e., is in the image of the functor U_{CA}. Before we continue, we introduce one more category, the category $\mathsf{CA}_\bullet$ of pointed convex algebras. Objects of $\mathsf{CA}_\bullet$ are pairs $(\mathbb{A}, a)$ where $\mathbb{A} = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{ca}})\rangle$ is an algebra in CA and $a \in A$ is a fixed element, point, in the carrier. A map $f: A \to B$ is a $\mathsf{CA}_\bullet$-morphism from a pointed convex algebra $(\mathbb{A}, a)$ to a pointed convex algebra $(\mathbb{B}, b)$ if and only if it is a convex algebra homomorphism from $\mathbb{A}$ to $\mathbb{B}$ and it preserves the designated point, i.e., $f(a) = b$. There is an obvious forgetful functor $U_\bullet: \mathsf{CA}_\bullet \to \mathsf{CA}$ that forgets the point, i.e., on objects $U_\bullet(\mathbb{A}, a) = \mathbb{A}$ and on morphisms it is the identity, $U_\bullet(f) = f$. Clearly, every algebra in CA with nonempty carrier is in the image of $U_\bullet$.

Proposition 3.6. *The category* $\mathsf{CA}_\bullet$ *is isomorphic to the category* PCA *of positively convex algebras. We have the following situation of categories and functors.*

$$\begin{tikzcd}
\mathsf{CA}_\bullet \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\cong"] \arrow[dr, "U_\bullet"'] & & \mathsf{PCA} \arrow[ll, bend left, "G"] \arrow[dl, "U_{\mathsf{CA}}"] \\
& \mathsf{CA} &
\end{tikzcd}$$

Proof. Let $\mathbb{A}$ be a convex algebra and $a \in A$. The functor F is given on morphisms by $F(f) = f$ and on objects by

$$F(\mathbb{A}, a) = \big\langle A, (\bar\tau_{A,a}\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\big\rangle$$

with

$$\bar\tau_{A,a}(p_i)_{i=1}^n(x_1, \ldots, x_n) := \tau_A(p_1, \ldots, p_n, \bar p)(x_1, \ldots, x_n, a),$$

for

$$\bar p := 1 - \sum_{i=1}^n p_i.$$

The extended projection law in $\mathbb{A}$ gives that

$$F(\mathbb{A}, a) = \big\langle A, (\bar\tau_{A,a}\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\big\rangle$$

satisfies (3.5). We still need to check it satisfies the PCA axioms. The projection axiom holds in $F(\mathbb{A}, a)$ as all operations involved belong to $\mathcal{T}_{\mathrm{ca}}$. To show the PCA-barycenter axiom, let $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$ and $(p_{ij})_{i=1}^m \in \mathcal{T}_{\mathrm{pca}}$, $i = 1, \ldots, n$, be given. Denote

$$\bar p := 1 - \sum_{i=1}^n p_i, \qquad \bar p_i := 1 - \sum_{j=1}^m p_{ij}, \quad i = 1, \ldots, n.$$

Then (δ_{ij} again denotes the Kronecker-delta)

$$(p_1,\dots,p_n,\bar{p}),\qquad (p_{i1},\dots,p_{im},\bar{p}_i),\quad \text{for } i=1,\dots,n, \text{ and } (\delta_{i,m+1})_{i=1}^{m+1}$$

all belong to $\mathcal{T}_{\rm ca}$, and hence we may apply the CA-barycenter axiom and use the projection axiom. This gives

$$\begin{aligned}
&\bar{\tau}_{A,a}(p_i)_{i=1}^n\big(\bar{\tau}_{A,a}(p_{1j})_{j=1}^m(x_1,\dots,x_m),\dots,\bar{\tau}_{A,a}(p_{nj})_{j=1}^m(x_1,\dots,x_m)\big)\\
&\quad=\sum_{i=1}^n p_i\bar{\tau}_{A,a}(p_{1j})_{j=1}^m(x_1,\dots,x_m)+\bar{p}a\\
&\quad=\sum_{i=1}^n p_i\left(\sum_{j=1}^m p_{ij}x_j+\bar{p}_ia\right)+\bar{p}a\\
&\quad=\sum_{i=1}^n p_i\left(\sum_{j=1}^m p_{ij}x_j\right)+\left(\sum_{i=1}^n p_i\bar{p}_i+\bar{p}\right)a\\
&\quad=\bar{\tau}_{A,a}\left(\sum_{i=1}^n p_ip_{ij}\right)_{j=1}^m(x_1,\dots,x_m).
\end{aligned}$$

The last equality holds since, due to (3.1),

$$\sum_{i=1}^n p_i\bar{p}_i+\bar{p}=1-\sum_{j=1}^m\left(\sum_{i=1}^n p_ip_{ij}\right).$$

Moreover, we have for arbitrary $x\in A$

$$0_A=\bar{\tau}_A(0)(x)=\tau_A(0,1)(x,a)=a. \tag{3.6}$$

A $\mathsf{CA}_\bullet$ morphism $f\colon(\mathbb{A},a)\to(\mathbb{B},b)$ is a PCA morphism from $F(\mathbb{A},a)$ to $F(\mathbb{B},b)$ since

$$\begin{aligned}
f\big(\bar{\tau}_{A,a}(p_i)_{i=1}^n(x_1,\dots,x_n)\big)&=f\big(\tau_A(p_1,\dots,p_n,\bar{p})(x_1,\dots,x_n,a)\big)\\
&=\tau_B(p_1,\dots,p_n,\bar{p})\big(f(x_1),\dots,f(x_n),f(a)\big)\\
&=\tau_B(p_1,\dots,p_n,\bar{p})\big(f(x_1),\dots,f(x_n),b\big)\\
&=\bar{\tau}_{B,b}(p_i)_{i=1}^n\big(f(x_1),\dots,f(x_n)\big).
\end{aligned}$$

Since $F(\mathbb{A},a)$ is an extension of $\mathbb{A}$ to PCA we have $U_{\mathsf{CA}}\circ F=U_\bullet$.

Let now $\mathbb{A}$ be in PCA. The functor G is defined on objects by $G(\mathbb{A})=(U_{\mathsf{CA}}(\mathbb{A}),0_A)\in\mathsf{CA}_\bullet$ and again leaves the morphisms unchanged, i.e., $G(f)=f$. Clearly, any PCA morphism $f\colon\mathbb{A}\to\mathbb{B}$ satisfies that $f(0_A)=0_B$ and is a CA morphism from $U_{\mathsf{CA}}(\mathbb{A})$ to $U_{\mathsf{CA}}(\mathbb{B})$. From the definition of G it is obvious that $U_\bullet\circ G=U_{\mathsf{CA}}$.

Finally, note that F and G are inverse to each other: For the morphisms, this is clear. For $\mathbb{A}$ in CA and $a\in A$ we have

$$G\circ F(\mathbb{A},a)=G\big(F(\mathbb{A},a)\big)\overset{(3.6)}{=}\big(U_{\mathsf{CA}}\big(F(\mathbb{A},a)\big),a\big)=\big(U_\bullet(\mathbb{A},a),a\big)=(\mathbb{A},a).$$

For $\mathbb{A}=\langle A,(\tau_A\alpha\mid\tau_A\alpha\in\mathcal{T}_{\rm pca})\rangle$ in PCA we have

$$F\circ G(\mathbb{A})=F\big(U_{\mathsf{CA}}(\mathbb{A}),0_A\big)=\mathbb{A}$$

since $U_{\mathsf{CA}}(\mathbb{A}) = \langle \mathbb{A}, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{ca}})\rangle$, $F(U_{\mathsf{CA}}(\mathbb{A}), 0_A) = \langle A, (\bar\tau_{A,0_A}\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$ and for $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$

$$\begin{aligned}\bar\tau_{A,0_A}(p_i)_{i=1}^n(x_1,\ldots,x_n) &= \tau_A(p_1,\ldots,p_n,\bar p)(x_1,\ldots,x_n,0_A)\\ &\overset{(*)}{=} \tau_A(p_1,\ldots,p_n)(x_1,\ldots,x_n)\end{aligned}$$

where $\bar p := 1 - \sum_{i=1}^n p_i$ and the equality marked by $(*)$ holds by Lemma 3.4(v). □

Remark 3.7. In concrete terms, Proposition 3.6 shows that any convex algebra $\mathbb{A}$ with nonempty carrier has an extension to PCA and the set of all possible extensions is in a bijective correspondence with the carrier A. Hence, any convex algebra with nonempty carrier is in the image of the functor U_{CA}; the size of the inverse image of $U_\bullet$ or U_{CA} of a single convex algebra equals the size of its carrier.

Proposition 3.6 enables us to relate the congruences of algebras in CA and PCA. Let $\mathbb{A}$ be an algebra in CA or PCA. Recall that an equivalence relation Θ on A is a congruence of $\mathbb{A}$ if whenever $(x_i, y_i) \in \Theta$, for $i \in \{1,\ldots,n\}$ then also $(\sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i) \in \Theta$ for $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{ca}}$ or $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$, respectively. Equivalently, congruences are kernels of homomorphisms: $\Theta \subseteq A \times A$ is a congruence if and only if there is an algebra $\mathbb{B}$ and a homomorphism $f: \mathbb{A} \to \mathbb{B}$ with

$$\Theta = \ker f = \big\{(x,y) \in A \times A \bigm| f(x) = f(y)\big\}.$$

In general categorical terms, congruences are kernel pairs. By $\mathrm{Con}_{\mathsf{CA}}\,\mathbb{A}$ or $\mathrm{Con}_{\mathsf{PCA}}\,\mathbb{A}$ we denote the sets of all CA- or PCA-congruences on $\mathbb{A}$, respectively.

Lemma 3.8. *Let $\mathbb{A}$ be an algebra in PCA. Then $\mathrm{Con}_{\mathsf{PCA}}\,\mathbb{A} = \mathrm{Con}_{\mathsf{CA}}\,U_{\mathsf{CA}}(\mathbb{A})$.*

Proof. The inclusion $\subseteq$ is clear as any PCA congruence of $\mathbb{A}$ is a CA congruence of $U_{\mathsf{CA}}(\mathbb{A})$. Formally, let $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\,\mathbb{A}$. Then $\Theta = \ker f$ for some PCA-homomorphism $f: \mathbb{A} \to \mathbb{B}$. But then $f = U_{\mathsf{CA}}(f): U_{\mathsf{CA}}(\mathbb{A}) \to U_{\mathsf{CA}}(\mathbb{B})$ and $\Theta = \ker(f)$ showing that $\Theta \in \mathrm{Con}_{\mathsf{CA}}\,U_{\mathsf{CA}}(\mathbb{A})$.

For the opposite inclusion, assume $\Theta \in \mathrm{Con}_{\mathsf{CA}}\,U_{\mathsf{CA}}(\mathbb{A})$. Then $\Theta = \ker(f)$ for a CA-homomorphism $f: U_{\mathsf{CA}}(\mathbb{A}) \to \mathbb{B}$ for some algebra $\mathbb{B}$ in CA. Consider now $G(\mathbb{A}) = (U_{\mathsf{CA}}(\mathbb{A}), 0_A)$ in $\mathsf{CA}_\bullet$. Let $b = f(0_A) \in B$. Then $f: G(\mathbb{A}) \to (\mathbb{B}, b)$ is a $\mathsf{CA}_\bullet$-homomorphism. Furthermore, $f = F(f): \mathbb{A} \to F(\mathbb{B}, b)$ is a PCA-homomorphism and $\Theta = \ker(f)$, showing that $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\,\mathbb{A}$. □

Absolutely convex algebras vs. positively convex algebras

We now compare absolutely convex algebras and positively convex algebras. The question is which positively convex algebras are in the image of the functor U_{PCA}. It turns out that not every positively convex algebra has this property.

Let $\mathbb{A} \in \mathsf{PCA}$. An algebra $\mathbb{A}^{\mathrm{aca}}$ in ACA is an extension of $\mathbb{A}$ to ACA if $U_{\mathsf{PCA}}(\mathbb{A}^{\mathrm{aca}}) = \mathbb{A}$. In order words, if $\mathbb{A} = \langle A, (\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$ and $\mathbb{A}^{\mathrm{aca}} = \langle A, (\bar\tau_A\alpha \mid \alpha \in \mathcal{T}_{\mathrm{aca}})\rangle$, then $\mathbb{A}^{\mathrm{aca}}$ is an extension of $\mathbb{A}$ to ACA if

$$\bar\tau_A\alpha = \tau_A\alpha, \quad \alpha \in \mathcal{T}_{\mathrm{pca}}. \tag{3.7}$$

We identify one more category and show its equivalence to ACA. Let $\mathsf{PCA}_{\circlearrowleft}$ be the category whose objects are pairs $(\mathbb{A}, \omega)$ where $\mathbb{A}$ is a positively convex algebra and ω is an involutory homomorphism ($\omega^2 = \mathrm{id}$) on $\mathbb{A}$ with the property that

$$\begin{aligned}&\tau_A(p_1,\ldots,p_n,q_1,\ldots,q_n)(x_1,\ldots,x_n,\omega x_1,\ldots,\omega x_n)\\ &\quad = \tau_A\big(p_1',\ldots,p_n',q_1',\ldots,q_n'\big)(x_1,\ldots,x_n,\omega x_1,\ldots,\omega x_n),\end{aligned} \tag{3.8}$$

whenever $n \in \mathbb{N}^+$,

$$(p_1, \dots, p_n, q_1, \dots, q_n), (p'_1, \dots, p'_n, q'_1, \dots, q'_n) \in \mathcal{T}_{\mathrm{pca}},$$
$$p_k - q_k = p'_k - q'_k, \quad k = 1, \dots, n.$$

A morphism f in $\mathsf{PCA}_\circlearrowleft$ from $(\mathbb{A}, \omega_A)$ to $(\mathbb{B}, \omega_B)$ is a PCA homomorphism from $\mathbb{A}$ to $\mathbb{B}$ that commutes with the involutory morphisms, i.e., $\omega_B \circ f = f \circ \omega_A$. There is an obvious forgetful functor $U_\circlearrowleft\colon \mathsf{PCA}_\circlearrowleft \to \mathsf{PCA}$ that forgets the involutory homomorphism, i.e., $U_\circlearrowleft(\mathbb{A}, \omega) = \mathbb{A}$.

