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Harald Woracek

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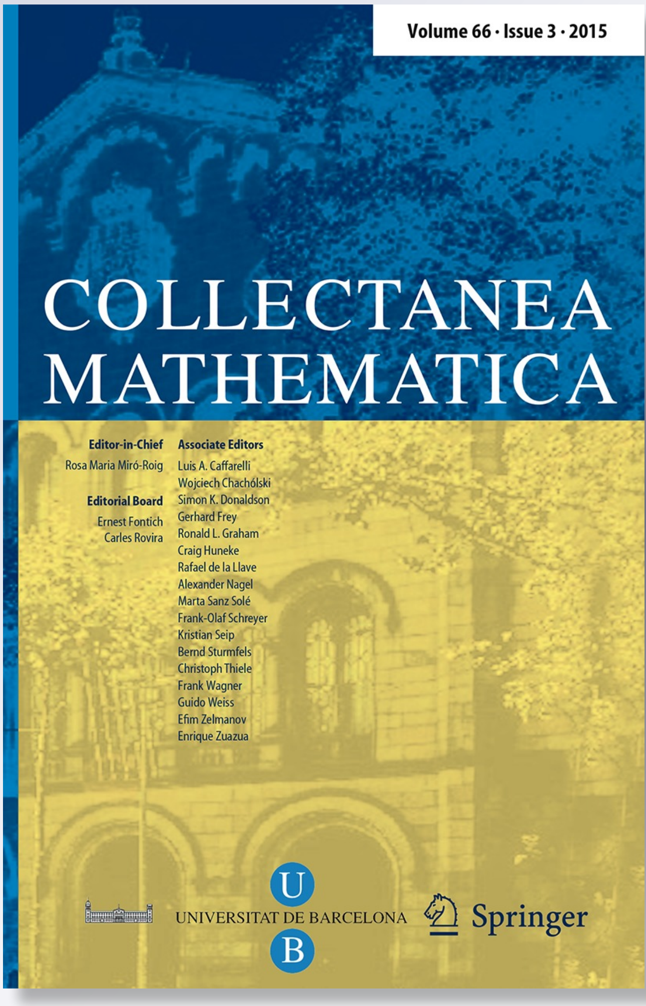

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Asymptotics of eigenvalues for a class of singular Kreĭn strings

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Abstract A Kreĭn string is (essentially) a pair $\mathsf{S}[L, m]$ where $0 < L \leq \infty$ and $m : [0, L) \to [0, \infty)$ is nondecreasing. Each string gives rise to an operator model, the Kreĭn-Feller differential operator $-D_m D_x$ acting in the space $L^2(dm)$. This operator has a selfadjoint realization which is nonnegative. Provided that $L + \lim_{x \to L} m(x) < \infty$, this realization has discrete spectrum and, when (λ_n) denotes the sequence of positive eigenvalues arranged increasingly, then

$$\lim \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi} \int_0^L \sqrt{m'(x)}\, dx \,.$$

We show that for a class of strings defined by a weaker growth restriction the spectrum is discrete, the integral on the right side is still finite, and the asymptotic behaviour of the eigenvalues is determined by the above formula.

Keywords Kreĭn string · Eigenvalue asymptotics

Mathematics Subject Classification (2010) Primary 34L20 · 46E22 · 47B50; Secondary 34L40 · 37J99

1 Introduction

A Kreĭn string is a pair, we denote it as $\mathsf{S}[L, m]$, which consists of a number L with $0 < L \leq \infty$ and a nonnegative and nondecreasing left-continuous function defined on $(-\infty, L]$ (or $(-\infty, \infty)$ if $L = \infty$) which is equal to 0 on $(-\infty, 0]$. Without loss of generality, we assume that m is not constant in any neighbourhood of L. Thereby, the number L models the length

H. Woracek (✉)
Institute for Analysis and Scientific Computing, Vienna University of Technology,
Wiedner Hauptstraße. 8-10/101, 1040 Wien, Austria
e-mail: harald.woracek@tuwien.ac.at

of the string, and m its mass-distribution. A string $\mathsf{S}[L, m]$ is called regular (or 'short' in the terminology of [6]), if both numbers L and $m(L) := \lim_{x \nearrow L} m(x)$ are finite. Otherwise, it is called singular (or 'long').

