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Comparison of Regularisation Methods Referring to a Multi-Pendulum Case Study

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Abstract: This paper compares several regularisation methods for differential-algebraic equation systems of high differential index by applying them to two model problems. Both problems are mechanical systems which are described by equation systems of index three, a pendulum and a double pendulum, where the latter furthermore shows chaotic behaviour. Some methods turn out to be not suitable at all for high index differential-algebraic equation systems while the quality of many regularisation methods depends on an adequate choice of parameters, implementation and the system itself to be solved with the respective method.

Keywords: DAE, Differential Index, Regularisation Methods, Mechanical System, Pendulum

1. INTRODUCTION

The description of mechanical systems in general leads to differential algebraic equation systems (DAEs) of high index. To solve these DAEs, different methods for regularisation or index reduction can be applied, see Hairer, and Wanner (2002). In this paper, six of those methods are compared by two case studies. The two considered case studies are mechanical systems of index three. On the one hand the equations of motion of a pendulum on a circular path in Cartesian coordinates are considered. On the other hand, the equations of motion of the double pendulum in Cartesian coordinates, which shows chaotic behaviour, are used. For the comparison of the considered regularisation methods, the obtained numerical solutions and the deviation from the constraint equations are taken into account. The presented methods are differentiation and substitution of the constraint equations (DaS C), the Baumgarte-Method (BM) (see Eich, and Hanke (1995)), the Pantelides algorithm (P), the orthogonal projection method (OP), the symmetric projection method (SP) (see Hairer (2000)) and transformation of the state space (SST).

2. CASE STUDIES

2.1 Pendulum

The equations of motion of a pendulum in Cartesian coordinates are given by

$$\begin{aligned} \dot{x} &= v_x \qquad \dot{y} = v_y \\ \dot{v}_x &= -Fx \qquad \dot{v}_y = \mathbf{g} - Fy \\ & x^2 + y^2 = 1, \end{aligned} \tag{1}$$

where F is the force and $\mathbf{g}$ is the gravitational acceleration, see Cellier, and Kofman (2006). The constraint equation of this system is given by $x^2 + y^2 - 1 = 0$. In the following the constraint equation and its derivatives with respect to t are considered

$$xv_x + yv_y = 0 \tag{2}$$

$$v_x^2 + v_y^2 - F(x^2 + y^2) + \mathbf{g}y = 0, \tag{3}$$

where it can be observed that from the second deriviative the force F can be obtained. This shows that the given DAE has differential index three.

All following simulations are done with MATLAB R2012b. The initial values for the presented scenario are $x = 1$, $y = 0$ and $v_x = v_y = F = 0$, where the initial value for the force F is not necessary for every method. Fig. 1 shows that the substitution of the constraint by its second derivative (3) is no suitable method for this problem as a clear numerical drift-off can be observed due to the loss of information by differentiation. The Baumgarte-Method

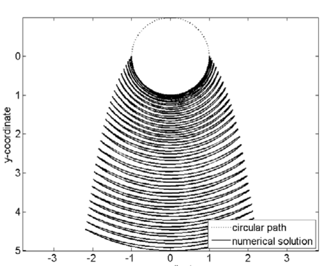

Fig. 1. Drift-off occuring after differentiation and substitution of the constraint, calculated with ode15s (MATLAB)

substitutes the constraint by a linear combination of the constraint and its derivatives, see (4).

$$\ddot{g} + 2\alpha\dot{g} + \beta^2 g = 0 \tag{4}$$

Due to the consideration of the original constraint in the new system, there is no loss of information. Nevertheless, the choice of suitable values for the parameters α and β can be challenging.

For the application of the Pantelides algorithm, four different systems have to be considered for different areas of

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the coordinate system due to the squares in the constraint, which has to be transformed to become an assignment for either x or y.
The results of the two projection methods both stay close to the circular path, but the positions themselves differ gravely. This is caused by a seemingly unbounded increase of speed with the orthogonal projection method, see Fig. 2. For this system, a global transformation - the common po-

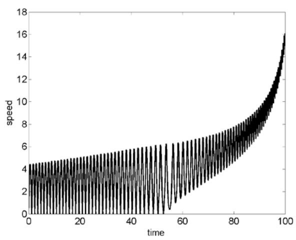


Fig. 2. Increase of speed with the orthogonal projection method (solved with an own implementation of the explicit Euler method)

lar coordinatisation, resulting in (5) - can be found. These equations represent an ordinary differential equation which can easily be solved with common ODE solvers.