Proposition 3.9. *The category* $\mathsf{PCA}_\circlearrowleft$ *is isomorphic to the category* ACA *of absolutely convex algebras. We have the following situation of categories and functors.*

$$\begin{tikzcd}
\mathsf{PCA}_\circlearrowleft \arrow[rr, bend left, "H"] \arrow[rr, phantom, "\cong"] \arrow[dr, "U_\circlearrowleft"'] & & \mathsf{ACA} \arrow[ll, bend left, "I"] \arrow[dl, "U_{\mathsf{PCA}}"] \\
& \mathsf{PCA} &
\end{tikzcd}$$

Proof. Let $\mathbb{A}$ be a positively convex algebra that admits an involutory homomorphism ω satisfying (3.8). The functor H is identity on morphisms and on objects it acts as

$$H(\mathbb{A}, \omega) = \big\langle A, (\bar{\tau}_{A,\omega}\alpha \mid \alpha \in \mathcal{T}_{\mathrm{aca}})\big\rangle$$

with

$$\bar{\tau}_{A,\omega}(p_i)_{i=1}^n(x_1, \dots, x_n) := \tau_A\big(p_1^+, \dots, p_n^+, p_1^-, \dots, p_n^-\big)(x_1, \dots, x_n, \omega x_1, \dots, \omega x_n), \tag{3.9}$$

where

$$p^+ := \max\{p, 0\}, \qquad p^- := -\min\{p, 0\}, \quad p \in \mathbb{R}. \tag{3.10}$$

Note that $p = p^+ - p^-$. Moreover, we have

$$\sum_{i=1}^n p_i^+ + \sum_{i=1}^n p_i^- = \sum_{i=1}^n |p_i|,$$

and hence the operation in $\mathcal{T}_{\mathrm{pca}}$ on the right side of (3.9) is well-defined. If $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}$, then $p_i^+ = p_i$ and $p_i^- = 0$ for all $i \in \{1, \dots, n\}$, and the extended projection law for $\mathbb{A}$ gives

$$\begin{aligned}
\bar{\tau}_{A,\omega}(p_i)_{i=1}^n(x_1, \dots, x_n) &= \tau_A(p_1, \dots, p_n, 0, \dots, 0)(x_1, \dots, x_n, \omega x_1, \dots, \omega x_n) \\
&= \tau_A(p_i)_{i=1}^n(x_1, \dots, x_n).
\end{aligned}$$

It is not difficult to see that the projection axiom holds for $H(\mathbb{A}, \omega)$; we skip the details. To check the ACA barycenter axiom, let $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{aca}}$ and $(p_{ij})_{j=1}^m \in \mathcal{T}_{\mathrm{aca}}$, $i = 1, \dots, n$, be given. First, compute

$$\begin{aligned}
&\omega\bar{\tau}_{A,\omega}(p_{ij})_{j=1}^m(x_1, \dots, x_m) \\
&\quad = \omega\tau_A\big(p_{i1}^+, \dots, p_{im}^+, p_{i1}^-, \dots, p_{im}^-\big)(x_1, \dots, x_m, \omega x_1, \dots, \omega x_m) \\
&\quad = \tau_A\big(p_{i1}^+, \dots, p_{im}^+, p_{i1}^-, \dots, p_{im}^-\big)(\omega x_1, \dots, \omega x_m, x_1, \dots, x_m) \\
&\quad = \tau_A\big(p_{i1}^-, \dots, p_{im}^-, p_{i1}^+, \dots, p_{im}^+\big)(x_1, \dots, x_m, \omega x_1, \dots, \omega x_m),
\end{aligned} \tag{3.11}$$

where the last equality holds by commutativity, Lemma 3.4(i), and the one but last from ω being involutory. Next we use the PCA barycenter axiom for $\mathbb{A}$ to compute

$$\begin{aligned}
&\bar{\tau}_{A,\omega}(p_i)_{i=1}^n\big(\bar{\tau}_{A,\omega}(p_{1j})_{j=1}^m(x_1,\dots,x_m),\dots,\bar{\tau}_{A,\omega}(p_{nj})_{j=1}^n(x_1,\dots,x_m)\big)\\
&= \sum_{i=1}^n p_i^+\left(\sum_{j=1}^m p_{i,j}^+ x_j + \sum_{j=1}^m p_{i,j}^- \omega x_j\right) + \sum_{i=1}^n p_i^-\omega\left(\sum_{j=1}^m p_{i,j}^+ x_j + \sum_{j=1}^m p_{i,j}^- \omega x_j\right)\\
&\overset{(3.11)}{=} \sum_{i=1}^n p_i^+\left(\sum_{j=1}^m p_{i,j}^+ x_j + \sum_{j=1}^m p_{i,j}^- \omega x_j\right) + \sum_{i=1}^n p_i^-\left(\sum_{j=1}^m p_{i,j}^- x_j + \sum_{j=1}^m p_{i,j}^+ \omega x_j\right)\\
&= \sum_{j=1}^m\left(\sum_{i=1}^n p_i^+p_{i,j}^+ + \sum_{i=1}^n p_i^-p_{i,j}^-\right)x_j + \sum_{j=1}^m\left(\sum_{i=1}^n p_i^+p_{i,j}^- + \sum_{i=1}^n p_i^-p_{i,j}^+\right)\omega x_j\\
&\overset{(*)}{=} \sum_{j=1}^m\left(\sum_{i=1}^n p_ip_{i,j}\right)^+ x_j + \sum_{j=1}^m\left(\sum_{i=1}^n p_ip_{i,j}\right)^- \omega x_j\\
&= \bar{\tau}_{A,\omega}\left(\sum_{i=1}^n p_ip_{i,j}\right)_{j=1}^m(x_1,\dots,x_m)
\end{aligned}$$

where the equality marked with $(*)$ follows from (3.8) since

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i^+p_{ij}^+ + \sum_{i=1}^n p_i^-p_{ij}^-\right) - \left(\sum_{i=1}^n p_i^+p_{ij}^- + \sum_{i=1}^n p_i^-p_{ij}^+\right)\\
&= \sum_{\substack{i=1\\ p_i\geq 0,\, p_{ij}\geq 0}}^n p_ip_{ij} + \sum_{\substack{i=1\\ p_i<0,\, p_{ij}<0}}^n p_ip_{ij} - \sum_{\substack{i=1\\ p_i\geq 0,\, p_{ij}<0}}^n p_i(-p_{ij}) - \sum_{\substack{i=1\\ p_i<0,\, p_{ij}\geq 0}}^n (-p_i)p_{ij}\\
&= \sum_{i=1}^n p_ip_{ij}\\
&= \left(\sum_{i=1}^n p_ip_{ij}\right)^+ - \left(\sum_{i=1}^n p_ip_{ij}\right)^-.
\end{aligned}$$

Hence, we have shown that $H(\mathbb{A},\omega)$ is an extension of $\mathbb{A}$ to ACA.

A $\mathsf{PCA}_\circlearrowleft$ morphism $f\colon(\mathbb{A},\omega_A)\to(\mathbb{B},\omega_B)$ is an ACA morphism from $H(\mathbb{A},\omega_A)$ to $H(\mathbb{B},\omega_B)$ as

$$\begin{aligned}
&f\big(\bar{\tau}_{A,\omega_A}(p_i)_{i=1}^n(x_1,\dots,x_n)\big)\\
&\quad= f\big(\tau_A\big(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-\big)(x_1,\dots,x_n,\omega_Ax_1,\dots,\omega_Ax_n)\big)\\
&\quad= \tau_B\big(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-\big)\big(f(x_1),\dots,f(x_n),f(\omega_Ax_1),\dots,f(\omega_Ax_n)\big)\\
&\quad= \tau_B\big(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-\big)\big(f(x_1),\dots,f(x_n),\omega_Bf(x_1),\dots,\omega_Bf(x_n)\big)\\
&\quad= \bar{\tau}_{B,\omega_B}(p_i)_{i=1}^n\big(f(x_1),\dots,f(x_n)\big).
\end{aligned}$$

Since $H(\mathbb{A},\omega)$ is an extension of $\mathbb{A}$ to ACA, we have $U_{\mathsf{PCA}}\circ H = U_\circlearrowleft$.

Let now $\mathbb{A} = \langle A,(\tau_A\alpha \mid \alpha\in\mathcal{T}_{\mathrm{aca}})\rangle$ be in ACA. The functor I is defined on objects by $I(\mathbb{A}) = (U_{\mathsf{PCA}}(\mathbb{A}),\bar{\omega}_A)\in\mathsf{PCA}_\circlearrowleft$ with $\bar{\omega}_A(x) = \tau_A(-1)x$. It leaves the morphisms unchanged, i.e., $I(f) = f$. All this is well defined since, for $(p_i)_{i=1}^n\in\mathcal{T}_{\mathrm{aca}}$, we have

$$\begin{aligned}\tau_A(-1)\big(\tau_A(p_i)_{i=1}^n(x_1,\dots,x_n)\big) &= \tau_A(-p_i)_{i=1}^n(x_1,\dots,x_n)\\ &= \tau_A(p_i)_{i=1}^n\big(\tau_A(-1)(x_1),\dots,\tau_A(-1)(x_n)\big).\end{aligned}$$

This shows that $\bar{\omega}_A$ is a PCA endomorphism of $U_{\mathsf{PCA}}(\mathbb{A})$ (it is even an ACA endomorphism of $\mathbb{A}$). Moreover, we have

$$\tau_A(-1)\big(\tau_A(-1)(x)\big) = \tau_A(1)(x) = x,$$

i.e., $\bar{\omega}_A$ is involutory. Eq. (3.8) will follow from

$$\begin{aligned}&\tau_A(p_1,\dots,p_n,q_1,\dots,q_n)(x_1,\dots,x_n,\bar{\omega}_A x_1,\dots,\bar{\omega}_A x_n)\\ &\quad = \tau_A(p_1-q_1,\dots,p_n-q_n)(x_1,\dots,x_n),\end{aligned} \tag{3.12}$$

whenever $(p_1,\dots,p_n,q_1,\dots,q_n)\in\mathcal{T}_{\mathrm{aca}}$. Note here that

$$\sum_{i=1}^n |p_i-q_i| \le \sum_{i=1}^n\big(|p_i|+|q_i|\big) = \sum_{i=1}^n |p_i| + \sum_{i=1}^n |q_i| \le 1,$$

and hence the operation written on the right-hand side is legitimate.

To see (3.12), we compute using the ACA barycenter axiom

$$\begin{aligned}\sum_{i=1}^n p_i x_i + \sum_{i=1}^n q_i\bar{\omega}_A(x_i) &= \sum_{i=1}^n p_i x_i + \sum_{i=1}^n q_i(-1)x_i\\ &= \sum_{i=1}^n (p_i-q_i)x_i.\end{aligned}$$

Finally, for any ACA homomorphism $f\colon\mathbb{A}\to\mathbb{B}$ and $x\in A$ we have

$$f\circ\bar{\omega}_A(x) = f\big(\tau_A(-1)x\big) = \tau_B(-1)f(x) = \bar{\omega}_B\circ f(x)$$

showing that f is a $\mathsf{PCA}_\circlearrowleft$ morphism from $I(\mathbb{A})$ to $I(\mathbb{B})$, as f is certainly a PCA homomorphism from $U_{\mathrm{pca}}(\mathbb{A})$ to $U_{\mathrm{pca}}(\mathbb{B})$. From the definition of I it is obvious that $U_\circlearrowleft\circ I = U_{\mathsf{PCA}}$.

It remains to show that H and I are inverse to each other: For the morphisms, this is clear. Let $(\mathbb{A},\omega_A)$ in $\mathsf{PCA}_\circlearrowleft$. Then

$$I\circ H(\mathbb{A},\omega) = I\big(H(\mathbb{A},\omega)\big) = \big(U_{\mathsf{PCA}}\big(H(\mathbb{A},\omega)\big),\bar{\omega}_A\big) = (\mathbb{A},\omega)$$

since in $H(\mathbb{A},\omega)$ we have

$$\bar{\tau}_{A,\omega}(-1)(x) = \tau_A(0,1)(x,\omega x) = \omega x$$

by the projection axiom. For $\mathbb{A} = \langle A,(\tau_A\alpha \mid \alpha\in\mathcal{T}_{\mathrm{aca}})\rangle$ we have $U_{\mathsf{PCA}}(\mathbb{A}) = \langle A,(\tau_A\alpha \mid \alpha\in\mathcal{T}_{\mathrm{pca}})\rangle$ and

$$H\circ I(\mathbb{A}) = H\big(U_{\mathsf{PCA}}(\mathbb{A}),\bar{\omega}_A\big) = \big\langle A,(\bar{\tau}_{A,\bar{\omega}_A}\alpha \mid \alpha\in\mathcal{T}_{\mathrm{aca}})\big\rangle = \mathbb{A}$$

since, as a consequence of the axioms,

$$\begin{aligned}\bar{\tau}_{A,\bar{\omega}_A}(p_i)_{i=1}^n(x_1,\dots,x_n) &= \tau_A\big(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-\big)(x_1,\dots,x_n,\bar{\omega}_A x_1,\dots,\bar{\omega}_A x_n)\\ &= \sum_{i=1}^n (p_i^+ - p_i^-)x_i = \sum_{i=1}^n p_i x_i = \tau_A(p_i)_{i=1}^n(x_1,\dots,x_n). \qquad \square\end{aligned}$$

As a consequence of Proposition 3.9, not every algebra in PCA is in the image of U_{PCA}, i.e., not every positively convex algebra has an extension to an absolutely convex algebra. Namely, there are positively convex algebras which do not admit an involutory endomorphism satisfying (3.8). Examples of such can already be found in euclidean space, as the following proposition (proven in [39]) shows.

Proposition 3.10. *Let $K \subseteq \mathbb{R}^n$ be a polytope with $0 \in K$, and let $\mathbb{K} = \langle K, (\tau_K \alpha \mid \alpha \in \mathcal{T}_{\mathrm{pca}})\rangle$ be the positively convex algebra with the operations given as the usual sum of vectors in $\mathbb{R}^n$*

$$\tau_K(p_i)_{i=1}^n(x_1,\dots,x_n) := \sum_{i=1}^n p_i x_i, \quad (p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{pca}}.$$

If $\mathbb{K}$ has an extension to ACA, then K has an even number of extremal points. $\square$

We close the discussion of ACA by making explicit the relationship between ACA congruences and PCA congruences. Denote by $\mathrm{Con}_{\mathsf{ACA}}\,\mathbb{A}$ the set of all congruences of $\mathbb{A}$ in ACA.

Lemma 3.11. *Let $\mathbb{A}$ in ACA. Then*

$$\mathrm{Con}_{\mathsf{ACA}}\,\mathbb{A} = \mathrm{Con}_{\mathsf{PCA}}\,U_{\mathsf{PCA}}(\mathbb{A}) \cap [\bar{\omega}_A]$$

where $[\bar{\omega}_A]$ denotes the set of $\bar{\omega}_A$-invariant relations on A, i.e.,

$$[\bar{\omega}_A] = \big\{\Theta \subseteq A\times A \;\big|\; (\bar{\omega}_A \times \bar{\omega}_A)\Theta \subseteq \Theta\big\}$$

for $\bar{\omega}_A = \tau_A(-1)$ as before.

Proof. If $\Theta \in \mathrm{Con}_{\mathsf{ACA}}\,\mathbb{A}$ then $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\,U_{\mathsf{PCA}}(\mathbb{A})$ and $(\bar{\omega}_A \times \bar{\omega}_A)\Theta \subseteq \Theta$ as $\bar{\omega}_A = \tau_A(-1)$ is an ACA operation. For the converse, let $\Theta \in \mathrm{Con}_{\mathsf{PCA}}\,U_{\mathsf{PCA}}(\mathbb{A})\cap[\bar{\omega}_A]$. Let $(p_i)_{i=1}^n \in \mathcal{T}_{\mathrm{aca}}$ and $(x_i,x_i') \in \Theta$, $i=1,\dots,n$, be given. Since $\mathbb{A} = H\circ I(\mathbb{A}) = H(U_{\mathsf{PCA}}(\mathbb{A}),\bar{\omega}_A)$, we have from (3.12) that

$$\tau_A(p_i)_{i=1}^n(x_1,\dots,x_n) = \tau_A\big(p_1^+,\dots,p_n^+,p_1^-,\dots,p_n^-\big)(x_1,\dots,x_n,\bar{\omega}_A x_1,\dots,\bar{\omega}_A x_n).$$

The same equation holds for x_i' instead of x_i. Since Θ is a PCA congruence on $U_{\mathsf{PCA}}(\mathbb{A})$, the operations on the right are operations of $U_{\mathsf{PCA}}(\mathbb{A})$, $x_i\Theta x_i'$, and by $\bar{\omega}_A$ invariance $\bar{\omega}_A x_i \Theta \bar{\omega}_A x_i'$, we see that indeed

$$\tau_A(p_i)_{i=1}^n(x_1,\dots,x_n)\Theta\tau_A(p_i)_{i=1}^n\big(x_1',\dots,x_n'\big). \qquad \square$$

Remark 3.12. We can also compare the corresponding homomorphisms in the situation of Lemma 3.11. Let $\mathbb{A}$ be an algebra in ACA. Then $f\colon\mathbb{A}\to\mathbb{B}$ is an ACA homomorphism if and only if $f\colon U_{\mathsf{PCA}}(\mathbb{A}) \to U_{\mathsf{PCA}}(\mathbb{B})$ is a PCA homomorphism and $\bar{\omega}_B\circ f = f\circ\bar{\omega}_A$.

4. Convex equivalences on polytopes

The finitely generated free algebras in CA, PCA, and ACA are polytopes. Therefore, in order to understand all finitely generated algebras in CA, PCA, and ACA, it suffices to describe the congruences of polytopes, which we do in this section.

Let $K \subseteq \mathbb{R}^n$ be a polytope. We consider the convex algebra $\mathbb{K}$ with carrier K and operations inherited from $\mathbb{R}^n$ (as in Proposition 3.10). The following property is a direct consequence of the definitions and Lemma 2.2 but it is an important observation for what follows.

Lemma 4.1. *An equivalence relation Θ on a polytope K is in* $\mathrm{Con}_{\mathsf{CA}}\,\mathbb{K}$ *if and only if it is convex as a subset of $K \times K \subseteq \mathbb{R}^n \times \mathbb{R}^n$, with operations defined component-wise.* $\square$

Throughout the paper, we denote

$$V_K := \mathcal{P}(\operatorname{ext} K) \setminus \{\emptyset\},$$

where $\mathcal{P}(M)$ denotes the power set of a set M, and consider V_K as a join-subsemilattice of the lattice $\mathcal{P}(\operatorname{ext} K)$. The elements of V_K (subsets of $\operatorname{ext} K$) represent facets of K: each facet is the convex hull of an element in V_K.