A string $\mathsf{S}[L, m]$ gives rise to an operator model, namely the Kreĭn-Feller differential operator $-D_m D_x$ acting in the space $L^2(dm)$. The eigenvalue equation of one of its selfadjoint realizations can be written as an integral boundary value problem in the form

$$\begin{cases} y'(x) + \int_{[0,x]} zy(u)\,dm(u) = 0, & x \in (-\infty, L)\,, \\ y'(0-) = 0, \quad \text{and} & y(L) = 0 \text{ if } L + m(L) < \infty\,, \end{cases} \tag{1.1}$$

where $z \in \mathbb{C}$ is the eigenvalue parameter. The operator $-D_m D_x$ arises when Fourier's method is applied to the partial differential equation

$$\frac{\partial}{\partial m(s)} \left(\frac{\partial v(s,t)}{\partial s} \right) - \frac{\partial^2}{\partial t^2} v(s,t) = 0\,.$$

Concerning physical interpretation, this equation describes the vibrations of an inhomogenous string with a free left endpoint, being stretched with unit tension on the interval $[0, L)$, and whose total mass on the interval $[0, x]$ equals $m(x)$.

The spectrum of the Kreĭn-Feller operator associated with a string $\mathsf{S}[L, m]$ is fully described by one analytic function, its principle Titchmarsh-Weyl coefficient q_S. In fact, a Fourier transform can be constructed which maps $-D_m D_x$ to multiplication in $L^2(\mu_S)$, where μ_S is the measure in the representation of q_S as a Cauchy integral.

The principles of the theory of strings, including direct and inverse spectral theorems, were established by M.G.Kreĭn in the early 1950s, cf. [18][1], see also [10][2]. For a presentation from a slightly different viewpoint we refer to [6].

Computing the asymptotics of eigenvalues by means of an integral involving mass-function or potential is a commonly done. Consider, for example, the case that $\mathsf{S}[L, m]$ is a regular string. Then the spectrum of $-D_m D_x$ is discrete. If (λ_n) denotes the sequence of positive eigenvalues arranged in increasing order, then[3]

$$\lim \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi} \int_0^L \sqrt{m'(x)}\,dx \tag{1.2}$$

Validity of this formula is a classical result which dates back to a paper of M.G.Kreĭn, cf. [17][1] see also [10, 11.8°][1]. It can be derived from a general formula computing the exponential type of the fundamental solution of a two-dimensional Hamiltonian system[4]. This approach has been used in [7, Theorem 8.1] where a complete proof of (1.2) is presented. A more direct method to compute exponential type for the de Branges space associated with a regular string, and thereby establish (1.2), is presented in [6, §6.3(6)].

The formula (1.2) has been extended to a certain class of singular strings in a paper by Kac: In [9, Theorem 5][1] it is stated that, under certain growth and smoothness assumptions

1 It contains the mentioned statement without a proof.

2 For the reason of physical interpretation, in this paper the principle Titchmarsh-Weyl coefficient is called 'coefficient of dynamic compliance'.

3 The limit of a finite sequence is tacitly understood as 0.

4 Also a classical result, see, e.g., [16], [4, Theorem X], or [7, Ch.VI, (6.5)].

on $\mathsf{S}[L, m]$, the formula

$$\lim \frac{n}{\sqrt{\lambda_n}} = \frac{2}{\pi} \int_0^L \frac{1}{\sqrt{m'_+(x)} + \sqrt{m'_-(x)}}\, dm(x) \tag{1.3}$$

holds where m'_+ and m'_- denote the one-sided derivatives of m. See also [1], where this formula was established under a stronger smoothness assumption. For equations of the form $-y''(x) = zV(x)y(x)$ the corresponding result has been shown for a wide range of potentials in [21]. See also [2], where in addition some other forms of equations are studied. The difference to the presently studied equation is that the string equation not only involves a pure second derivative $-D_x D_x$ but the derivative $-D_m D_x$. Of course, under some smoothness conditions equations of these forms can be tranformed into each other by performing a Liouville transform (one certainly sufficient condition being that m is absolutely continuous and m' is positive a.e.). However, if m behaves singularly (meaning compared to Lebesgue measure), making a Liouville transform is not possible. Also one should say that, due to the form of the Liouville transform, it is by no means clear how conditions on the mass-distribution m reflect in conditions on the corresponding potential V and vice versa.

In the present paper we show that the formula (1.2), including finiteness of the integral on its right side, is valid for a class of singular strings defined by a pure growth condition. See Definitions 3.1 and 3.2 for the definition of the class under consideration, and Theorem 3.5 for the result itself. Our proof proceeds via Pontryagin space theory and mimicks the mentioned approach via Hamiltonian systems: The crucial idea is that for a string $\mathsf{S}[L, m]$ belonging to the considered class the function $zq_S(z^2)$ can be written as the quotient of two entire functions which are entries of the fundamental solution of a Hamiltonian system with an inner singularity. Then the recent result [20, Theorem 4.1] is applied to compute the exponential type of these entire functions. Finally, some computations and routine complex analysis lead to the desired formula.