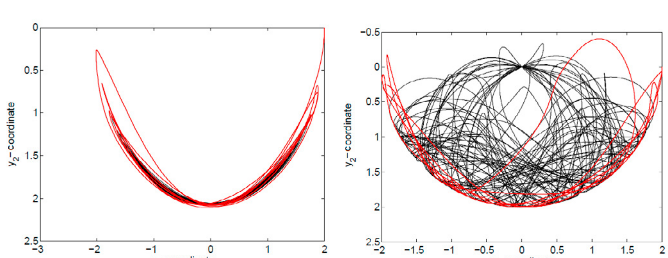
$$\begin{aligned}\dot{\varphi} &= \eta \\ \dot{\eta} &= g\cos\varphi.\end{aligned} \tag{5}$$

2.2 Double Pendulum

The second case study is the double pendulum, where the equations of motion in Cartesian coordinates are given by

$$\begin{array}{ll}\dot{x}_1 = v_{x_1} & \dot{v}_{x_1} = -F_1 x_1 - F_2(x_1 - x_2) \\ \dot{y}_1 = v_{y_1} & \dot{v}_{y_1} = \mathbf{g} - F_1 y_1 - F_2(y_1 - y_2) \\ \dot{x}_2 = v_{x_2} & \dot{v}_{x_2} = -F_2(x_2 - x_1) \\ \dot{y}_2 = v_{y_2} & \dot{v}_{y_2} = \mathbf{g} - F_2(y_2 - y_1) \\ x_1^2 + y_1^2 = 1 & (x_1 - x_2)^2 + (y_1 - y_2)^2 = 1,\end{array} \tag{6}$$

where F_1 and F_2 are forces and $\mathbf{g}$ is the gravitational acceleration. The constraint equations are given by $x_1^2 + y_1^2 - 1 = 0$ and $(x_1 - x_2)^2 + (y_1 - y_2)^2 - 1 = 0$. Deriving the constraint equations two times with respect to the time t shows that the system has differential index three in analogy to section 2.1.
The method of differentiation and substitution of the constraint equations again causes a grave numerical drift-off. Figure 3 shows that the Baumgarte method provides different results for different values of α and β. The



(a) $\alpha = \beta = 10$ (b) $\alpha = \beta = 100$

Fig. 3. Results of the Baumgarte-Method for different values of α and β, calculated with ode45 (MATLAB)

Pantelides algorithm results in sixteen equation systems of 22 equations each, which represents a very complex description of the given problem. With the orthogonal projection method, an unbounded increase of speed can again be observed, while the symmetric projection method delivers quite reasonable results but takes a long time to simulate due to iteration in each step.
Similar to the pendulum, a global transformation can also be found for the double pendulum, see (7).

$$\begin{array}{ll}x_1 = \cos\varphi_1 & x_2 = \cos\varphi_1 + \cos\varphi_2 \\ y_1 = \sin\varphi_1 & y_2 = \sin\varphi_1 + \sin\varphi_2 \\ v_{x_1} = -\eta_1\sin\varphi_1 & v_{x_2} = -\eta_1\sin\varphi_1 - \eta_2\sin\varphi_2 \\ v_{y_1} = \eta_1\cos\varphi_1 & v_{y_2} = \eta_1\cos\varphi_1 + \eta_2\cos\varphi_2\end{array} \tag{7}$$

2.3 Results

In Table 1 the maximal error (deviations to the circular paths) and the computing time of all methods until 100 seconds simulation time are shown. It becomes clear that

	pendulum		double pendulum		
method	max err e	t(s)	max err e_1	max err e_2	t(s)
DaS C	24.731	1.1	1.008	9.495	4.9
$\alpha \neq \beta$	$1.909 \cdot 10^{-5}$	1.7	$1.612 \cdot 10^{-4}$	$1.605 \cdot 10^{-4}$	11.7
$\alpha = \beta$	$2.464 \cdot 10^{-4}$	0.7	$2.242 \cdot 10^{-5}$	0.013	3.8
P	$3.345 \cdot 10^{-4}$	8.5	$1.550 \cdot 10^{-4}$	$2.677 \cdot 10^{-4}$	21.0
OPM	$5.551 \cdot 10^{-16}$	76.2	$4.441 \cdot 10^{-16}$	$6.662 \cdot 10^{-16}$	166.6
SPM	$2.701 \cdot 10^{-8}$	71.7	$2.538 \cdot 10^{-7}$	$4.764 \cdot 10^{-7}$	129.6
SST	$2.221 \cdot 10^{-16}$	0.6	$2.221 \cdot 10^{-16}$	$2.221 \cdot 10^{-16}$	3.7

Table 1. Maximal error and simulation time

differentiation and substitution of the constraint and the orthogonal projection method (in spite of staying close to the circular path, compare Fig. 2) lead to unreasonable results while state space transformation proves to be the most suitable for the given problem.

3. CONCLUSION

Differentiating and substituting the constraint equations turns out to be not suitable, therefore other approaches for solving the case studies are necessary. The orthogonal projection method has problems with the correct positions due to the increasing speed and hence does not provide reasonable results. The Baumgarte-Method shows good results if the parameters are chosen well. The Pantelides algorithm has the disadvantage of a complex implementation and the used ode-solver has problems to solve the equation systems. In contrast to the orthogonal projection method the symmetric projection method has bounded speed and leads to reasonable results. The last method is the state space transformation which can be done globally for both case studies. All in all, it can be stated that the most suitable method depends highly on the given problem.

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