We denote by $\operatorname{Sub}\mathbb{R}^n$ the set of all linear subspaces of $\mathbb{R}^n$, and consider $\operatorname{Sub}\mathbb{R}^n$ as being ordered by inclusion.

In the following definition we introduce the crucial concepts for describing $\mathrm{Con}_{\mathsf{CA}}\,\mathbb{K}$. Recall that, due to Lemmas 3.8 and 3.11, this immediately yields a description of $\mathrm{Con}_{\mathsf{PCA}}\,\mathbb{K}$ and $\mathrm{Con}_{\mathsf{ACA}}\,\mathbb{K}$ (provided that $\mathbb{K}$ has an extension to PCA or ACA, respectively, which we here denote the same).

Definition 4.2. Let $K \subseteq \mathbb{R}^n$ be a polytope and let $\Theta \in \mathrm{Con}_{\mathsf{CA}}\,\mathbb{K}$. We define a map $\varphi_\Theta\colon V_K \to \operatorname{Sub}\mathbb{R}^n$ by

$$\varphi_\Theta(Y) = \operatorname{span}\{x_2 - x_1 \mid x_1, x_2 \in \breve{\mathrm{co}}Y,\ x_1 \Theta x_2\}, \quad Y \in V_K.$$

We define a graph $\mathcal{G}_\Theta$ with vertices V_K and (undirected) edges E_Θ given by

$$\{Y_1, Y_2\} \in E_\Theta \quad \Longleftrightarrow \quad \Theta \cap (\breve{\mathrm{co}}Y_1 \times \breve{\mathrm{co}}Y_2) \neq \emptyset, \quad Y_1, Y_2 \in V_K.$$

We denote by $\approx_\Theta$ the equivalence relation on V_K defined as

$$Y_1 \approx_\Theta Y_2 \quad \Longleftrightarrow \quad Y_1 \text{ and } Y_2 \text{ are connected by a path in } \mathcal{G}_\Theta.$$

The equivalence classes of $\approx_\Theta$ are the connected components of $\mathcal{G}_\Theta$, see, e.g., [8]. Note that there is always an edge in $\mathcal{G}_\Theta$ connecting a vertex Y with itself, even if $|Y| = 1$, due to the definition of $\breve{\mathrm{co}}Y$.

The intuition behind the graph E_Θ and the linear subspaces $\varphi_\Theta(Y)$ is as follows. As mentioned, each element (vertex) in V_K represents a facet of K. The subspace $\varphi_\Theta(Y)$ describes the congruence within the interior of the facet represented by Y. For example, if Y contains three elements, then the facet it represents is a triangle. The theorems below will show that if $\varphi_\Theta(Y)$ is the zero subspace, then all points in the interior of this triangle are separate classes of Θ; if $\varphi_\Theta(Y)$ is a one dimensional subspace, then there are infinitely many parallel classes in the interior of the triangle, all in the direction of $\varphi_\Theta(Y)$; finally if $\varphi_\Theta(Y)$ is a two dimensional subspace, then all points in the triangle belong to a single class of Θ. The edges E_Θ of the graph $\mathcal{G}_\Theta$ describe the Θ-connectivity between the facets. For example, if $y_1, y_2, y_3 \in \operatorname{ext} K$, then $\{y_1\}$ and $\{y_2, y_3\}$ in V_K are connected by an edge of $\mathcal{G}_\Theta$ if y_1 is Θ-related to some point in the open line segment connecting y_2 and y_3.

Within this section we give and prove three theorems that make this intuition precise and give a complete description of the congruence lattice $\mathrm{Con}_{\mathsf{CA}}\,\mathbb{K}$. Before, we present several auxiliary results that are essential for our proofs. From now on, unless stated otherwise, let K be a polytope and Θ a convex congruence of $\mathbb{K}$.

Geometric consequences of convexity

We start with a definition of the useful concept of a perspective.

Definition 4.3. For each $z \in K$ we denote by $\Phi_z\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ the map defined as

$$\Phi_z(s,x) := sz + (1-s)x, \quad s \in [0,1],\ x \in \mathbb{R}^n.$$

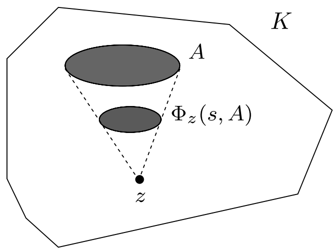
Based on geometric intuition, we refer to Φ_z as *the perspective with center z*.

Since K is convex and $z \in K$, we have $\Phi_z(s,K) \subseteq K$ for any $s \in [0,1]$. Moreover, $\Phi_z(0,.)$ is the identity map, and $\Phi_z(1,.)$ is the constant map with value z.

The following observation is simple but important, and we further refer to it as perspective invariance. Intuitively speaking, perspective invariance means that a congruence class cannot split and distribute over several different classes when moved with a perspective.

Lemma 4.4. *Let $A \subseteq K$ be an equivalence class of Θ. Then, for each $z \in K$ and $s \in [0,1]$, there exists an equivalence class $A_{z,s} \subseteq K$ of Θ with*

$$\Phi_z(s,A) \subseteq A_{z,s}.$$



Perspective invariance implies in particular that each equivalence class of Θ is convex.

Proof. Let $x_1, x_2 \in K$ with $x_1 \Theta x_2$, $z \in K$, and $s \in [0,1]$ be given. Since $z\Theta z$ and $(x,y) \mapsto sx + (1-s)y$ is an operation of $\mathbb{K} \in \mathsf{CA}$, we have

$$\big(\Phi_z(s,x_1), \Phi_z(s,x_2)\big) = \big(sz + (1-s)x_1, sz + (1-s)x_2\big) \in \Theta.$$

To deduce that equivalence classes are convex, let $x_1, x_2 \in K$ with $x_1 \Theta x_2$, and $s \in [0,1]$ be given. Perspective invariance with $z := x_1$ gives

$$\big(x_1, sx_1 + (1-s)x_2\big) = \big(\Phi_{x_1}(s,x_1), \Phi_{x_1}(s,x_2)\big) \in \Theta. \qquad \square$$

Also the next fact is simple but useful.

Lemma 4.5.

(*i*) *Let* $Y_1, Y_2 \in V_K$ *be given. Then*

$$\forall z \in \breve{\mathrm{co}} Y_1,\ x \in \breve{\mathrm{co}} Y_2 : \Phi_z(s,x) \in \breve{\mathrm{co}}(Y_1 \cup Y_2), \quad s \in (0,1).$$

(*ii*) *Let* $Y \in V_K$ *be given. Then*

$$\forall z \in \mathrm{co}\, Y,\ x \in \breve{\mathrm{co}} Y : \Phi_z(s,x) \in \breve{\mathrm{co}} Y, \quad s \in [0,1).$$
$$\forall z \in \breve{\mathrm{co}} Y,\ x \in \mathrm{co}\, Y : \Phi_z(s,x) \in \breve{\mathrm{co}} Y, \quad s \in (0,1].$$

Proof. For the proof of (*i*) write

$$z = \sum_{y \in Y_1} \lambda^1_y y, \quad \lambda^1_y \in (0,1],\ \sum_{y \in Y_1} \lambda^1_y = 1,$$
$$x = \sum_{y \in Y_2} \lambda^2_y y, \quad \lambda^2_y \in (0,1],\ \sum_{y \in Y_2} \lambda^2_y = 1.$$

Then

$$\Phi_z(s,x) = s\Big(\sum_{y \in Y_1} \lambda^1_y y\Big) + (1-s)\Big(\sum_{y \in Y_2} \lambda^2_y y\Big)$$
$$= \sum_{y \in Y_1 \setminus Y_2} s\lambda^1_y y + \sum_{y \in Y_1 \cap Y_2} \big(s\lambda^1_y + (1-s)\lambda^2_y\big) y + \sum_{y \in Y_2 \setminus Y_1} (1-s)\lambda^2_y y.$$

All coefficients in these sums are positive and they sum up to 1. Hence, $\Phi_z(s,x) \in \breve{\mathrm{co}}(Y_1 \cup Y_2)$. Item (*ii*) follows in the same manner. □

Also the property shown in the next lemma will be used in several instances.

Lemma 4.6. *Let* $Y \in V_K$, $|Y| \geq 2$, *and let* $x_1, x_2 \in \breve{\mathrm{co}} Y$, $x_1 \neq x_2$. *Consider the map* $\Gamma: \mathbb{R} \to \mathrm{aff}\, Y$ *given by*

$$\Gamma(t) := t x_2 + (1-t) x_1, \quad t \in \mathbb{R}.$$

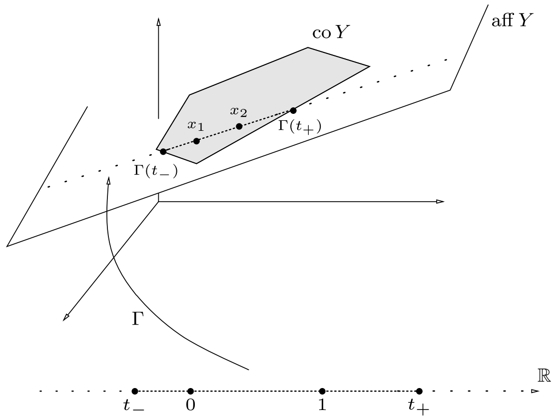
Clearly, $\Gamma(\mathbb{R}) = \mathrm{aff}\{x_1, x_2\}$ *is the line containing* x_1 *and* x_2.[3] *Then*

$$\Phi_{\Gamma(r)}\big(s, \Gamma(t)\big) = \Gamma\big(sr + (1-s)t\big), \quad r,t \in \mathbb{R},\ s \in [0,1]. \tag{4.1}$$

There exist numbers $t_- < 0$ *and* $t_+ > 1$, *such that*

$$\Gamma^{-1}(\breve{\mathrm{co}} Y) = (t_-, t_+), \qquad \Gamma^{-1}(\mathrm{co}\, Y) = [t_-, t_+]$$

[3] Note that Γ actually depends on the points x_1 and x_2 but we prefer a light, overloaded notation.



Proof. To show (4.1), we compute

$$\begin{aligned}\Phi_{\Gamma(r)}\big(s,\Gamma(t)\big) &= s\Gamma(r)+(1-s)\Gamma(t)\\ &= s\big(rx_2+(1-r)x_1\big)+(1-s)\big(tx_2+(1-t)x_1\big)\\ &= \big(sr+(1-s)t\big)x_2+\big(s(1-r)+(1-s)(1-t)\big)x_1\\ &= \big(sr+(1-s)t\big)x_2+\big(1-\big(sr+(1-s)t\big)\big)x_1=\Gamma\big(sr+(1-s)t\big).\end{aligned}$$

Recall that $\mathbb{R}$ and $\operatorname{aff} Y$ inherit the euclidean topology from $\mathbb{R}^n$. Moreover they also inherit the euclidean metric from $\mathbb{R}^n$. Also, a line is continuous, i.e., Γ is a continuous function. Consider the set $\Gamma^{-1}(\breve{\text{co}}Y)$. Since Γ is continuous and $\breve{\text{co}}Y$ is an open subset of $\operatorname{aff} Y$ by Lemma 2.5, this set is an open subset of $\mathbb{R}$. Since Γ is linear and $\breve{\text{co}}Y$ is convex, cf. Section 2, it is convex. We will now show that $\Gamma^{-1}(\breve{\text{co}}Y)$ is bounded. Note that $\Gamma(t)=x_1+t(x_2-x_1)$. Therefore, for $\|\cdot\|$ denoting the euclidean norm, using the downward triangle inequality, we get

$$\big\|\Gamma(t)\big\|\geq\big|\big\|t(x_2-x_1)\big\|-\|x_1\|\big|\geq|t|\cdot\|x_2-x_1\|-\|x_1\|.$$

Hence, for each positive real number R, if $|t|>\frac{R+\|x_1\|}{\|x_2-x_1\|}$, then $\|\Gamma(t)\|>R$. This shows that the inverse image by Γ of a bounded set in $\operatorname{aff} Y$ is a bounded set in $\mathbb{R}$. Now, since $\breve{\text{co}}Y\subseteq\operatorname{co} Y$ and the latter is bounded by Lemma 2.4, we get that $\Gamma^{-1}(\breve{\text{co}}Y)$ is bounded. Note that in $\mathbb{R}$ a set is open, convex, and bounded if and only if it is a finite open interval. Hence

$$\Gamma^{-1}(\breve{\text{co}}Y)=(t_-,t_+)$$

with some numbers $t_-,t_+\in\mathbb{R}$. Since $\Gamma(0)=x_1\in\breve{\text{co}}Y$ and $\Gamma(1)=x_2\in\breve{\text{co}}Y$, we must have $t_-<0$ and $t_+>1$.

Again by Lemma 2.4, $\operatorname{co} Y$ is the closure of $\breve{\text{co}}Y$ and hence continuity of Γ implies that $\Gamma^{-1}(\operatorname{co} Y)$ is closed. Since $\Gamma^{-1}(\breve{\text{co}}Y)\subseteq\Gamma^{-1}(\operatorname{co} Y)$ and $[t_-,t_+]$ is the closure of $\Gamma^{-1}(\breve{\text{co}}Y)=(t_-,t_+)$, we also have $[t_-,t_+]\subseteq\Gamma^{-1}(\operatorname{co} Y)$.

To show the opposite inclusion, let $t\in\mathbb{R}$ with $\Gamma(t)\in\operatorname{co} Y$ be given. If $t\in[0,1]$, we have $\Gamma(t)\in\operatorname{co}\{x_1,x_2\}\subseteq\breve{\text{co}}Y$, and hence $t\in(t_-,t_+)$. Next, consider the case that $t>1$. Choose $t'\in(1,t)$ and set $s:=\frac{t'}{t}$. Then $s\in(0,1)$ and

$$\begin{aligned}\Phi_{\Gamma(t)}(s,x_1) &= s\Gamma(t)+(1-s)x_1=s\big(tx_2+(1-t)x_1\big)+(1-s)x_1\\ &= (st)x_2+(1-st)x_1=t'x_2+\big(1-t'\big)x_1=\Gamma\big(t'\big).\end{aligned}$$

By Lemma 4.5, we have $\Gamma(t') \in \check{\mathrm{co}}Y$. Thus $t' \in (t_-, t_+)$. Since t' was arbitrary in $(1,t)$, we have $(1,t) \subseteq (t_-, t_+)$ and hence $[1,t] \subseteq [t_-, t_+]$ showing $t \in [t_-, t_+]$. The case that $t < 0$ is analogous. □

The structure of $\mathcal{G}_\Theta$

In order to understand the structure of $\mathcal{G}_\Theta$, we need one preparatory result.

Lemma 4.7. *Let $Y_1, Y_2 \in V_K$, and assume that $Y_1 \subseteq Y_2$. If $\{Y_1, Y_2\} \in E_\Theta$, i.e.,*

$$\exists x_1 \in \check{\mathrm{co}}Y_1,\ x_2 \in \check{\mathrm{co}}Y_2 : x_1 \Theta x_2,$$

then actually

$$\forall x_1' \in \check{\mathrm{co}}Y_1,\ \exists x_2' \in \check{\mathrm{co}}Y_2 : x_1' \Theta x_2'.$$

Proof. If $Y_1 = Y_2$, we can always choose $x_2' := x_1'$. If $|Y_1| = 1$, also $|\check{\mathrm{co}}Y_1| = 1$, and hence the assertion is trivial. Note that in general the assertion cannot be obtained in a trivial way, as $\check{\mathrm{co}}Y_1 \cap \check{\mathrm{co}}Y_2 = \emptyset$ if $Y_1 \subset Y_2$. Think for example of $|Y_1| = 2$ and $|Y_2| = 3$ when $\check{\mathrm{co}}Y_2$ is the interior of a triangle and $\check{\mathrm{co}}Y_1$ is the interior of one of its sides.