2 Some notation and facts

In the present paper, we use without further notice the classical theory of two-dimensional Hamiltonian systems

$$y'(x) = zJH(x)y(x), \quad x \in [s_-, s_+), \tag{2.1}$$

with H being positive semidefinite and locally integrable on $[s_-, s_+)$. Here J denotes the signature matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and z is a complex parameter, the eigenvalue parameter. For notation and a compilation of the basic properties of Hamiltonian systems (in particular Weyl theory) from an up-to-date viewpoint, we refer the reader to [8]. A more classical reference would be, e.g., [7].

2.1 Strings vs. Hamiltonian systems

Strings and Hamiltonian systems are related in various ways; we use the following fact established, e.g., in [11].

Proposition 2.1 *Let a string $\mathsf{S}[L, m]$ be given. Denote $\mu(x) := x + m(x)$, then the Lebesgue measure dx and the Borel measure dm are both absolutely continuous with respect to $d\mu$.*

We define a Hamiltonian H_d *on the interval* $I_d := [0, \infty)$ *as*

$$H_d(x) := \begin{cases} \begin{pmatrix} \frac{dx}{d\mu}(x) & 0 \\ 0 & \frac{dm}{d\mu}(x) \end{pmatrix} , & x \in \operatorname{ran} \mu \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , & x \in [0, \infty) \setminus \operatorname{ran} \mu \end{cases}$$

Then the Weyl coefficient q_{H_d} *of the Hamiltonian system*

$$y'(x) = zJH_d(x)y(x), \quad x \in [0, \infty) ,$$

and the principal Titchmarsh-Weyl coefficient q_S *of* $\mathsf{S}[L, m]$ *are related as*

$$q_{H_d}(z) = zq_S(z^2) . \tag{2.2}$$

Conversely, each diagonal Hamiltonian $H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ defined on some interval $[s_-, s_+)$ gives rise to a string. Denote

$$\check{v}(t) := \int_{s_-}^{t} h_1(x)\, dx, \quad \hat{v}(t) := \int_{s_-}^{t} h_2(x)\, dx, \qquad t \in [s_-, s_+] ,$$

and let $\hat{\rho}$ and $\check{\rho}$ be the left-continuous right inverses of $\hat{v}$ and $\check{v}$, respectively. Explicitly, this is

$$\hat{\rho}(y) := \inf \big\{x \in [s_-, s_+] : \hat{v}(x) = y\big\}, \quad y \in [0, \hat{v}(s_+)], \tag{2.3}$$

$$\check{\rho}(y) := \inf \big\{x \in [s_-, s_+] : \check{v}(x) = y\big\}, \quad y \in [0, \check{v}(s_+)] . \tag{2.4}$$

Then the pair consisting of $L := \check{v}(s_+)$ and $m(x) := (\hat{v} \circ \check{\rho})(x)$, $x \in [0, L]$, constitutes a string.

The fact that these constructions are converse to each other, follows by comparing [11, Sect.4], in particular the relations '(4.1–4.3)' and '(4.4), (4.6), (4.8)', with the notation introduced above.

2.2 Hamiltonian systems with inner singularities.

The classical theory of the 'positive definite' Eq. (2.1), can be generalized to an indefinite (Pontryagin space) setting. In [13] this more general situation is introduced and studied, and an operator model acting in a Pontryagin space is constructed; direct and inverse spectral theorems are established in [14] and [15]. Thereby:

- ⋆ The Hamiltonian H is permitted to have a finite number of inner singularities (inner points of $[s_-, s_+)$ where H is not locally integrable). Such points contribute to the equation by means of interface conditions connecting before and after the singularity as well as by an action concentrated in the singularity.
- ⋆ The class of Nevanlinna functions (appearing as the totality of all Weyl coefficients of Eq. (2.1) with singular right endpoint) is subsituted by the class $\mathcal{N}_{<\infty}$ of generalized Nevanlinna functions in the sense of [19].
- ⋆ The class of J-contractive entire matrix functions (appearing as monodromy matrices of Eq. (2.1) with regular right endpoint) is substituted by the class $\mathcal{M}_{<\infty}$ of all entire matrix functions for which the kernel

$$H_W(w, z) := \frac{W(z)JW(w)^* - J}{z - \overline{w}}$$

has a finite number of negative squares.
⋆ The fundamental matrix solution $W(x;z)$, $x \in [s_-, s_+)$, of the Eq. (2.1) is substituted by a maximal chain $W_{\mathfrak{h}}$ of matrices belonging to the class $\mathcal{M}_{<\infty}$ (for the definition of this class, see, e.g., [14, Sect.3.a, Sect.3.b]).

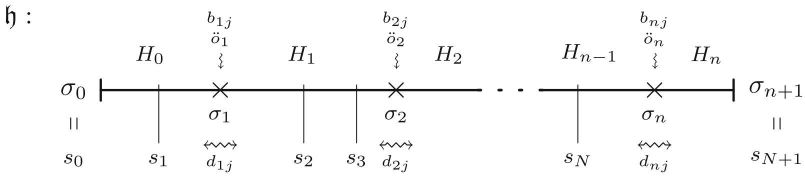
The formal definition of a 'general Hamiltonian' is rather complicated and would require more preparation, cf. [13, Definition 8.1]. In view of our present needs, we content ourselves with the following intuitive description of a regular general Hamiltonian $\mathfrak{h}$: It is given by the data

1. Points $s_-, s_+ \in \mathbb{R} \cup \{+\infty\}$, $s_- < s_+$; the interval on which the system acts. Points $\sigma_1, \ldots, \sigma_n \in (s_-, s_+)$ with $s_- < \sigma_1 < \ldots < \sigma_n < s_+$; the singularities of the system.
2. Hamiltonians H_i defined on $[s_-, \sigma_1)$, (σ_i, σ_{i+1}), and $(\sigma_n, s_+]$, respectively, which are integrable up to s_- and s_+, but are not integrable towards the singularities. Locally at singularities, the functions H_i are subject to certain growth restrictions, weaker than integrability. To unify notation, we set

$$H(x) := \begin{cases} H_1(x) & , \quad x \in [s_-, \sigma_1) \\ H_i(x) & , \quad x \in (\sigma_i, \sigma_{i+1}),\ i = 1, \ldots, n-1 \\ H_{n+1}(x) & , \quad x \in (\sigma_n, s_+] \end{cases}$$

3. Numbers $\ddot{o}_1, \ldots, \ddot{o}_n \in \mathbb{N} \cup \{0\}$ and $b_{i,1}, \ldots, b_{i,\ddot{o}_i+1} \in \mathbb{R}$, $i = 1, \ldots, n$; these numbers model a contribution to the equation which is concentrated at the singularity.
4. Numbers $d_{i,0}, \ldots, d_{i,2\Delta_i - 1} \in \mathbb{R}$, where Δ_i is a measure for the growth of H towards the singularity σ_i; these numbers model the local interaction between the Hamiltonians before and after the singularity. A finite subset $E = \{s_0, \ldots, s_{N+1}\}$ of $\{s_-, s_+\} \cup \bigcup_{i=0}^{n} (\sigma_i, \sigma_{i+1})$; the points of this set in the vicinity of a singularity make quantitatively precise what 'local interaction' means.

One can picture the situation as follows:



With a regular general Hamiltonian there is associated a family $W_{\mathfrak{h}}(x;z)$, $x \in [s_-, s_+] \setminus \{\sigma_1, \ldots, \sigma_n\}$, of entire matrix functions belonging to the class $\mathcal{M}_{<\infty}$, cf. [14, Definition 5.3]. This family $W_{\mathfrak{h}}$ is a solution of the differential equation

$$\begin{cases} \frac{\partial}{\partial x} W_{\mathfrak{h}}(x;z) J = z W_{\mathfrak{h}}(x;z) H(x), \quad x \in [s_-, s_+] \setminus \{\sigma_1, \ldots, \sigma_n\} \\ W_{\mathfrak{h}}(s_-;z) = I, \end{cases}$$

cf. [14, Corollary 5.6]. Note that the above equation is an initial value problem only on the interval $[s_-, \sigma_1)$, hence $W_{\mathfrak{h}}$ is determined by the Hamiltonian function H only on this interval; how to 'jump over singularities' depends on the other parameters of $\mathfrak{h}$. In analogy with the classical case (no singularities), we refer to the family $W_{\mathfrak{h}}$ as the fundamental solution associated with $\mathfrak{h}$, and to $W_{\mathfrak{h}}(s_+; .)$ as the monodromy matrix of $\mathfrak{h}$.

With each singularity σ_i of $\mathfrak{h}$ there is associated a function q_i of class $\mathcal{N}_{<\infty}$, the intermediate Weyl coefficient of $\mathfrak{h}$ at σ_i. It is defined as ($\tau \in \mathbb{R} \cup \{\infty\}$)

$$q_i(z) := \lim_{x \to \sigma_i} W_{\mathfrak{h}}(x;z) \star \tau\,, \tag{2.5}$$

where we denote $(W_{\mathfrak{h}}(x;z) = (W_{\mathfrak{h},ij}(x;z))_{i,j=1}^2)$

$$W_{\mathfrak{h}}(x;z) \star \tau := \frac{W_{\mathfrak{h},11}(x;z)\tau + W_{\mathfrak{h},12}(x;z)}{W_{\mathfrak{h},21}(x;z)\tau + W_{\mathfrak{h},22}(x;z)} .$$

The fact that the limit (2.5) exists and represents a function of class $\mathcal{N}_{<\infty}$ is [12, Proposition 5.1, Theorem 5.6].