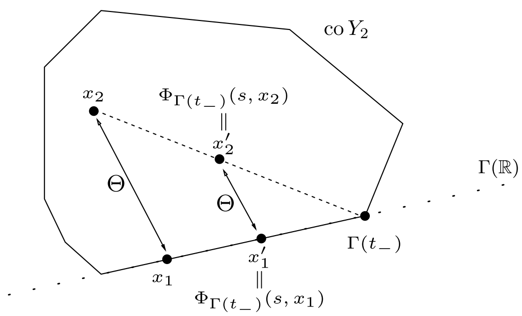
Assume that $Y_1 \subset Y_2$, $|Y_1| > 1$, and choose $x_1 \in \check{\mathrm{co}}Y_1$ and $x_2 \in \check{\mathrm{co}}Y_2$ with $x_1 \Theta x_2$. Let $x_1' \in \check{\mathrm{co}}Y_1$ be given. If $x_1' = x_1$, we can take $x_2' := x_2$. Hence, assume that $x_1' \neq x_1$. Consider the line Γ containing the points x_1', x_1, i.e.,

$$\Gamma(t) := t x_1 + (1-t) x_1', \quad t \in \mathbb{R}.$$

By Lemma 4.6, we have $\Gamma^{-1}(\check{\mathrm{co}}Y_1) = (t_-, t_+)$ and $\Gamma^{-1}(\mathrm{co}\,Y_1) = [t_-, t_+]$ with some $t_- < 0$ and $t_+ > 1$. Set $s := \frac{1}{1-t_-}$, then $s \in (0,1)$ and $st_- + (1-s)\cdot 1 = 0$. By (4.1) since $\Gamma(1) = x_1$ we have

$$\Phi_{\Gamma(t_-)}(s, x_1) = \Gamma\big(st_- + (1-s)\cdot 1\big) = \Gamma(0) = x_1'.$$

Take $x_2' := \Phi_{\Gamma(t_-)}(s, x_2)$. By perspective invariance and the above equality, we get $x_1' \Theta x_2'$. Since $\Gamma(t_-) \in \mathrm{co}\,Y_1 \subseteq \mathrm{co}\,Y_2$ and $x_2 \in \check{\mathrm{co}}Y_2$, Lemma 4.5(*ii*) implies $x_2' = \Phi_{\Gamma(t_-)}(s, x_2) \in \check{\mathrm{co}}Y_2$, which completes the proof.



□

Corollary 4.8. *Let $Y_1, Y_2, Y_3 \in V_K$, and assume that $\{Y_1, Y_2\}, \{Y_2, Y_3\} \in E_\Theta$. If $Y_2 \subseteq Y_3$, then also $\{Y_1, Y_3\} \in E_\Theta$.*

Proof. Choose $x_1 \in \check{\mathrm{co}}Y_1$ and $x_2 \in \check{\mathrm{co}}Y_2$ with $x_1 \Theta x_2$. According to Lemma 4.7 there exists $x_3 \in \check{\mathrm{co}}Y_3$ with $x_2 \Theta x_3$. Transitivity gives $x_1 \Theta x_3$, and we see that $\{Y_1, Y_3\} \in E_\Theta$. □

Behavior of Θ inside of $\check{\mathrm{co}} Y$

Let $Y \in V_K$. Our aim is to describe $\Theta \cap (\check{\mathrm{co}} Y \times \check{\mathrm{co}} Y)$. If Y contains only one element, then $\check{\mathrm{co}} Y = Y$ and clearly $\Theta \cap (\check{\mathrm{co}} Y \times \check{\mathrm{co}} Y) = Y \times Y$. Hence, assume that $|Y| \geq 2$.

The main observation is that an equivalence class of Θ which contains two different points of $\check{\mathrm{co}} Y$ must already stretch all the way to the boundary of $\check{\mathrm{co}} Y$. This is the analogue of [30, Theorem 1.3], where a similar result was established for congruences of absolutely convex algebras. However, the proof given there does not immediately carry over to the presently considered situation, since we are bound to operations in $\mathcal{T}_{\mathrm{ca}}$, i.e., true convex combinations.

Lemma 4.9. *Let $x_1, x_2 \in \check{\mathrm{co}} Y$, $x_1 \neq x_2$, with $x_1 \Theta x_2$. Then*

$$\mathrm{aff}\{x_1, x_2\} \cap \check{\mathrm{co}} Y \subseteq [x_1]_\Theta,$$

where $[x_1]_\Theta$ denotes the Θ-equivalence class of x_1.

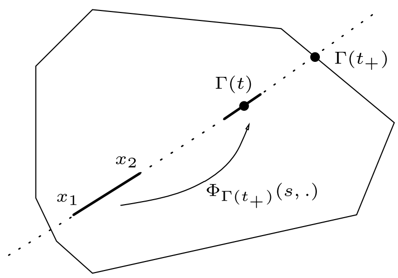
Proof. Consider the line containing the points x_1 and x_2, i.e., the map

$$\Gamma(t) := t x_2 + (1-t) x_1, \quad t \in \mathbb{R},$$

and let t_-, t_+ be as given by Lemma 4.6. Recall that $\mathrm{aff}\{x_1, x_2\} = \Gamma(\mathbb{R})$. As a first step we show that, for each $t \in (t_-, t_+)$, the set $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \check{\mathrm{co}} Y)$ contains an open neighborhood of t.

Consider first the case that $t \geq \frac{1}{2}$. Choose $s \in [0, 1)$ such that $s t_+ + (1-s)\frac{1}{2} = t$. This is doable since under the assumption for t we have $t_+ > \frac{1}{2}$. Then by (4.1)

$$\begin{aligned}
\Phi_{\Gamma(t_+)}(s, x_1) &= \Phi_{\Gamma(t_+)}\big(s, \Gamma(0)\big) = \Gamma(s t_+), \\
\Phi_{\Gamma(t_+)}(s, x_2) &= \Phi_{\Gamma(t_+)}\big(s, \Gamma(1)\big) = \Gamma\big(s t_+ + (1-s)\big), \\
\Phi_{\Gamma(t_+)}\left(s, \frac{1}{2}(x_1 + x_2)\right) &= \Phi_{\Gamma(t_+)}\left(s, \Gamma\left(\frac{1}{2}\right)\right) = \Gamma(t).
\end{aligned}$$



Projective invariance implies

$$\Gamma(s t_+) \Theta \Gamma\big(s t_+ + (1-s)\big).$$

Note that $\mathrm{co}(Y + z) = \mathrm{co}(Y) + z$ and for any linear map f it holds that $f(\mathrm{co}\, Y) = \mathrm{co}(f(Y))$. Since the map $t \mapsto \Gamma(t) - x_1$ is linear, we have

$$\Gamma\big([s t_+, s t_+ + (1-s)]\big) = \mathrm{co}\big\{\Gamma(s t_+), \Gamma\big(s t_+ + (1-s)\big)\big\} \tag{4.2}$$

and we conclude from convexity of equivalence classes that

$$\Gamma\big([st_+, st_+ + (1-s)]\big) \subseteq \big[\Gamma(st_+)\big]_\Theta.$$

Since $st_+ < t < st_+ + (1-s)$, in particular $\Gamma(t) \in [\Gamma(st_+)]_\Theta$ and so $[\Gamma(st_+)]_\Theta = [\Gamma(t)]_\Theta$. It follows that

$$t \in \big(st_+, st_+ + (1-s)\big) \subseteq \Gamma^{-1}\big(\big[\Gamma(t)\big]_\Theta\big).$$

We have $t_- < 0 \le st_+$ and (since $t_+ > 1$) $st_+ + (1-s) < st_+ + (1-s)t_+ = t_+$. Therefore,

$$\Gamma\big((st_+, st_+ + (1-s))\big) \subseteq \Gamma\big((t_-, t_+)\big) \subseteq \breve{\operatorname{co}} Y.$$

Hence, we have found an open neighborhood of t which is contained in $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\operatorname{co}} Y)$. The case that $t \le \frac{1}{2}$ is treated in the same way, using the perspective $\Phi_{\Gamma(t_-)}$.

From this it is easy to deduce that actually for each $t \in (t_-, t_+)$ the set $\Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\operatorname{co}} Y)$ is open: Let $t' \in \Gamma^{-1}([\Gamma(t)]_\Theta \cap \breve{\operatorname{co}} Y)$. Then $\Gamma(t') \in [\Gamma(t)]_\Theta \cap \breve{\operatorname{co}} Y$ and hence, by Lemma 4.6, $t' \in (t_-, t_+)$ and $\Gamma(t')\Theta\Gamma(t)$. Then, by what we showed above, there exists an open neighborhood U of t' with

$$U \subseteq \Gamma^{-1}\big(\big[\Gamma(t')\big]_\Theta \cap \breve{\operatorname{co}} Y\big).$$

However, $[\Gamma(t')]_\Theta = [\Gamma(t)]_\Theta$.

Let $I \subseteq (t_-, t_+)$ be such that $\{\Gamma(t) \mid t \in I\}$ is a set of class representatives of the equivalence relation $\Theta \cap [\Gamma((t_-, t_+)) \times \Gamma((t_-, t_+))]$. Recall that $(t_-, t_+) = \Gamma^{-1}(\breve{\operatorname{co}} Y)$ by Lemma 4.6. We have,

$$(t_-, t_+) = \bigcup_{t \in I} \big[\Gamma^{-1}\big(\big[\Gamma(t)\big]_\Theta\big) \cap (t_-, t_+)\big],$$

where the "$\supseteq$" inclusion is obvious, and the other one is a consequence of $\bigcup_{t \in I}([\Gamma(t)]_\Theta) \supseteq \Gamma(t_-, t_+)$ and properties of the inverse image Γ^{-1}.

Now, the sets $\Gamma^{-1}([\Gamma(t)]_\Theta) \cap (t_-, t_+)$, for $t \in I$, are all nonempty (each contains t), disjoint (since the corresponding equivalence classes are disjoint), and open (by the above shown). Since (t_-, t_+) is connected, it must be that $|I| = 1$. This just says that all of $\Gamma((t_-, t_+))$ is contained in a single class of Θ. On the other hand,

$$(t_-, t_+) = \Gamma^{-1}(\breve{\operatorname{co}} Y) = \Gamma^{-1}\big(\Gamma(\mathbb{R}) \cap \breve{\operatorname{co}} Y\big) = \Gamma^{-1}\big(\operatorname{aff}\{x_1, x_2\} \cap \breve{\operatorname{co}} Y\big)$$

and hence

$$\Gamma\big((t_-, t_+)\big) = \Gamma\big(\Gamma^{-1}\big(\operatorname{aff}\{x_1, x_2\} \cap \breve{\operatorname{co}} Y\big)\big) = \operatorname{aff}\{x_1, x_2\} \cap \breve{\operatorname{co}} Y$$

where the last equality holds since $\operatorname{aff}\{x_1, x_2\} \cap \breve{\operatorname{co}} Y$ is contained in the image of Γ. The statement of the lemma is now a direct consequence of this as $x_1 \in \operatorname{aff}\{x_1, x_2\} \cap \breve{\operatorname{co}} Y$. $\square$

The forward theorem

We are now in position to present our main result, the description of a given convex congruence on a polytope in terms of the function φ_Θ and graph $\mathcal{G}_\Theta$. This theorem will be accompanied by a converse construction resulting in Theorem 4.11 below.

Theorem 4.10. *Let K be a polytope in $\mathbb{R}^n$ and let $\Theta \in \mathrm{Con}_{\mathsf{CA}}\, \mathbb{K}$. Then*

(*i*) *The map φ_Θ is monotone.*
(*ii*) *Let $\mathcal{C}$ be a component of the graph $\mathcal{G}_\Theta$. Then $\mathcal{C}$ contains a largest element with respect to inclusion. Denoting this largest element by $Y(\mathcal{C})$, we have $\{Y, Y(\mathcal{C})\} \in E_\Theta$, $Y \in \mathcal{C}$.*
(*iii*) *The connectivity relation $\approx_\Theta$ is a congruence of the join-semilattice V_K.*
(*iv*) *Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$ and $Y(\mathcal{C})$ its largest element. Then*

$$\varphi_\Theta(Y) = \varphi_\Theta\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y \quad \text{for } Y \in \mathcal{C}, \tag{4.3}$$

$$\big[\check{\mathrm{co}} Y + \varphi_\Theta\big(Y(\mathcal{C})\big)\big] \cap \check{\mathrm{co}} Y(\mathcal{C}) \neq \emptyset \quad \text{for } Y \in \mathcal{C}. \tag{4.4}$$

Set

$$Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \check{\mathrm{co}} Y, \tag{4.5}$$

then the congruence Θ can be recovered from φ_Θ and $\mathcal{G}_\Theta$ as

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi_\Theta\big(Y(\mathcal{C})\big)\big\}. \tag{4.6}$$

Let us point out that reconstructing Θ by means of the formula (4.6) only requires knowledge of the classes of $\approx_\Theta$ and the values of φ_Θ on the respective largest elements of these classes.

Proof. Our schedule is as follows. We first prove (*i*) and (*ii*), then the representation (4.6), and only then we establish (*iii*) and (*iv*).

(*i*) We exploit perspective invariance to show that φ_Θ is monotone. Let $Y_1, Y_2 \in V_K$ with $Y_1 \subseteq Y_2$ be given. If $Y_1 = Y_2$, there is nothing to prove. Hence, assume that $Y_1 \subset Y_2$.

Let $w \in \varphi_\Theta(Y_1)$, and write according to the definition of φ_Θ

$$w = \sum_{k=1}^{m} \lambda_k \big(x_2^k - x_1^k\big),$$

with some

$$\lambda_k \in \mathbb{R}, \qquad x_1^k, x_2^k \in \check{\mathrm{co}} Y_1, \qquad x_1^k \Theta x_2^k, \quad k = 1, \ldots, m.$$

Set $z := \frac{1}{|Y_2 \setminus Y_1|} \sum_{y \in Y_2 \setminus Y_1} y$, and fix $s \in (0, 1)$. Note that the assumption $Y_1 \subset Y_2$ ensures that z is well defined. Perspective invariance gives

$$\Phi_z\big(s, x_1^k\big) \Theta \Phi_z\big(s, x_2^k\big), \quad k = 1, \ldots, m.$$

By Lemma 4.5(*i*), since $z \in \check{\mathrm{co}}(Y_2 \setminus Y_1)$ and $Y_2 = (Y_2 \setminus Y_1) \cup Y_1$ by the assumption, we have $\Phi_z(s, x_1^k), \Phi_z(s, x_2^k) \in \check{\mathrm{co}} Y_2$, and hence

$$w' := \sum_{k=1}^{m} \lambda_k \big(\Phi_z\big(s, x_2^k\big) - \Phi_z\big(s, x_1^k\big)\big) \in \varphi_\Theta(Y_2).$$

However,

$$\Phi_z\big(s, x_2^k\big) - \Phi_z\big(s, x_1^k\big) = (1-s)\big(x_2^k - x_1^k\big),$$

and hence $w' = (1-s)w$. Since $\varphi_\Theta(Y_2)$ is a vector subspace, it follows that also $w \in \varphi_\Theta(Y_2)$. $\square$

(ii) We proceed in three steps. The first step is to show that

$$\{Y_1, Y_2\} \in E_\Theta \quad \Rightarrow \quad \{Y_1, Y_1 \cup Y_2\}, \{Y_2, Y_1 \cup Y_2\} \in E_\Theta$$

To this end, choose $x_1 \in \check{\text{co}} Y_1$ and $x_2 \in \check{\text{co}} Y_2$ with $x_1 \Theta x_2$. By convexity of equivalence classes, we have

$$\frac{1}{2}(x_1 + x_2)\Theta x_j, \quad j = 1, 2,$$

and since $\Phi_{x_1}(\frac{1}{2}, x_2) = \frac{1}{2}(x_1 + x_2)$, Lemma 4.5$(i)$ gives $\frac{1}{2}(x_1 + x_2) \in \check{\text{co}}(Y_1 \cup Y_2)$ which shows the above property.

Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$. The second step is to show that

$$Y_1, Y_2 \in \mathcal{C} \quad \Rightarrow \quad \exists Y_3 \in \mathcal{C} : Y_1 \cup Y_2 \subseteq Y_3, \{Y_1, Y_3\}, \{Y_2, Y_3\} \in E_\Theta$$

To established existence of Y_3 with the required properties, we use induction on the length of a path in $\mathcal{G}_\Theta$ connecting Y_1 and Y_2.

If Y_1 and Y_2 can be connected by a path of length 1, i.e., if $\{Y_1, Y_2\} \in E_\Theta$, then set $Y_3 := Y_1 \cup Y_2$. By the property shown in the first step, we have $\{Y_1, Y_3\}, \{Y_2, Y_3\} \in E_\Theta$. This also implies that $Y_3 \in \mathcal{C}$.