A basic tool for the present considerations is the following statement which has been established in [20, Theorem 4.1]. It says that the formula to compute exponential type af a monodromy matrix, which is known from the classical case, remains valid also in the indefinite situation.

Theorem 2.2 *Let $\mathfrak{h}$ be a regular general Hamiltonian, and let $W_{\mathfrak{h}}(s_+;.)=(W_{\mathfrak{h},ij}(s_+;.))_{i,j=1}^2$ be the monodromy matrix of $\mathfrak{h}$. Then the entries $W_{\mathfrak{h},ij}(s_+;.)$ are entire functions of finite exponential type. Their exponential types* $\operatorname{et} W_{\mathfrak{h},ij}(s_+;.)$, *$i,j = 1,2$, are all equal and can be computed from $\mathfrak{h}$ by means of the formula*

$$\operatorname{et} W_{\mathfrak{h},ij}(s_+;.) = \int_{s_-}^{s_+} \sqrt{\det H(x)}\, dx .$$

Writing this formula includes the statement that the integral on the right side is finite.

3 Eigenvalue asymptotics

The class of strings under consideration in the present paper is defined by a, recursively computable, growth condition.

Definition 3.1 Let $\mathsf{S}[L,m]$ be a string.

(i) Assume that $L=\infty$. We denote by Θ_m the operator whose domain $\operatorname{dom}\Theta_m$ consists of all measureable functions $f:[0,\infty)\to\mathbb{C}$ with

$$f\in L^1_{\mathrm{loc}}([0,\infty)),\quad \int_0^x f(t)\,dt\in L^1(dm),$$

and which acts as

$$(\Theta_m f)(x) := \int_{[x,\infty)} \Big(\int_0^\xi f(s)\,ds\Big)\,dm(\xi),\quad x\in[0,\infty),\ f\in\operatorname{dom}\Theta_m .$$

(ii) Assume that $L<\infty$. We denote by Θ_L the operator whose domain $\operatorname{dom}\Theta_L$ consists of all measureable functions $f:[0,L)\to\mathbb{C}$ with

$$f\in L^1_{\mathrm{loc}}(dm),\quad \int_0^x f(t)\,dm(t)\in L^1(dx),$$

and which acts as

$$(\Theta_L f)(x) := \int_x^L \Big(\int_{[0,\xi)} f(s)\,dm(s)\Big)\,d\xi,\quad x\in[0,L),\ f\in\operatorname{dom}\Theta_L .$$

Definition 3.2 Let $\mathsf{S}[L,m]$ be a string. We say that $\mathsf{S}[L,m]$ is of Pontryagin type, if one of the following holds:

(i) $L=\infty$, $\int_0^\infty x\,dm(x)<\infty$, and for some $n\in\mathbb{N}_0$ we have $\Theta_m^n 1\in L^2([0,\infty))$.

(ii) $L < \infty$, $\int_0^L m(x)\,dx < \infty$, and for some $n \in \mathbb{N}_0$ we have $\Theta_L^n 1 \in L^2(dm)$.

Remark 3.3 According to [23, Remark 2.23] and [10, 11.9°], the respective first conditions in Definition 3.2 could be substituted by the much weaker conditions $\lim_{x\to\infty} x(m(\infty) - m(x)) = 0$ in (i), and $\lim_{x\to L}(L-x)m(x) = 0$ in (ii). The relevant part of this remark, however, is based on a reasoning which has not been carried out in detail (and is far beyond the scope of the present manuscript). Hence, we do not use it in our present exposition.

It is obvious that each regular string satisfies (ii), in fact, with $n = 0$. However, the class of Pontryagin type strings also contains many singular strings. For example, if $L < \infty$ and $\int_{s_-}^{s_+} m(x)^2\,dx < \infty$, then (ii) holds with $n = 1$. An illustrative concrete example is the following (modelled after [23, Example 3.15]).

Example 3.4 Let

$$\alpha \in (1,2) \setminus \left\{\tfrac{4n+1}{2n+1} : n \in \mathbb{N}\right\},$$

set

$$L := 1, \qquad m(x) := (1-x)^{1-\alpha} - 1,\ x \in [0,1),$$

and consider the string $\mathsf{S}[L, m]$. Clearly, $\int_0^L m(x)\,dx < \infty$. Moreover, a straightforward induction shows that

$$(\Theta_L^n 1)(x) \in \operatorname{span}\Big(\big\{(1-x)^{2n-n\alpha}\big\} \cup \big\{(1-x)^\beta : \beta \geq 1\big\}\Big), \quad n \in \mathbb{N}.$$

From this it follows that $\Theta_L^n 1 \in L^2(dm)$ if and only if $\alpha < \frac{4n+1}{2n+1}$, and hence

$\alpha \in$	$\min\{n \in \mathbb{N} : \Theta_L^n 1 \in L^2(dm)\}$
$\left(1, \frac{5}{3}\right)$	1
$\left(\frac{5}{3}, \frac{9}{5}\right)$	2
$\left(\frac{9}{5}, \frac{13}{7}\right)$	3
$\vdots$	$\vdots$

The result we are aiming for in the present paper, can now be formulated and proved.

Theorem 3.5 *Let $\mathsf{S}[L, m]$ be a string of Pontryagin type, and denote by (λ_n) the (finite or infinite) sequence of positive eigenvalues of the Kreĭn-Feller differential operator $-D_m D_x$. Then*

$$\lim \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi} \int_0^L \sqrt{m'(x)}\,dx,$$

with the integral on the right side being finite. In particular, the asymptotic behaviour of the sequence of eigenvalues depends only on the absolutely continuous part of m.

The proof of this result is a, to our taste appealing, application of Pontryagin space theory. The idea is to 'prolongue' the generically singular Hamiltonian H_d associated with $\mathsf{S}[L, m]$ to a regular indefinite Hamiltonian $\mathfrak{h}$ in such a way that exponential type can be computed from m and determines the asymptotics of the poles of q_S.

Proof (of Theorem 3.5; The core argument) Due to the conditions in Definition 3.2, at least one of $L < \infty$ and $m(L) < \infty$ holds. If both of these numbers are finite, the string is regular, and hence the assertion of the theorem reduces to the classical case. Thus we may for the rest of the proof assume that either $L < \infty, m(L) = \infty$ or $L = \infty, m(L) < \infty$.

Write $H_d = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. If $L < \infty$, we have $\int_0^\infty h_1(x)\,dx < \infty$. If $m(L) < \infty$, we have $\int_0^\infty h_2(x)\,dx < \infty$. In the first case, set $\phi(H_d) := 0$, in the second $\phi(H_d) := \frac{\pi}{2}$.

For technical reasons, choose a reparameterization $\tilde{H}_d$ of H_d which is defined on the interval $(0, 1)$. Let h_+ be a positive and locally integrable function on $(1, 2]$ with $\int_1^2 h_+(x)\,dx = \infty$, and denote by H_+ the function (here we denote $\xi_\phi := (\cos\phi, \sin\phi)^T$)

$$H_+(x) := h_+(x)\xi_{\phi(H_d)+\frac{\pi}{2}}\xi^T_{\phi(H_d)+\frac{\pi}{2}}, \quad x \in (1, 2]\,.$$

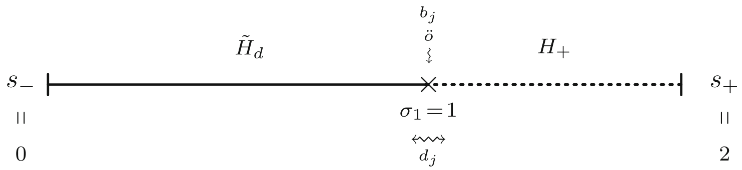
Moreover, set $s_0 := \inf\{x \in [0, 1] :\ \tilde{H}_d|_{(x,1)} = \operatorname{tr}\tilde{H}_d\xi_{\phi(H_d)+\frac{\pi}{2}}\xi^T_{\phi(H_d)+\frac{\pi}{2}}\}$.

By [23, Theorems 6.4 and 4.1], the following data constitutes a regular general Hamiltonian $\mathfrak{h}$:

$$n = 1,\ s_- = 0, \sigma_1 = 1, s_+ = 2, \quad \tilde{H}_d, H_+, \quad \ddot{o} = 0, b_j = 0, d_j = 0,$$

$$E := \begin{cases} \{0, 2\} & , \quad s_0 = 1 \\ \{0, s_0, 2\} & , \quad s_0 < 1 \end{cases}$$

$\mathfrak{h}$:



Denote by $W_\mathfrak{h}(x; z)$, $x \in [0, 2]$, the fundamental solution of $\mathfrak{h}$. Since $\tilde{H}_d$ is just a reparameterization of H_d, the intermediate Weyl coefficient q_1 of $\mathfrak{h}$ at the singularity 1 can be computed as ($\tau \in \mathbb{R} \cup \{\infty\}$ arbitrary)

$$q_1(z) = \lim_{x \nearrow 1} W_\mathfrak{h}(x; z) \star \tau = q_{H_d}(z)\,.$$

On the other hand, the function q_1 can also be obtained as a limit from above. Since we chose the Hamiltonian H_+ in a very simple form, this limit can be computed easily from the monodromy matrix of $\mathfrak{h}$. In fact, since $W_\mathfrak{h}$ is a solution of the differential equation $\frac{\partial}{\partial x} W_\mathfrak{h}(x; z)J = zW_\mathfrak{h}(x; z)H_+(x)$, $x \in (1, 2]$,

$$W_\mathfrak{h}(x; z) = W_\mathfrak{h}(2; z) \cdot \begin{cases} \begin{pmatrix} 1 & -l(x)z \\ 0 & 1 \end{pmatrix}, & \phi(H_d) = \frac{\pi}{2} \\ \begin{pmatrix} 1 & 0 \\ l(x)z & 1 \end{pmatrix}, & \phi(H_d) = 0 \end{cases}$$

where $l(x) := \int_x^2 h_+(t)\,dt$. We obtain that (remember that the value of the limit in the definition of a Weyl coefficient is independent of the choice of parameter τ)

$$q_1(z) = \lim_{x \searrow 1} W_\mathfrak{h}(x; z) \star \begin{cases} \infty, & \phi(H_d) = \frac{\pi}{2} \\ 0, & \phi(H_d) = 0 \end{cases} = \begin{cases} \frac{W_\mathfrak{h}(2;z)_{11}}{W_\mathfrak{h}(2;z)_{21}}, & \phi(H_d) = \frac{\pi}{2} \\ \frac{W_\mathfrak{h}(2;z)_{12}}{W_\mathfrak{h}(2;z)_{22}}, & \phi(H_d) = 0 \end{cases}$$

This shows that the poles of q_{H_d} coincide with the zeros of either $W_\mathfrak{h}(2; z)_{21}$ or $W_\mathfrak{h}(2; z)_{22}$.

Consider the entire function

$$A(z) := \begin{cases} W_{\mathfrak{h}}(2;z)_{21}, & \phi(H_d) = \frac{\pi}{2} \\ W_{\mathfrak{h}}(2;z)_{22}, & \phi(H_d) = 0 \end{cases}$$

The exponential type of A can be computed as

$$\text{et}\, A = \int_0^1 \sqrt{\det \tilde{H}_d(t)}\, dt + \int_1^2 \sqrt{\det H_+(t)}\, dt = \int_0^\infty \sqrt{\det H_d(x)}\, dx\,, \tag{3.1}$$

cf. Theorem 2.2. □

The rest is routine; we follow the proof of the regular case. However, since this is not accurately elaborated in the existing literature, we provide the necessary arguments in detail.

Proof (of Theorem 3.5; Finishing arguments).

Step 1; Some complex analysis: The function A is an entire function which takes real values along the real axis, and has no zeros off the real axis. Moreover, it is of bounded type in both half planes $\mathbb{C}^+$ and $\mathbb{C}^-$, for an explicit argument see, e.g., [20, Proposition 2.7]. By [5, Problem 34], it is of Polya class, and hence [5, Theorem 7] implies that A is a canonical product. Since A is of bounded type in $\mathbb{C}^+$, the condition (I) formulated in [3, p.137] holds. Hence [3, Theorem 8.21], together with [22, Theorems 6.18 and 6.15], gives

$$\lim_{r\to\infty} \frac{n(r)}{r} = \frac{2}{\pi} \lim_{y\to+\infty} \frac{1}{y} \log|A(iy)| = \frac{2}{\pi}\, \text{et}\, A\,,$$

where we denote $n(r) := \#\{z \in \mathbb{C} :\ A(z) = 0, |z| \le r\}$.

Since H_d is diagonal, the Weyl coefficient q_{H_d} is an odd function, see, e.g., [5, Problem 181]. In particular, its poles are located symetrically with respect to the origin. We conclude that

$$n(r) := 2 \cdot \#\{x > 0 :\ A(x) = 0, x \le r\} + \begin{cases} 0, & \phi(H_d) = 0 \\ 1, & \phi(H_d) = \frac{\pi}{2} \end{cases}$$

Denote by (x_n) the sequence of poles of q_{H_d} located on the positive real half-axis and arranged increasingly. Referring, e.g., to [3, Lemma 1.5.1], we then have

$$\lim_{n\to\infty} \frac{n}{x_n} = \frac{1}{2} \lim_{r\to\infty} \frac{n(r)}{r}\,.$$

Consider the sequence (λ_n) of nonzero eigenvalues of $-D_m D_x$ arranged increasingly. The relation (2.2) shows that $\lambda_n = x_n^2$. Putting together the above relations with (3.1), we see that

$$\lim \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi} \int_0^\infty \sqrt{\det H_d(x)}\, dx\,. \tag{3.2}$$

Step 2; Rewriting the integral: We write $H_d = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ and use the notation introduced in §2.a. Moreover, we assume without loss of generality that $h_1(x) = 0$ whenever x belongs to the closure of an indivisible interval[5] of type $\frac{\pi}{2}$; this can always be achieved by redefining H_d on a set of Lebesgue measure zero.