For the inductive step, let $m \in \mathbb{N}^+$ be given, and assume that Y_1 and Y_2 can be connected by a path of length $m + 1$. This means that there exist

$$Y'_0, \ldots, Y'_{m+1} \in V_K, \quad Y'_0 = Y_1, \quad Y'_{m+1} = Y_2 \quad \text{with } \big\{Y'_k, Y'_{k+1}\big\} \in E_\Theta$$

for $k \in \{0, \ldots, m\}$. Clearly, all Y'_k belong to $\mathcal{C}$. The inductive hypothesis provides a vertex $Y' \in \mathcal{C}$ with

$$Y_1 \cup Y'_m \subseteq Y', \qquad \big\{Y_1, Y'\big\}, \big\{Y'_m, Y'\big\} \in E_\Theta.$$

Now, since $Y'_m \subseteq Y'$ and $\{Y_2, Y'_m\}, \{Y'_m, Y'\} \in E_\Theta$, from Corollary 4.8, we get $\{Y_2, Y'\} \in E_\Theta$. From the property shown in the first step, we get $\{Y_2, Y_2 \cup Y'\}, \{Y', Y_2 \cup Y'\} \in E_\Theta$. Another application of Corollary 4.8, this time since $\{Y_1, Y'\}, \{Y', Y_2 \cup Y'\} \in E_\Theta$ and obviously $Y' \subseteq Y_2 \cup Y'$, gives that also $\{Y_1, Y_2 \cup Y'\} \in E_\Theta$. Thus the element $Y_3 := Y_2 \cup Y'$ has all required properties. This finishes the proof of the second step.

Let $Y \in \mathcal{C}$. If there exists $Y' \in \mathcal{C}$ with $Y' \nsubseteq Y$, the property shown in the second step gives an element $Y'' \in \mathcal{C}$ with $Y \cup Y' \subseteq Y''$ and $Y \subset Y''$ (since $Y \subset Y \cup Y'$). Hence, Y is not maximal in $\mathcal{C}$. We conclude that if an element is maximal in $\mathcal{C}$, then it must also be the largest in $\mathcal{C}$. Since $\mathcal{C}$ is finite (the whole graph is finite), there certainly exists a maximal element. Let Y_0 be such. Then, since it is the largest, we have

$$Y_0 \supseteq \bigcup_{Y \in \mathcal{C}} Y,$$

and hence $Y_0 = \bigcup_{Y \in \mathcal{C}} Y$.

Let $Y \in \mathcal{C}$ be given. By the property shown in the second step, applied to Y and Y_0, we obtain $Y_3 \in \mathcal{C}$ with

$$Y \cup Y_0 \subseteq Y_3, \qquad \{Y, Y_3\}, \{Y_0, Y_3\} \in E_\Theta.$$

However, since Y_0 is the largest element of $\mathcal{C}$, it follows that $Y_3 = Y_0$. Hence, we have shown that $\{Y, Y_0\} \in E_\Theta$. $\square$

Next we present a description of Θ inside of $\check{\mathrm{co}}Y$. We will show that

$$\Theta \cap (\check{\mathrm{co}}Y \times \check{\mathrm{co}}Y) = \big\{(x_1, x_2) \in \check{\mathrm{co}}Y \times \check{\mathrm{co}}Y : x_2 - x_1 \in \varphi_\Theta(Y)\big\}. \tag{4.7}$$

The inclusion "$\subseteq$" is immediate from the definition of φ_Θ, Definition 4.2. We have to show the reverse inclusion. Let $x_1, x_2 \in \check{\mathrm{co}}Y$ be given, and assume that $x_2 - x_1 \in \varphi_\Theta(Y)$. If $x_1 = x_2$, there is nothing to prove. Hence, assume in addition that $x_1 \neq x_2$. Note that this implies that $|Y| > 1$.

Since $x_2 - x_1 \in \varphi_\Theta(Y)$, we can write

$$x_2 - x_1 = \sum_{i=1}^{m} \lambda_i \big(x_2^i - x_1^i\big),$$

with some $\lambda_i \in \mathbb{R}$ and $x_1^i, x_2^i \in \check{\mathrm{co}}Y$, $x_1^i \Theta x_2^i$. Clearly, we may assume that always $x_1^i \neq x_2^i$.

Choose $\varepsilon > 0$, and define elements z_k for $k = 0, \dots, m$

$$z_k := x_1 + \varepsilon \sum_{i=1}^{k} \lambda_i \big(x_2^i - x_1^i\big).$$

Note that for $k = 1, \dots, m$ we have

$$z_k = z_{k-1} + \varepsilon \lambda_k \big(x_2^k - x_1^k\big).$$

Then $z_0 = x_1$ and it is easy to see that

$$z_k \in \operatorname{aff} Y, \quad k = 0, \dots, m.$$

By Lemma 2.5, $\check{\mathrm{co}}Y$ is an open subset of $\operatorname{aff} Y$. Now from $x_1 \in \check{\mathrm{co}}Y$, we can make the choice of $\varepsilon > 0$ such that

$$z_k \in \check{\mathrm{co}}Y, \quad k = 0, \dots, m.$$

We will next show that

$$z_{k-1} \Theta z_k, \quad k = 1, \dots, m.$$

Let $k \in \{1, \dots, m\}$ be given. First consider the case that $x_1^k = z_{k-1}$. Then we have

$$z_k = x_1^k + \varepsilon \lambda_k \big(x_2^k - x_1^k\big) \in \operatorname{aff}\big\{x_1^k, x_2^k\big\} \cap \check{\mathrm{co}}Y.$$

By Lemma 4.9, the set on the right side is contained in $[x_1^k]_\Theta$, and thus $z_k \Theta z_{k-1}$.

Assume now that $x_1^k \neq z_{k-1}$. Let Γ be the line containing the points z_{k-1} and x_1^k, that is

$$\Gamma(t) := t x_1^k + (1 - t) z_{k-1}, \quad t \in \mathbb{R}.$$

Let t_-, t_+ be as in Lemma 4.6. Choose $s \in (0, 1)$ such that $st_- + (1 - s) = 0$ (this is always possible, namely $s = \frac{1}{1 - t_-}$ is such). Then, using Lemma 4.6,

$$\Phi_{\Gamma(t_-)}\big(s, x_1^k\big) = \Phi_{\Gamma(t_-)}\big(s, \Gamma(1)\big) = \Gamma\big(st_- + (1 - s) \cdot 1\big) = \Gamma(0) = z_{k-1}.$$

By perspective invariance

$$\Phi_{\Gamma(t_-)}(s, x_2^k)\Theta z_{k-1}.$$

Also

$$\Phi_{\Gamma(t_-)}(s, x_2^k) - z_{k-1} = \Phi_{\Gamma(t_-)}(s, x_2^k) - \Phi_{\Gamma(t_-)}(s, x_1^k) = (1-s)(x_2^k - x_1^k).$$

It follows that

$$z_k = z_{k-1} + \frac{\varepsilon\lambda_k}{1-s}\big(\Phi_{\Gamma(t_-)}(s, x_2^k) - z_{k-1}\big) \in \operatorname{aff}\big\{\Phi_{\Gamma(t_-)}(s, x_2^k), z_{k-1}\big\} \cap \check{\text{co}} Y \subseteq [z_{k-1}]_\Theta,$$

where the last inclusion holds by Lemma 4.9 since $\Phi_{\Gamma(t_-)}(s, x_2^k) \in \check{\text{co}} Y$ by Lemma 4.5 using also Lemma 4.6, $\Phi_{\Gamma(t_-)}(s, x_2^k)\Theta z_{k-1}$ as shown above, and $\Phi_{\Gamma(t_-)}(s, x_2^k) \neq z_{k-1}$ since $x_2^k \neq x_1^k$ as assumed above, and we again have $z_k \Theta z_{k-1}$.

By transitivity we have $z_0 \Theta z_m$, i.e. $x_1 \Theta z_m$, and another application of Lemma 4.9 gives

$$x_2 = x_1 + \sum_{i=1}^{m} \lambda_i\big(x_2^i - x_1^i\big) = x_1 + \frac{1}{\varepsilon}(z_m - x_1) \in \operatorname{aff}\{x_1, z_m\} \cap \check{\text{co}} Y \subseteq [x_1]_\Theta,$$

where again $x_1 \neq z_m$ by the made assumption $x_1 \neq x_2$. This completes the proof of (4.7). $\square$

Next we show the following auxiliary statement:
Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$ and $Y(\mathcal{C})$ its largest element, and let $Z(\mathcal{C})$ be as in (4.5). Then

$$\Theta \cap \big(Z(\mathcal{C}) \times Z(\mathcal{C})\big) = \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi_\Theta\big(Y(\mathcal{C})\big)\big\}. \tag{4.8}$$

As a first step, we show: If $x \in Z(\mathcal{C})$, $x' \in \check{\text{co}} Y(\mathcal{C})$, and $x\Theta x'$, then $x - x' \in \varphi_\Theta(Y(\mathcal{C}))$.

If $x = x'$, this is trivial. Hence, assume that $x \neq x'$. Let $z := \frac{1}{2}(x + x')$. Since equivalence classes of Θ are convex, we have $z\Theta x'$.

Recall that $Y(\mathcal{C}) = \bigcup_{Y\in\mathcal{C}} Y$ and $\check{\text{co}}$ is monotone, implying that $Z(\mathcal{C}) \subseteq \check{\text{co}} Y(\mathcal{C}) \subseteq \operatorname{co} Y(\mathcal{C})$. Hence $x \in \operatorname{co} Y(\mathcal{C})$ and $x' \in \check{\text{co}} Y(\mathcal{C})$, which by Lemma 4.5 gives $z \in \check{\text{co}} Y(\mathcal{C})$ as $z = \Phi_x(\frac{1}{2}, x')$. Eq. (4.7) now implies that $z - x' \in \varphi_\Theta(Y(\mathcal{C}))$. It follows that

$$x - x' = 2\big(z - x'\big) \in \varphi_\Theta\big(Y(\mathcal{C})\big)$$

which completes the first step.

We now prove the inclusion "$\subseteq$" in (4.8). Let $x_1, x_2 \in Z(\mathcal{C})$ with $x_1\Theta x_2$ be given. Since, by the already proved (*ii*), for any $Y \in \mathcal{C}$ we have $\{Y, Y(\mathcal{C})\} \in E_\Theta$, we can choose $x_1', x_2' \in \check{\text{co}} Y(\mathcal{C})$ with $x_1\Theta x_1'$ and $x_2\Theta x_2'$. Transitivity implies $x_1'\Theta x_2'$, and (4.7) thus gives $x_1' - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$. By the property shown in the first step, also $x_1 - x_1', x_2 - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$, and together

$$x_2 - x_1 = \big(x_2 - x_2'\big) + \big(x_2' - x_1'\big) + \big(x_1' - x_1\big) \in \varphi_\Theta\big(Y(\mathcal{C})\big).$$

Finally, we show the inclusion "$\supseteq$" in (4.8). This is done by reversing the argument in the previous paragraph. Let $x_1, x_2 \in Z(\mathcal{C})$ with $x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))$ be given. Again choose $x_1', x_2' \in \check{\text{co}} Y(\mathcal{C})$ with $x_1\Theta x_1'$ and $x_2\Theta x_2'$. Again from the property shown in the first step, we have $x_1 - x_1', x_2 - x_2' \in \varphi_\Theta(Y(\mathcal{C}))$. Hence, also

$$x_2' - x_1' = \big(x_2' - x_2\big) + (x_2 - x_1) + \big(x_1 - x_1'\big) \in \varphi_\Theta\big(Y(\mathcal{C})\big),$$

and (4.7) implies that $x_1'\Theta x_2'$. Transitivity gives $x_1\Theta x_2$, showing (4.8). $\square$

We are now in position to establish the representation (4.6) of Θ, which is then used to prove (*iii*) and (*iv*). The main observation to make is that

$$K = \bigcup_{Y \in V_K} \check{\mathrm{co}} Y = \bigcup_{\mathcal{C}} \bigcup_{Y \in \mathcal{C}} \check{\mathrm{co}} Y.$$

Hence, we can write K as the disjoint union

$$K = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} Z(\mathcal{C})$$

where disjointness is easy to check: Assume $x \in Z(\mathcal{C}) \cap Z(\mathcal{C}')$. Then $x \in \check{\mathrm{co}} Y$ for some $Y \in \mathcal{C}$ and $x \in \check{\mathrm{co}} Y'$ for some $Y' \in \mathcal{C}'$. From the reflexivity of Θ we have $(x, x) \in \Theta$ and from the definition of $\mathcal{G}_\Theta$ this yields $\{Y, Y'\} \in E_\Theta$, which implies that $\mathcal{C} = \mathcal{C}'$.

As a consequence, Θ is the disjoint union

$$\Theta = \bigcup_{\substack{\mathcal{C}, \mathcal{C}' \text{ components} \\ \text{of } \mathcal{G}_\Theta}} \left[\Theta \cap \left(Z(\mathcal{C}) \times Z\left(\mathcal{C}'\right)\right)\right].$$

With a similar argument as above, the definition of $\mathcal{G}_\Theta$ ensures that

$$\Theta \cap \left(Z(\mathcal{C}) \times Z\left(\mathcal{C}'\right)\right) = \emptyset, \quad \mathcal{C} \neq \mathcal{C}',$$

and hence

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ component} \\ \text{of } \mathcal{G}_\Theta}} \left[\Theta \cap \left(Z(\mathcal{C}) \times Z(\mathcal{C})\right)\right].$$

The desired representation (4.6) follows now directly from (4.8). □

(*iii*) Let $Y_1, Y_2 \in V_K$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the components which contain Y_1 and Y_2, respectively. Moreover, let $\mathcal{C}$ be the component which contains $Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)$. We show that then $Y_1 \cup Y_2 \in \mathcal{C}$. This suffices for the proof that $\approx_\Theta$ is a congruence on the join-semilattice V_K since if $Y_1 \approx_\Theta Y_1'$ and $Y_2 \approx_\Theta Y_2'$, i.e., $Y_1' \in \mathcal{C}_1$ and $Y_2' \in \mathcal{C}_2$, then both $Y_1 \cup Y_2$ and $Y_1' \cup Y_2'$ are in $\mathcal{C}$ showing that $Y_1 \cup Y_2 \approx_\Theta Y_1' \cup Y_2'$.

For $j = 1, 2$ choose

$$x_j \in \check{\mathrm{co}} Y_j, \qquad x_j' \in \check{\mathrm{co}} Y(\mathcal{C}_j) \quad \text{with } x_j \Theta x_j'.$$

Since Θ is convex, we have

$$\left(\frac{1}{2}\left(x_1 + x_1'\right), \frac{1}{2}\left(x_2 + x_2'\right)\right) = \frac{1}{2}\left(x_1, x_1'\right) + \frac{1}{2}\left(x_2, x_2'\right) \in \Theta.$$

By Lemma 4.5 we have

$$\Phi_{x_1}\left(\frac{1}{2}, x_2\right) = \frac{1}{2}(x_1 + x_2) \in \check{\mathrm{co}}(Y_1 \cup Y_2)$$

$$\Phi_{x_1'}\left(\frac{1}{2}, x_2'\right) = \frac{1}{2}\left(x_1' + x_2'\right) \in \check{\mathrm{co}}\left(Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)\right)$$

and hence $Y_1 \cup Y_2 \approx_\Theta Y(\mathcal{C}_1) \cup Y(\mathcal{C}_2)$. This just means that $Y_1 \cup Y_2 \in \mathcal{C}$. □

(*iv*) Let $\mathcal{C}$ be a component of $\mathcal{G}_\Theta$, $Y \in \mathcal{C}$, and $Y(\mathcal{C})$ the largest element of $\mathcal{C}$.

If $|Y| = 1$, we have $\varphi_\Theta(Y) = \{0\}$ and $\operatorname{dir} Y = \{0\}$. Hence, in this case, equality (4.3) holds trivially. Assume that $|Y| \geq 2$. The inclusion '$\subseteq$' in (4.3) follows since φ_Θ is monotone, $Y \subseteq Y(\mathcal{C})$, and

$$\varphi_\Theta(Y) \subseteq \operatorname{span}(\check{\mathrm{co}} Y - \check{\mathrm{co}} Y) = \operatorname{dir} Y,$$

where the last equality follows by Lemma 2.6. To show the reverse inclusion, let $u \in \varphi_\Theta(Y(\mathcal{C})) \cap \operatorname{dir} Y$ be given. Choose $x_1 \in \check{\mathrm{co}} Y$, let $\varepsilon > 0$, and set $x_2 := x_1 + \varepsilon u$. Since $u \in \operatorname{dir} Y$ and $x_1 \in \operatorname{aff} Y$, we have $x_1 + \operatorname{span}\{u\} \subseteq \operatorname{aff} Y$. Since $\check{\mathrm{co}} Y$ is an open subset of $\operatorname{aff} Y$, we can choose $\varepsilon > 0$ so small that $x_2 \in \check{\mathrm{co}} Y$. We have $x_2 - x_1 = \varepsilon u \in \varphi_\Theta(Y(\mathcal{C}))$, and hence (4.6) implies that $x_1 \Theta x_2$. This is enough to conclude that $u = \frac{1}{\varepsilon} \cdot \varepsilon u = \frac{1}{\varepsilon}(x_2 - x_1) \in \varphi_\Theta(Y)$.

It remains to prove (4.4). To this end, remember that $\{Y, Y(\mathcal{C})\} \in E_\Theta$, i.e., there exist $x_1 \in \check{\mathrm{co}} Y$, $x_2 \in \check{\mathrm{co}} Y(\mathcal{C})$, with $x_1 \Theta x_2$. By (4.6), we have $x_2 - x_1 \in \varphi_\Theta(Y(\mathcal{C}))$, and hence

$$x_2 \in \big[\check{\mathrm{co}} Y + \varphi_\Theta\big(Y(\mathcal{C})\big)\big] \cap \check{\mathrm{co}} Y(\mathcal{C}).$$

The proof of Theorem 4.10 is now complete. $\square$

The converse construction

Convex congruences of a polytope K can be constructed from certain join-semilattice congruences of V_K. The following theorem provides this converse construction.

Theorem 4.11. *Let K be a polytope in $\mathbb{R}^n$. Let $\sim$ be a congruence relation of the join-semilattice V_K with the property that each congruence class $\mathcal{C}$ of $\sim$ contains a largest element, say $Y(\mathcal{C})$. Moreover, let*

$$\varphi\colon \big\{Y(\mathcal{C}) \bigm| \mathcal{C} \textit{ class of } \sim\big\} \longrightarrow \operatorname{Sub} \mathbb{R}^n$$

be a monotone map such that, for each class $\mathcal{C}$ of $\sim$,

$$\varphi\big(Y(\mathcal{C})\big) \subseteq \operatorname{dir} Y(\mathcal{C}), \qquad \big[\check{\mathrm{co}} Y + \varphi\big(Y(\mathcal{C})\big)\big] \cap \check{\mathrm{co}} Y(\mathcal{C}) \neq \emptyset \quad \textit{for } Y \in \mathcal{C}.$$

Then there exists a unique congruence $\Theta \in \operatorname{Con}_{\mathsf{CA}} \mathbb{K}$ such that

$$\approx_\Theta = \sim, \qquad \varphi_\Theta\big(Y(\mathcal{C})\big) = \varphi\big(Y(\mathcal{C})\big) \quad \textit{for } \mathcal{C} \textit{ a class of } \sim.$$

This congruence Θ can be computed from $\sim$ and φ by means of the formula

$$\Theta = \bigcup_{\substack{\mathcal{C} \text{ class} \\ \text{of } \sim}} \big\{(x_1, x_2) \in Z(\mathcal{C}) \times Z(\mathcal{C}) : x_2 - x_1 \in \varphi\big(Y(\mathcal{C})\big)\big\}, \tag{4.9}$$

where again $Z(\mathcal{C}) := \bigcup_{Y \in \mathcal{C}} \check{\mathrm{co}} Y$. Its associated function φ_Θ is given as

$$\varphi_\Theta(Y) = \varphi\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y \quad \textit{for } Y \in \mathcal{C}, \tag{4.10}$$

and the set of edges E_Θ of its associated graph $\mathcal{G}_\Theta$ is given as

$$\{Y_1, Y_2\} \in E_\Theta \quad \Longleftrightarrow \quad \big(Y_1 \sim Y_2 \wedge \big[\check{\mathrm{co}} Y_1 + \varphi\big(Y\big([Y_1]_\sim\big)\big)\big] \cap \check{\mathrm{co}} Y_2 \neq \emptyset\big) \tag{4.11}$$

where $[Y_1]_\sim$ denotes the equivalence class of Y_1.

Proof. Let a relation $\sim$ and a map φ as in the statement of Theorem 4.11 be given. For each equivalence class $\mathcal{C}$ of $\sim$, we denote its largest element by $Y(\mathcal{C})$ and set

$$Z(\mathcal{C}) := \bigcup_{Y\in\mathcal{C}} \breve{\mathrm{co}}Y.$$

Consider the relation on K defined as

$$\Theta = \bigcup_{\substack{\mathcal{C}\text{ class}\\ \text{of }\sim}} \big\{(x_1,x_2)\in Z(\mathcal{C})\times Z(\mathcal{C})\ \big|\ x_2-x_1\in\varphi\big(Y(\mathcal{C})\big)\big\}.$$

The main step is to show that the relation Θ is a convex congruence on K, i.e., $\Theta\in\mathrm{Con}_{\mathsf{CA}}\,\mathbb{K}$, and

$$\approx_\Theta=\sim,\qquad \varphi_\Theta\big(Y(\mathcal{C})\big)=\varphi\big(Y(\mathcal{C})\big)\quad\text{for }\mathcal{C}\text{ class of }\sim.$$

First of all, Θ is an equivalence relation as a direct consequence of $\varphi(Y(\mathcal{C}))$ being a linear subspace.

Next, notice the following: If $x\in\breve{\mathrm{co}}Y$, $x'\in\breve{\mathrm{co}}Y'$, and $x\Theta x'$, then x and x' must both belong to the same of the sets $Z(\mathcal{C})$, and hence $Y\sim Y'$.

To show that Θ is a convex congruence, let $(x_1,x_1'),(x_2,x_2')\in\Theta$ and $s\in(0,1)$ be given. Choose Y_j,Y_j', $j=1,2$, such that $x_j\in\breve{\mathrm{co}}Y_j, x_j'\in\breve{\mathrm{co}}Y_j'$. Then $Y_j\sim Y_j'$, and hence also

$$Y_1\cup Y_2\sim Y_1'\cup Y_2'.$$

Let $\mathcal{C}$ be the class of $\sim$ which contains $Y_1\cup Y_2$. Lemma 4.5 gives

$$x:=sx_1+(1-s)x_2\in\breve{\mathrm{co}}(Y_1\cup Y_2),\qquad x':=sx_1'+(1-s)x_2'\in\breve{\mathrm{co}}\big(Y_1'\cup Y_2'\big),$$

and hence $x,x'\in Z(\mathcal{C})$.

Let $\mathcal{C}_j$ be the class which contains Y_j. Since $Y_j\sim Y(\mathcal{C}_j)$, it follows that $Y_1\cup Y_2\sim Y(\mathcal{C}_1)\cup Y(\mathcal{C}_2)$, and hence $Y(\mathcal{C}_1)\cup Y(\mathcal{C}_2)\subseteq Y(\mathcal{C})$. Since φ is monotone, we conclude that

$$\varphi\big(Y(\mathcal{C}_j)\big)\subseteq\varphi\big(Y(\mathcal{C})\big),\quad j=1,2.$$

We compute

$$x'-x=s\big(x_1'-x_1\big)+(1-s)\big(x_2'-x_2\big)\in\varphi\big(Y(\mathcal{C}_1)\big)+\varphi\big(Y(\mathcal{C}_2)\big)\subseteq\varphi\big(Y(\mathcal{C})\big),$$

and this shows that $(x,x')\in\Theta$ and therefore Θ is a convex congruence.

Next, we show that $\varphi_\Theta(Y(\mathcal{C}))=\varphi(Y(\mathcal{C}))$ whenever $\mathcal{C}$ is a class of $\sim$. By the definition of Θ, we have

$$\big\{x_2-x_1\ \big|\ x_1,x_2\in\breve{\mathrm{co}}Y(\mathcal{C}),x_1\Theta x_2\big\}\subseteq\varphi\big(Y(\mathcal{C})\big),$$

and hence $\varphi_\Theta(Y(\mathcal{C}))\subseteq\varphi(Y(\mathcal{C}))$. Conversely, let $u\in\varphi(Y(\mathcal{C}))$ be given. Choose $x_1\in\breve{\mathrm{co}}Y(\mathcal{C})\subseteq\mathrm{aff}\,Y(\mathcal{C})$, let $\varepsilon>0$, and set $x_2:=x_1+\varepsilon u$. Since $\varphi(Y(\mathcal{C}))\subseteq\mathrm{dir}\,Y(\mathcal{C})$, we have $x_2\in\mathrm{aff}\,Y(\mathcal{C})$. Since $\breve{\mathrm{co}}Y(\mathcal{C})$ is an open subset of $\mathrm{aff}\,Y(\mathcal{C})$, we may choose $\varepsilon>0$ so small that $x_2\in\breve{\mathrm{co}}Y(\mathcal{C})$. Then, by the definition of Θ, we have $x_1\Theta x_2$. It follows that $u=\frac{1}{\varepsilon}(x_2-x_1)\in\varphi_\Theta(Y(\mathcal{C}))$.

In order to establish the inclusion "$\approx_\Theta\subseteq\sim$", it is enough to show that $\{Y_1,Y_2\}\in E_\Theta$ implies $Y_1\sim Y_2$. This, however, is clear from the note made in the beginning of this proof. We next prove the reverse inclusion. First, let one element $Y\in V_K$ be given, and denote by $\mathcal{C}$ the class of $\sim$ which contains Y, i.e., $\mathcal{C}=[Y]_\sim$. By the hypothesis that $[\breve{\mathrm{co}}Y+\varphi(Y(\mathcal{C}))]\cap\breve{\mathrm{co}}Y(\mathcal{C})\neq\emptyset$, we can choose $x_1\in\breve{\mathrm{co}}Y$, $u\in\varphi(Y(\mathcal{C}))$, and $x_2\in\breve{\mathrm{co}}Y(\mathcal{C})$, such that $x_2 = x_1 + u$. By the definition of Θ, we have $x_1 \Theta x_2$. This shows that $\{Y, Y(\mathcal{C})\} \in E_\Theta$. Let now $Y_1, Y_2 \in V_K$ with $Y_1 \sim Y_2$, and denote by $\mathcal{C}$ the class of $\sim$ which contains Y_1 (and hence also Y_2). By what we just showed, $\{Y_1, Y(\mathcal{C})\}, \{Y_2, Y(\mathcal{C})\} \in E_\Theta$. This implies that $Y_1 \approx_\Theta Y_2$.

The fact that Θ can be computed by means of (4.9) is just the above definition of Θ. The fact that Θ is unique, is clear from (4.6). The remaining assertions of Theorem 4.11 now follow easily, as shown below.

Let $Y \in V_K$, and let $\mathcal{C}$ be the class of $\sim$ which contains Y. Using (4.3), we obtain

$$\varphi_\Theta(Y) = \varphi_\Theta\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y = \varphi\big(Y(\mathcal{C})\big) \cap \operatorname{dir} Y,$$

showing (4.10). For (4.11), it is enough to note (again) that our definition of Θ ensures

$$\forall Y_1, Y_2 \in V_K. \quad \{Y_1, Y_2\} \in E_\Theta \;\Rightarrow\; Y_1 \sim Y_2$$

and

$$\forall Y_1, Y_2 \in \mathcal{C}. \quad \{Y_1, Y_2\} \in E_\Theta \;\Leftrightarrow\; \big[\breve{\text{co}} Y_1 + \varphi\big(Y(\mathcal{C})\big)\big] \cap \breve{\text{co}} Y_2 \neq \emptyset$$

which is easy to show unfolding the definitions. □

Remark 4.12. Note that for any join-semilattice congruence $\sim$ with largest element in every class (as in Theorem 4.11) there is at least one possible choice for the assignment φ in Theorem 4.11. Namely, $\varphi(Y(\mathcal{C})) = \operatorname{dir} Y(\mathcal{C})$ satisfies all conditions.

We do not have a simple description of all join-semilattices congruences which are admissible in the sense of Theorem 4.11. However, some examples are easily obtained. Let $K \subseteq \mathbb{R}^n$, $|K| > 1$, be a polytope, and let $Y_0 \subseteq \operatorname{ext} K$. Define an equivalence relation $\sim$ on V_K by specifying its equivalence classes to be

$$\mathcal{C}_y := \big\{\{y\}\big\}, \quad y \in Y_0, \qquad \mathcal{C}_0 := V_K \setminus \bigcup_{y \in Y_0} \mathcal{C}_y.$$

Clearly, each of these classes contains a largest element $Y(\mathcal{C})$ (for $\mathcal{C}_0$ it is $Y(\mathcal{C}_0) = \operatorname{ext} K$), and it is straightforward to verify that $\sim$ is a congruence of the join-semilattice V_K.

Let us now illustrate the results on a (toy) example. Another example describing the congruences of the free absolutely convex algebra with two generators is given in [30].

Example 4.13. Let $Y := \{0, 1\}$ and $K := \operatorname{co} Y$ in $\mathbb{R}$. Then $K = [0, 1]$ and $\mathbb{K}$ is in CA. We will show, using Theorem 4.10 and Theorem 4.11, that there are exactly five CA-congruences of $\mathbb{K}$. These are:

$$\begin{aligned}
\Theta_1 &= \Delta \\
\Theta_2 &= \big\{(0,0), (1,1)\big\} \cup (0,1) \times (0,1) \\
\Theta_3 &= \big\{(0,0)\big\} \cup (0,1] \times (0,1] \\
\Theta_4 &= [0,1) \times [0,1) \cup \big\{(1,1)\big\}, \quad \text{and} \\
\Theta_5 &= [0,1] \times [0,1].
\end{aligned}$$

We have $V_K = \{\{0\}, \{1\}, \{0,1\}\}$. To ease the notation we write

$$\mathbf{0} = \{0\}, \qquad \mathbf{1} = \{1\}, \qquad \mathbf{01} = \{0,1\},$$

hence $V_K = \{\mathbf{0}, \mathbf{1}, \mathbf{01}\}$. Next we list all join-semilattice congruences of V_K with the property that each class has a largest element. There are four such, given by their partitions:

$$\begin{aligned} V_K/\sim_1 &= \big\{\{\mathbf{0}\}, \{\mathbf{1}\}, \{\mathbf{01}\}\big\} \\ V_K/\sim_2 &= \big\{\{\mathbf{0}\}, \{\mathbf{1}, \mathbf{01}\}\big\} \\ V_K/\sim_3 &= \big\{\{\mathbf{1}\}, \{\mathbf{0}, \mathbf{01}\}\big\} \\ V_K/\sim_4 &= \big\{\{\mathbf{0}, \mathbf{1}, \mathbf{01}\}\big\}. \end{aligned}$$

Note that $\mathbb{R}$ has only two vector subspaces, the trivial ones, $\operatorname{Sub}\mathbb{R} = \{\mathbf{0}, \mathbb{R}\}$. Furthermore, $\operatorname{dir}\mathbf{0} = \mathbf{0}$, $\operatorname{dir}\mathbf{1} = \mathbf{0}$, and $\operatorname{dir}\mathbf{01} = \mathbb{R}$.

For each of the join-semilattice congruences we need to consider all monotone maps φ mapping the largest elements of each class to $\operatorname{Sub}\mathbb{R}$ and satisfying the conditions $\varphi(Y(\mathcal{C})) \subseteq \operatorname{dir} Y(\mathcal{C})$ and $[\check{\text{co}}Y + \varphi(Y(\mathcal{C}))] \cap \check{\text{co}}Y(\mathcal{C}) \neq \emptyset$ for $Y \in \mathcal{C}$.

Consider $\sim_1 = \Delta$. The second condition here is always satisfied, and the first implies that $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathbf{1}) = \mathbf{0}$. Hence, there are two possibilities for defining φ, namely

$$\varphi_1 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbf{0}) \quad \text{and} \quad \varphi_2 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R}).$$

From (4.9) we then get two convex congruences on K and these are exactly Θ_1 and Θ_2. Consider next $\sim_2$. Due to the conditions on φ, there is a unique possibility to define this map, namely

$$\varphi_3 = (\mathbf{0} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R})$$

and (4.9) then gives Θ_3. The case $\sim_3$ is symmetric. Again there is a unique possibility to define the map φ, leading to

$$\varphi_4 = (\mathbf{1} \mapsto \mathbf{0}, \mathbf{01} \mapsto \mathbb{R})$$

and Θ_4. Finally, consider $\sim_4$. There is a unique map

$$\varphi_4 = (\mathbf{01} \mapsto \mathbb{R})$$

which satisfies the conditions imposed on φ leading to Θ_5. By Theorem 4.10, there are no other convex congruences on K.