[5] An interval (a, b) is called indivisible for a Hamiltonian H, if H is of the form $h(x)\xi_\phi \xi_\phi^T$, $x \in (a, b)$ a.e., with some scalar function $h(x)$ and an angle ϕ which is independent of x. If (a, b) is indivisible, ϕ is called the type of (a, b).

Consider the function $g : [0, L] \to [0, \infty]$ defined as

$$g(y) := \begin{cases} \sqrt{\frac{h_2(\check{\rho}(y))}{h_1(\check{\rho}(y))}}, & h_1(\check{\rho}(y)) \neq 0, \\ 0, & h_1(\check{\rho}(y)) = 0. \end{cases}$$

Then, clearly, g is measurable and nonnegative. Since $\check{v} : [0, \infty] \to [0, L]$ is absolutely continuous, surjective, and $\check{v}' = h_1$ a.e., we have

$$\int_0^L g(y)\, dy = \int_0^\infty (g \circ \check{v})(x) h_1(x)\, dx\,. \tag{3.3}$$

To further rewrite this integral, consider a point $x \in (0, \infty)$ such that $h_1(x) \neq 0$. Then x does not belong to the closure of an indivisible interval of type $\frac{\pi}{2}$, and hence $(\check{\rho} \circ \check{v})(x) = x$. It follows that

$$(g \circ \check{v})(x) h_1(x) = \sqrt{\frac{h_2(\check{\rho}(\check{v}(x)))}{h_1(\check{\rho}(\check{v}(x)))}} \cdot h_1(x) = \sqrt{h_2(x) h_1(x)}\,. \tag{3.4}$$

If $x \in (0, \infty)$ is such that $h_1(x) = 0$, this equality trivially holds. Hence,

$$\int_0^\infty (g \circ \check{v})(x) h_1(x)\, dx = \int_0^\infty \sqrt{h_2(x) h_1(x)}\, dx = \int_0^\infty \sqrt{\det H_d(x)}\, dx\,.$$

Step 3; Computing $m'(x)$: Set

$$\begin{aligned} \hat{M} &:= \big\{x \in (0, \infty) : \ \hat{v}'(x) \text{ does not exist, or } \hat{v}'(x) \neq h_2(x)\big\}\,, \\ \check{M} &:= \big\{x \in (0, \infty) : \ \check{v}'(x) \text{ does not exist, or } \check{v}'(x) \neq h_1(x)\big\}\,, \\ \check{E} &:= \big\{y \in (0, L) : \ \check{\rho}'(y) \text{ does not exist}\big\}\,, \end{aligned}$$

and $A := \check{v}(\hat{M}) \cup \check{v}(\check{M}) \cup \check{E}$. This is a Lebesgue zero set, since $\check{v}$ and $\hat{v}$ are absolutely continuous, have derivatives a.e. equal to h_1 and h_2, respectively, and $\check{\rho}$ is monotone.

Let $y \in (0, L) \setminus A$ be fixed. Then, note that $\check{\rho}$ is injective and certainly continuous at y,

$$\begin{aligned} \lim_{y' \to y} \tfrac{\hat{v}(\check{\rho}(y')) - \hat{v}(\check{\rho}(y))}{\check{\rho}(y') - \check{\rho}(y)} &= h_2(\check{\rho}(y))\,, \\ \lim_{y' \to y} \tfrac{y' - y}{\check{\rho}(y') - \check{\rho}(y)} = \lim_{y' \to y} \tfrac{\check{v}(\check{\rho}(y')) - \check{v}(\check{\rho}(y))}{\check{\rho}(y') - \check{\rho}(y)} &= h_1(\check{\rho}(y))\,. \end{aligned}$$

Since $y \notin \check{E}$, we must have $h_1(\check{\rho}(y)) > 0$. It follows that

$$m'(y) = (\hat{v} \circ \check{\rho})'(y) = \frac{h_2(\check{\rho}(y))}{h_1(\check{\rho}(y))} = g(y)^2\,.$$

Together with (3.3) and (3.4), thus

$$\int_0^\infty \sqrt{\det H_d(x)}\, dx = \int_0^L \sqrt{m'(y)}\, dy\,.$$

By (3.2), the desired formula follows. □

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