The order of the congruence lattice

Our third theorem shows that also the order relation on the congruence lattice $\operatorname{Con}_{\mathsf{CA}}\mathbb{K}$ can be characterized in terms of φ_Θ and $\mathcal{G}_\Theta$. This is a simple consequence of the representation (4.6).

Theorem 4.14. *Let K be a polytope in $\mathbb{R}^n$. If $\Theta_1, \Theta_2 \in \operatorname{Con}_{\mathsf{CA}}\mathbb{K}$, then*

$$\Theta_1 \subseteq \Theta_2 \quad \Longleftrightarrow \quad (E_{\Theta_1} \subseteq E_{\Theta_2} \wedge \varphi_{\Theta_1} \leq \varphi_{\Theta_2})$$

where $\leq$ denotes the pointwise order by inclusion.

Proof. The implication "$\Rightarrow$" is immediate from the definition of E_{Θ_j} and φ_{Θ_j}. Conversely, assume that $E_{\Theta_1} \subseteq E_{\Theta_2}$ and $\varphi_{\Theta_1} \leq \varphi_{\Theta_2}$. The former implies that each component $\mathcal{C}_2$ of $\mathcal{G}_{\Theta_2}$ is the union of components of $\mathcal{G}_{\Theta_1}$. This implies that

$$Z(\mathcal{C}_2) = \bigcup_{\substack{\mathcal{C}_1 \text{ comp. of } \mathcal{G}_{\Theta_1} \\ \mathcal{C}_1 \subseteq \mathcal{C}_2}} Z(\mathcal{C}_1).$$

Using this fact, the representation (4.6) for Θ_1 and for Θ_2, monotonicity of φ_{Θ_1} and φ_{Θ_2}, and the assumption $\varphi_{\Theta_1} \leq \varphi_{\Theta_2}$, we compute

$$\begin{aligned}
\Theta_1 &= \bigcup_{\substack{\mathcal{C}_1 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_1}}} \big\{(x_1,x_2) \in Z(\mathcal{C}_1) \times Z(\mathcal{C}_1) : x_2 - x_1 \in \varphi_{\Theta_1}\big(Y(\mathcal{C}_1)\big)\big\} \\
&\subseteq \bigcup_{\substack{\mathcal{C}_2 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_2}}} \Bigg[\bigcup_{\substack{\mathcal{C}_1 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_1} \\ \mathcal{C}_1 \subseteq \mathcal{C}_2}} \big\{(x_1,x_2) \in Z(\mathcal{C}_1) \times Z(\mathcal{C}_1) : x_2 - x_1 \in \varphi_{\Theta_2}\big(Y(\mathcal{C}_2)\big)\big\} \Bigg] \\
&\subseteq \bigcup_{\substack{\mathcal{C}_2 \text{ comp.} \\ \text{of } \mathcal{G}_{\Theta_2}}} \big\{(x_1,x_2) \in Z(\mathcal{C}_2) \times Z(\mathcal{C}_2) : x_2 - x_1 \in \varphi_{\Theta_2}\big(Y(\mathcal{C}_2)\big)\big\} \\
&= \Theta_2. \qquad \square
\end{aligned}$$

5. Finitely presentable (positively, absolutely) convex algebras

We have already mentioned in the introduction that the free algebra $F_n(\mathsf{CA})$ with n generators is the standard $(n-1)$-simplex in $\mathbb{R}^n$. Moreover, $F_n(\mathsf{PCA})$ is the n-dimensional simplex in $\mathbb{R}^n$ generated by the point 0 and the unit vectors $e_1, \ldots, e_n$, cf. Example 2.8, and $F_n(\mathsf{ACA})$ is the octahedron in $\mathbb{R}^n$ centered at 0 and having the $2n$ corners $\{\pm e_i :, \ i = 1, \ldots, n\}$, cf. Example 2.9. Hence, all these free algebras are polytopes (with the CA, PCA, ACA vector operations, respectively) whose congruences we have fully described. Using this description, we can prove that all congruences of finitely generated algebras in CA,PCA, ACA are finitely generated.

First, we recall the necessary notions: An algebra $\mathbb{A}$ in a variety $\mathcal{V}$ is finitely generated if it is a quotient (under a congruence) of a free algebra $F_n(\mathcal{V})$ with n generators for some natural number n. Equivalently, $\mathbb{A}$ is finitely generated if it has a finite number of generators, i.e., there is a finite subset X of the carrier A with the property that every element of A can be expressed using the operations on elements of X, notation $\mathbb{A} = \langle X \rangle_{\mathcal{V}}$. A congruence Θ on an algebra $\mathbb{A}$ with carrier A is finitely generated if there exists a finite subset $R \subseteq A \times A$ such that Θ is the smallest congruence which contains R, notation $\Theta = \langle R \rangle$.

Next, we recall a universal algebra property that the mentioned result relies on.

Lemma 5.1. *Let $\mathcal{V}$ be a variety of algebras. The following two statements are equivalent:*

(1) Every congruence of any finitely generated algebra in $\mathcal{V}$ is finitely generated.
(2) For any two congruences Θ_1 and Θ_2 of a free algebra $F_n(\mathcal{V})$ with $\Theta_1 \subseteq \Theta_2$, there exists a finite subset $R \subseteq \Theta_2$ with the property $\Theta_2 = \langle \Theta_1 \cup R \rangle$.

Proof. Assume (1) and let Θ_1 and Θ_2 be as in (2) with $F_n(\mathcal{V})$ the free algebra in $\mathcal{V}$ with n generators. Consider the canonical projection $\pi : F_n(\mathcal{V}) \to F_n(\mathcal{V})/\Theta_1$ with $\ker \pi = \Theta_1$. Clearly, π is a surjective homomorphism and $\pi \times \pi$ induces an order isomorphism

$$\pi \times \pi : \big\{\Theta \in \operatorname{Con}_{\mathcal{V}} F_n(\mathcal{V}) \bigm| \Theta_1 \subseteq \Theta\big\} \to \operatorname{Con}_{\mathcal{V}}\big(F_n(\mathcal{V})/\Theta_1\big).$$

As $F_n(\mathcal{V})/\Theta_1$ is finitely generated, by (1) there exists a finite set $R \subseteq (F_n(\mathcal{V})/\Theta_1) \times (F_n(\mathcal{V})/\Theta_1)$ such that $(\pi \times \pi)(\Theta_2) = \langle R \rangle$. Choose $R_2 \subseteq \Theta_2$ with $|R_2| = |R|$ and $(\pi \times \pi)(R_2) = R$. We will show that $\Theta_2 = \langle \Theta_1 \cup R_2 \rangle$

showing (2). Let $\Theta \in \mathrm{Con}_{\mathcal{V}}(F_n(\mathcal{V}))$ and let $\Theta_1 \cup R_2 \subseteq \Theta$. Then $R \subseteq (\pi \times \pi)(\Theta)$ and hence $(\pi \times \pi)(\Theta_2) \subseteq (\pi \times \pi)(\Theta)$. We will show that then also $\Theta_2 \subseteq \Theta$. Let $(x_1, x_2) \in \Theta_2$. Then $(\pi \times \pi)(x_1, x_2) \in (\pi \times \pi)(\Theta)$, so there exists a pair $(y_1, y_2) \in \Theta$ with $(\pi \times \pi)(x_1, x_2) = (\pi \times \pi)(y_1, y_2)$, i.e., $\pi(x_1) = \pi(y_1)$ and $\pi(x_2) = \pi(y_2)$. Since $\ker \pi = \Theta_1$, we get $(x_1, y_1) \in \Theta_1, (x_2, y_2) \in \Theta_1$. Then

$$(x_1, x_2) \in \Theta_1 \circ \Theta \circ \Theta_1 \subseteq \Theta \circ \Theta \circ \Theta \subseteq \Theta.$$

Assume now that (2) holds and let $\mathbb{A}$ be a finitely generated algebra in $\mathcal{V}$ with carrier A and $\Theta \in \mathrm{Con}_{\mathcal{V}}(\mathbb{A})$. Let $\hat{\pi}\colon A \to A/\Theta$ be the corresponding surjective homomorphism with $\ker \hat{\pi} = \Theta$. Choose a positive natural number n and a congruence $\Theta_1 \in \mathrm{Con}_{\mathcal{V}}(F_n(\mathcal{V}))$ such that $\mathbb{A} \cong F_n(\mathcal{V})/\Theta_1$. Let $\pi\colon F_n(\mathcal{V}) \to \mathbb{A}$ be the corresponding surjective homomorphism with $\ker \pi = \Theta_1$. Let $\Theta_2 = (\pi \times \pi)^{-1}(\Theta)$. We have the following situation

$$F_n(\mathcal{V}) \xrightarrow{\pi} F_n(\mathcal{V})/\Theta_1 \xrightarrow{\hat{\pi}} \big(F_n(\mathcal{V})/\Theta_1\big)/\Theta$$

and

$$\begin{aligned} (x_1, x_2) \in (\pi \times \pi)^{-1}(\Theta) \quad &\Leftrightarrow \quad (\pi \times \pi)(x_1, x_2) \in \Theta \\ &\Leftrightarrow \quad \hat{\pi} \circ \pi(x_1) = \hat{\pi} \circ \pi(x_2) \\ &\Leftrightarrow \quad (x_1, x_2) \in \ker(\hat{\pi} \circ \pi). \end{aligned}$$

Hence, $\Theta_2 = \ker(\hat{\pi} \circ \pi)$ and therefore $\Theta_2 \in \mathrm{Con}_{\mathcal{V}}(F_n(\mathcal{V}))$. Moreover,

$$\Theta_1 = \ker \pi \subseteq \ker(\hat{\pi} \circ \pi) = \Theta_2.$$

Choose, by (2), a finite subset $R_2 \subseteq \Theta_2$ such that $\Theta_2 = \langle \Theta_1 \cup R_2 \rangle$ and let $R = (\pi \times \pi)(R_2)$. Clearly, R is finite and $R \subseteq \Theta$. If $\hat{\Theta} \in \mathrm{Con}_{\mathcal{V}}(\mathbb{A})$ and $R \subseteq \hat{\Theta}$, then $\Theta_1 \cup R_2 \subseteq (\pi \times \pi)^{-1}(\hat{\Theta})$ and hence, since as above $(\pi \times \pi)^{-1}(\hat{\Theta})$ is a congruence, we get $(\pi \times \pi)^{-1}(\Theta) = \Theta_2 \subseteq (\pi \times \pi)^{-1}(\hat{\Theta})$. Since $\pi \times \pi$ is surjective, $\Theta \subseteq \hat{\Theta}$. This proves that $\Theta = \langle R \rangle$, i.e., (1) holds. □

Lemma 5.2. *Let K be a polytope in $\mathbb{R}^n$ and $\mathbb{K}$ the corresponding convex algebra in* CA. *Let $\Theta_1, \Theta_2 \in \mathrm{Con}_{\mathsf{CA}} \mathbb{K}$ and $\Theta_1 \subseteq \Theta_2$. Then there is a finite subset $R \subseteq \Theta_2$ such that $\Theta_2 = \langle \Theta_1 \cup R \rangle$.*

Proof. Let $\mathbb{K}$, Θ_1, and Θ_2 be as in the assertion of the lemma. We construct R as follows:
(i) For each edge $\{Y_1, Y_2\} \in E_{\Theta_2} \setminus E_{\Theta_1}$ choose a pair $(a, b) \in \Theta_2 \cap (\check{\mathrm{co}} Y_1 \times \check{\mathrm{co}} Y_2)$. Take this pair into R.
(ii) For each $Y \in V_K$, choose pairs $(a_i, b_i) \in \Theta_2 \cap (\check{\mathrm{co}} Y \times \check{\mathrm{co}} Y)$, $i = 1, \dots, \dim \varphi_{\Theta_2}(Y) - \dim \varphi_{\Theta_1}(Y)$, such that

$$\varphi_{\Theta_1}(Y) + \mathrm{span}\big\{b_i - a_i \;\big|\; i = 1, \dots, \dim \varphi_{\Theta_2}(Y) - \dim \varphi_{\Theta_1}(Y)\big\} = \varphi_{\Theta_2}(Y). \tag{5.1}$$

Take these pairs into R.

The set R is finite, as the graph is finite and we work in a finite-dimensional space. Let $\Theta \in \mathrm{Con}_{\mathsf{CA}} \mathbb{K}$, and assume that $\Theta_1 \cup R \subseteq \Theta$. Then $E_{\Theta_1} \subseteq E_\Theta$ since $\Theta_1 \subseteq \Theta$ (using Theorem 4.14), and $E_{\Theta_2} \setminus E_{\Theta_1} \subseteq E_\Theta$, since $R \subseteq \Theta$. Hence $E_{\Theta_2} \subseteq E_\Theta$.

Also, by Theorem 4.14 since $\Theta_1 \subseteq \Theta$, we have $\varphi_{\Theta_1}(Y) \subseteq \varphi_\Theta(Y)$, for all $Y \in V_K$, and since R contains the pairs with (5.1), also $\varphi_{\Theta_2}(Y) \subseteq \varphi_\Theta(Y)$, for all $Y \in V_K$. By Theorem 4.14, $\Theta_2 \subseteq \Theta$. Hence, $\Theta_2 = \langle \Theta_1 \cup R \rangle$. □

Theorem 5.3. *Let $\mathcal{V}$ be one of the equational classes* CA, PCA, *or* ACA, *and let $\mathbb{A} \in \mathcal{V}$ be finitely generated. Then every congruence on $\mathbb{A}$ is finitely generated.*

Proof. Consider first CA. Let $n \in \mathbb{N}^+$. $F_n(\mathsf{CA})$ is a polytope with the usual vector operations. Let $\Theta_1, \Theta_2 \in \mathrm{Con}_{\mathsf{CA}} F_n(\mathsf{CA})$ with $\Theta_1 \subseteq \Theta_2$. By Lemma 5.2, $\Theta_2 = \langle \Theta_1 \cup R \rangle$ for a finite set $R \subseteq \Theta_2$. By Lemma 5.1, all congruences of all finitely generated convex algebras are finitely generated.

Consider now PCA. Let $\mathbb{A}$ be a finitely generated algebra in PCA and $\Theta \in \mathrm{Con}_{\mathsf{PCA}} \mathbb{A}$. It is important to note that $U_{\mathsf{CA}}(\mathbb{A})$ is also finitely generated in CA. Namely, if $\mathbb{A} = \langle X \rangle_{\mathsf{PCA}}$ for a finite set X, then $U_{\mathsf{CA}}(\mathbb{A}) = \langle X \cup \{0_A\} \rangle_{\mathsf{CA}}$. Moreover, from Lemma 3.8 we get that $\Theta \in \mathrm{Con}_{\mathsf{CA}} U_{\mathsf{CA}}(\mathbb{A})$ and hence Θ is finitely generated.

Similarly for ACA. Let $\mathbb{A}$ be a finitely generated algebra in ACA and $\Theta \in \mathrm{Con}_{\mathsf{ACA}} \mathbb{A}$. Again we remark that $U_{\mathsf{PCA}}(\mathbb{A})$ is finitely generated in PCA: If $\mathbb{A} = \langle X \rangle_{\mathsf{ACA}}$ for a finite set X, then $U_{\mathsf{PCA}}(\mathbb{A}) = \langle X \cup \bar{X} \rangle_{\mathsf{PCA}}$ where $\bar{X} = \{(-1)x \mid x \in X\}$. Now from Lemma 3.11, $\Theta \in \mathrm{Con}_{\mathsf{PCA}} U_{\mathsf{PCA}}(\mathbb{A})$ and again Θ is finitely generated. $\square$

Let us now recall the (algebraic) definition of a finitely presentable algebra, see, e.g., [1, 3.10]. The connection to the categorical concept of presentability is made in [1, Theorem 3.12], see also [2, Corollary 11.33].

Definition 5.4. Let $\mathcal{V}$ be an equational class, and let $\mathbb{A} \in \mathcal{V}$. Let $F_X(\mathcal{V})$ be the free algebra in $\mathcal{V}$ with the set X as free generators. A presentation of $\mathbb{A}$ is a pair (X, R_X) where X is a set and R_X is a subset of $F_X(\mathcal{V}) \times F_X(\mathcal{V})$ such that $A \cong F_X(\mathcal{V})/\Theta$ where $\Theta = \langle R_X \rangle$.

An algebra $\mathbb{A}$ is finitely presentable if there exists a presentation (X, R_X) of $\mathbb{A}$ with both X and R_X finite.

Obviously, an algebra is finitely generated if and only if there exists a presentation (X, R_X) with X finite. Hence, trivially, each finitely presentable algebra is finitely generated. Having shown the previous theorem, we obtain as an immediate consequence that for CA, PCA, and ACA also the converse holds.

Corollary 5.5. *Let $\mathcal{V}$ be one of the equational classes CA, PCA, or ACA, and let $\mathbb{A} \in \mathcal{V}$. If $\mathbb{A}$ is finitely generated, then $\mathbb{A}$ is also finitely presentable. Hence a convex, positively convex, or absolutely convex algebra is finitely presentable if and only if it is finitely generated.* $\square$

Remark 5.6. A sufficient condition in order that each finitely generated algebra of an equational class is finitely presentable, appeared recently in a categorical context. The condition is that the subclass of all finitely generated algebras of the equational class is closed under kernel pairs, cf. [4, Lemma 3.19]. Formulated algebraically, it means that each congruence of any finitely generated algebra $\mathbb{A}$ is finitely generated as a subalgebra of $\mathbb{A} \times \mathbb{A}$.

In the sequel we will compare this sufficient condition for "finitely generated $\Rightarrow$ finitely presentable" with the sufficient condition that each congruence of a free finitely generated algebra is finitely generated as a congruence. In general, the condition that each congruence of a finitely generated algebra is finitely generated as a subalgebra of the product algebra is at least as strong as the condition that each congruence of a free finitely generated algebra is finitely generated as a congruence. We state this fact in the next lemma. The proof is direct by unfolding the definitions.

Lemma 5.7. *Let $\mathbb{A}$ be an algebra in a variety $\mathcal{V}$ and Θ a congruence on $\mathbb{A}$. If $\Theta = \langle R \rangle_{\mathcal{V}}$ for a set $R \subseteq \Theta$, then $\Theta = \langle R \rangle$. Hence, if Θ is finitely generated as a subalgebra of $\mathbb{A} \times \mathbb{A}$, then Θ is finitely generated as a congruence.* $\square$

Moreover, for $\mathcal{V}$ being any of CA, PCA, ACA, the condition of finitely generated algebras being closed under kernel pairs is stronger than the condition that each congruence of a free algebra with finitely many generators is finitely generated as a congruence. Namely, for $\mathcal{V}$ being any of CA, PCA, ACA, the free algebras $F_n(\mathcal{V})$ for $n \in \mathbb{N}^+$ (except for $F_1(\mathsf{CA})$ which contains only one element) always contain congruence relations which are not finitely generated as a subalgebra of $F_n(\mathcal{V}) \times F_n(\mathcal{V})$, see Example 5.10 below. This follows immediately from the next proposition, where we characterize the congruences on a polytope $\mathbb{K}$ that are finitely generated as subalgebras of $\mathbb{K} \times \mathbb{K}$, showing that a "kernel pair argument" cannot be applied in these equational classes. Before we can prove this fact, we need an auxiliary result.

Lemma 5.8. *Let K be a polytope in $\mathbb{R}^n$, and let $\Theta \in \operatorname{Con}_{\mathsf{CA}} \mathbb{K}$. Then*

$$\operatorname{Clos} \Theta = \big\{(x_1, x_2) \in K \times K : x_2 - x_1 \in \varphi_\Theta(\operatorname{ext} K)\big\}.$$

Proof. Denote the set on the right side of the desired equality as Θ_0. By the representation (4.6) and monotonicity of φ_Θ, we have $\Theta \subseteq \Theta_0$.

It is easy to show that Θ_0 is closed (the limit of any converging sequence of elements of Θ_0 is in Θ_0) since K is closed and $\varphi_\Theta(\operatorname{ext} K)$ is closed as a linear subspace.

Let $(x_1, x_2) \in \Theta_0$ be given. Choose a point $z \in \check{\operatorname{co}}(\operatorname{ext} K)$. Then, since $x_1, x_2 \in \operatorname{co}(\operatorname{ext} K)$, we obtain from Lemma 4.5 that $\Phi_z(s, x_1), \Phi_z(s, x_2) \in \check{\operatorname{co}}(\operatorname{ext} K)$, $s \in (0, 1)$, and we have

$$\Phi_z(s, x_2) - \Phi_z(s, x_1) = (1 - s)(x_2 - x_1) \in \varphi_\Theta(\operatorname{ext} K).$$

Now using (4.6), since $\operatorname{ext} K$ is a vertex of $\mathcal{G}_\Theta$ and hence in some connected component $\mathcal{C}$, and φ_Θ is monotone, we get

$$\big(\Phi_z(s, x_1), \Phi_z(s, x_2)\big) \in \Theta, \quad s \in (0, 1).$$

Letting s tend to 0, $\Phi_z(s, x_1)$ tends to x_1 and $\Phi_z(s, x_2)$ to x_2 and it follows that $(x_1, x_2) \in \operatorname{Clos} \Theta$. Hence $\Theta_0 \subseteq \operatorname{Clos} \Theta$ and therefore $\operatorname{Clos} \Theta = \Theta_0$. $\square$

This lemma tells us, in particular, that Θ is closed if and only if $\Theta = \Theta_0$.

Proposition 5.9. *Let $\mathcal{V}$ be one of* CA, PCA, ACA. *Let K be a polytope in $\mathbb{R}^n$ such that $\mathbb{K}$ with the corresponding vector operations is in $\mathcal{V}$. Let $\Theta \in \operatorname{Con}_{\mathcal{V}} \mathbb{K}$. Then Θ is finitely generated as a $\mathcal{V}$-subalgebra of $\mathbb{K} \times \mathbb{K}$ if and only if Θ is closed as a subset of $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Assume that $\Theta = \langle R \rangle_{\mathcal{V}}$ for a finite set $R \subseteq K \times K$. We distinguish the following cases:

- For $\mathcal{V} = \mathsf{CA}$, $\Theta = \operatorname{co} R$.
- For $\mathcal{V} = \mathsf{PCA}$, $\Theta = \operatorname{co}(R \cup \{0_K\})$.
- For $\mathcal{V} = \mathsf{ACA}$, $\Theta = \operatorname{co}(R \cup \bar{R})$.

Here again $\bar{R} = \{(-1)r \mid r \in R\}$. In any case, Θ is a polytope in $\mathbb{R}^{2n}$ and thus in particular closed.

For the opposite direction, assume first that $\Theta \in \operatorname{Con}_{\mathsf{CA}} \mathbb{K}$ and that Θ is closed as a subset of $\mathbb{R}^n \times \mathbb{R}^n$. We will now show that Θ is finitely generated as a CA-subalgebra of $\mathbb{K} \times \mathbb{K}$. Set $d := n - \dim \varphi_\Theta(\operatorname{ext} K)$, and choose a linear and surjective map $\pi: \mathbb{R}^n \to \mathbb{R}^d$ with $\operatorname{Ker} \pi = \varphi_\Theta(\operatorname{ext} K)$. Note that this means

$$\begin{aligned} (x_1, x_2) \in \ker \pi \quad &\Leftrightarrow \quad \pi(x_1) = \pi(x_2) \quad \Leftrightarrow \quad x_2 - x_1 \in \big\{x \mid \pi(x) = 0\big\} \\ &\Leftrightarrow \quad x_2 - x_1 \in \operatorname{Ker} \pi \quad \Leftrightarrow \quad x_2 - x_1 \in \varphi_\Theta(\operatorname{ext} K). \end{aligned}$$

We denote by Δ the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Since Θ is closed, by Lemma 5.8 and the above, we have

$$\Theta = \operatorname{Clos} \Theta = (K \times K) \cap \ker \pi = (K \times K) \cap (\pi \times \pi)^{-1}(\Delta).$$

The set $K \times K$ is a compact (since K is compact) and convex subset of $\mathbb{R}^{2n}$, and one can easily show unfolding the definition of an extremal point that

$$\operatorname{ext}(K \times K) = \operatorname{ext} K \times \operatorname{ext} K,$$

so $\operatorname{ext}(K \times K)$ is a finite set. Hence, $K \times K$ is a polytope in $\mathbb{R}^{2n}$.

Now we will employ some non-trivial geometric arguments from [13]. The diagonal Δ is a linear subspace, and hence can be written as a finite intersection of halfspaces, i.e., it is a polyhedral set in the sense of [13, §2.6]. Since $(\pi \times \pi)$ is linear and surjective, the inverse image of a halfspace is again a halfspace. Hence, $(\pi \times \pi)^{-1}(\Delta)$ is again a polyhedral set.

As an intersection of a polytope with a polyhedral set, Θ is a polytope, cf. [13, §3.1]. Hence, Θ has only finitely many extremal points and $\Theta = \operatorname{co}(\operatorname{ext}\Theta)$. This means that the finite set $\operatorname{ext}\Theta$ generates Θ as a CA-subalgebra of $\mathbb{K} \times \mathbb{K}$.

It remains to consider PCA and ACA. Assume first that $\mathbb{K} \in \mathsf{PCA}$. Assume that $\Theta \in \operatorname{Con}_{\mathsf{PCA}} \mathbb{K}$ and is closed. Then, by Lemma 3.8 also $\Theta \in \operatorname{Con}_{\mathsf{CA}} U_{\mathsf{CA}}(\mathbb{K})$, and hence by what we have proven so far Θ is finitely generated as a CA-subalgebra of $U_{\mathsf{CA}}(\mathbb{K}) \times U_{\mathsf{CA}}(\mathbb{K})$. This means $\Theta = \langle R\rangle_{\mathsf{CA}}$ for a finite set R, and we have $\Theta \subseteq \langle R\rangle_{\mathsf{PCA}}$. Since Θ is a PCA-congruence on $\mathbb{K}$, $\langle R\rangle_{\mathsf{PCA}} \subseteq \Theta$. Hence Θ is finitely generated as a PCA-subalgebra from $\mathbb{K} \times \mathbb{K}$. Analogously, let $\mathbb{K} \in \mathsf{ACA}$ and assume that Θ is closed and $\Theta \in \operatorname{Con}_{\mathsf{ACA}} \mathbb{K}$. Then by Lemma 3.11 also $\Theta \in \operatorname{Con}_{\mathsf{CA}} U_{\mathsf{PCA}} U_{\mathsf{CA}}(\mathbb{K})$, and hence Θ is finitely generated as a CA-subalgebra of $U_{\mathsf{PCA}} U_{\mathsf{CA}}(\mathbb{K}) \times U_{\mathsf{PCA}} U_{\mathsf{CA}}(\mathbb{K})$. The same arguments as for PCA yield that Θ is finitely generated as an ACA-subalgebra of $\mathbb{K} \times \mathbb{K}$. □

Example 5.10. Let $K \subseteq \mathbb{R}^n$ be a polytope with $|\operatorname{ext} K| \geq 2$, and let Y_0 be a nonempty and proper subset of $\operatorname{ext} K$. Consider the join-semilattice congruence $\sim$ on V_K defined by specifying its equivalence classes to be

$$\mathcal{C}_y := \big\{\{y\}\big\}, \quad y \in Y_0, \qquad \mathcal{C}_0 := V_K \setminus \bigcup_{y \in Y_0} \mathcal{C}_y,$$

and the map $\varphi : \{Y(\mathcal{C}) | \mathcal{C} \text{ class of } \sim\} \to \operatorname{Sub} \mathbb{R}^n$ defined as

$$\varphi\big(Y(\mathcal{C}_y)\big) := \{0\}, \quad y \in Y_0, \qquad \varphi\big(Y(\mathcal{C}_0)\big) := \operatorname{dir}(\operatorname{ext} K),$$

cf. Remark 4.12. Applying Theorem 4.11, we obtain that the relation

$$\Theta := \big\{(x,x) \mid x \in Y_0\big\} \cup \big\{(x_1,x_2) \in Z(\mathcal{C}_0) \times Z(\mathcal{C}_0) \mid x_2 - x_1 \in \operatorname{dir}(\operatorname{ext} K)\big\}$$

is a congruence of $\mathbb{K}$, the convex algebra with carrier K and the usual vector space operations of $\mathbb{R}^n$.

Let $y_0 \in Y_0$. Since y_0 is an extremal point, it cannot be an element of $Z(\mathcal{C}_0)$. However, it can be approximated by elements from $Z(\mathcal{C}_0)$: Choose $y \in (\operatorname{ext} K) \setminus Y_0$, and set $x_\varepsilon := \varepsilon y + (1-\varepsilon) y_0$, $\varepsilon \in (0,1]$. Then one can check from the definitions that $(x_\varepsilon, y) \in \Theta$ and $\lim_{\varepsilon \downarrow 0}(x_\varepsilon, y) = (y_0, y) \notin \Theta$, which shows that Θ is not closed.

This example applies immediately to the free algebras $F_n(\mathsf{CA})$, $n \geq 2$, and $F_n(\mathsf{PCA})$, $n \in \mathbb{N}^+$, and shows that they contain congruences which are not finitely generated as subalgebras. For $F_n(\mathsf{ACA})$, choose Y_0 symmetric around the zero vector.

Remark 5.11. The question whether or not every finitely generated algebra of an equational class if finitely presented is classical. Some previously known examples where the answer is positive are

- *Commutative groups:* Due to the classification of finitely generated commutative groups.
- *Semimodules over a Noetherian semiring*: Due to a "kernel pair argument", cf. [4, Proposition 2.6].

- *Commutative semigroups:* This is essentially a particular case of the previous item (units can be adjoined easily), but has a longer history. It was first shown by L. Rédei, cf. [31, Satz 72] or [7, §9.3]. A short proof based on Hilbert's basis theorem (i.e., a "Noetherian" argument) is given in [11].

There are many equational classes where the answer is negative. For example, the equational class of all groups: Not every finitely generated group is finitely presented. In fact, there exist only countable many non-isomorphic finitely presentable groups, but already $2^{\aleph_0}$ non-isomorphic 2-generator groups. The (probably) simplest example of a finitely generated but not finitely presented group is the standard wreath product $\mathbb{Z} \operatorname{wr} \mathbb{Z}$, cf. [33, §14.1]. The question whether a specific finitely generated group is finitely presented may be involved, see for example [32], [21], or [14]. We also would like to draw the readers attention to [23, Example 1.17], where an example of a Grothendieck category (hence an abelian category) is presented which does not possess any nontrivial finitely presented objects.

6. Conclusion

We fully describe the congruence lattice of a polytope in finite-dimensional euclidean space when considered as a convex algebra. The free finitely generated convex algebras are polytopes, hence our results provide a description of all congruences of free finitely generated convex algebras. As finitely generated algebras are quotients (under congruences) of free finitely generated algebras, we can describe the congruence lattice of finitely generated algebras in the varieties of convex algebras. Moreover, we see that each finitely generated convex algebra is finitely presented. The proofs are algebraic in their nature and use the geometry of euclidean space.

We show that the equational classes of positively and absolutely convex algebras (and their respective congruence lattices) are closely related with convex algebras. Using this relation, similar structure results for these equational classes follow.

As mentioned before, congruences of absolutely convex algebras (including the infinitary case) were studied in [30]. The authors treat finitary and infinitary absolutely convex algebras with the same methods and provide interesting results. On the other hand, [30] does not provide a full description of the congruence lattice of absolutely convex algebras nor deals with the problem of finite presentability. The main novelty in our approach is to formalize the connection between facets of different dimensions using a graph. The finiteness of this graph is crucial to our results, we do not see a direct way to extend our approach to the infinitary case. It is an interesting direction for future work to see if our methods can be combined with those of [30].

The problems of convex structures, discussed in this and other papers, have since more than six decades arisen in many different fields: physics, chemistry, probability theory, game theory, economics, and mathematics. Our personal interest in convexity was awaken by the fact that convex algebras happen to be the Eilenberg–Moore algebras of certain monads, used in the modeling of probabilistic transition systems. In particular, we believe that the knowledge that finitely generated and finitely presentable convex algebras coincide might be helpful in solving some open problems about probabilistic transition systems.

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