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Hardy space infinite elements for time harmonic wave equations with phase and group velocities of different signs

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Abstract We consider time harmonic wave equations in cylindrical wave-guides with physical solutions for which the signs of group and phase velocities differ. The perfectly matched layer methods select modes with positive phase velocity, and hence they yield stable, but unphysical solutions for such problems. We derive an infinite element method for a physically correct discretization of such wave-guide problems which is based on a Laplace transform in propagation direction. In the Laplace domain the space of transformed solutions can be separated into a sum of a space of incoming and a space of outgoing functions where both function spaces are Hardy spaces of a curved domain. The Hardy space is constructed such that it contains a simple and convenient Riesz basis with small condition numbers. In this paper the new method is only discussed for a one-dimensional fourth order model problem. Exponential

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convergence is shown. The method does not use a modal separation and works on an interval of frequencies. Numerical experiments confirm exponential convergence.

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1 Introduction

This paper was initiated by the study of infinite elastic wave-guides (see e.g. [1,7]). For finite element simulations the computational domain has to be truncated, and some type of transparent boundary condition has to be imposed. Frequently the perfectly matched layer (PML) method is used, which selects modes of positive phase velocity ω/k, where ω is the frequency and k the wave-number in the direction of radiation. However, a plot of the dispersion curves $k(\omega)$ (see Fig. 1) shows that there are modes with negative group velocity $\partial\omega/\partial k$ and positive phase velocity ω/k for certain frequencies ω. Typically physical modes are characterized by a positive group velocity. Therefore, standard PML fails in such cases as it picks modes with positive phase velocity. For the same reason standard Hardy space infinite elements (see [12,13]) fail.

For the particular problem of a two dimensional elastic wave-guide methods based on precomputation and separation of the problematic mode have been suggested [4,19]. However, these methods have some disadvantages. In particular, the precomputation step has to be repeated for each frequency ω, and these methods cannot be used in a straightforward manner for the solution of eigenvalue problems where ω is unknown.

The purpose of this paper is to derive a method which is independent of the frequency in an interval of frequencies. First ideas in this direction were made in [8,9]. The method takes advantage of the greater flexibility of the Hardy space formulation compared to PML and uses a Hardy space of a curved domain to accommodate for the special structure of the admissible wave-numbers. Our method is applicable to time harmonic wave equations in cylindrical wave-guides for which the pattern of admissible and inadmissible wave-numbers has the same structure as in this paper on a certain frequency interval. In particular, it can be used for the computation of

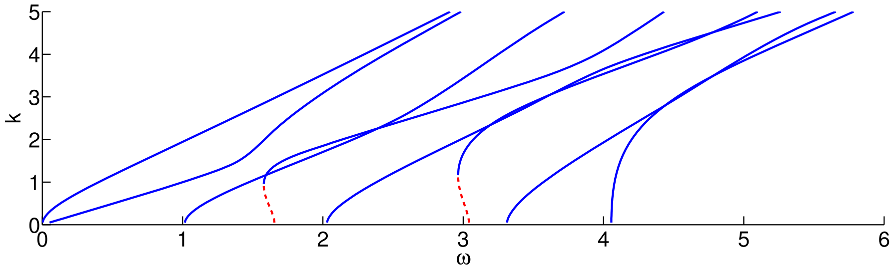


Fig. 1 The first four symmetric and first four antisymmetric dispersion curves of a two-dimensional semi-infinite elastic wave-guide with width $H = 1$, density $\rho = 1$, Young's module $E = 1$, and Poisson ratio $\nu = 0.2$. Modes corresponding to the *blue solid* part of the dispersion curves have positive phase velocity ω/k and positive group velocity $\partial\omega/\partial k$. Modes corresponding to the *red dashed* part of the dispersion curve have positive phase and negative group velocity (color figure online)

eigenvalues and resonances. We have successfully applied it to the elastic wave-guide problem mentioned above using the high-order finite element code Netgen/NGSolve [17,18], see the software module ngs-waves [15]. Details will be reported elsewhere. Here we will explain our method only for probably the simplest time harmonic wave equation which exhibits the phenomenon of group and phase velocities of different signs, a fourth order ordinary differential equation.

The remainder of this paper is organized as follows: in Sect. 2 we introduce our model problem, motivate a modal radiation condition and formulate an equivalent pole condition in the Laplace domain using a Hardy space of a curved domain. In the subsequent section we derive a variational formulation of our model problem in the curved Hardy space (Theorem 3.4). In Sect. 4 we specify our choices of the basis and the precise curve characterizing the curved Hardy space and show that our basis is a Riesz basis (Theorem 4.8). With these preparations we can prove exponential convergence of the proposed method in Sect. 5. Numerical experiments in Sect. 6 confirm the convergence rates, show small condition numbers and illustrate the dependence of our methods on the parameters, before we end with some conclusions. Our paper contains two appendices: Results on standard and non-standard Hardy spaces are collected in A.1 and results on Toeplitz operators in A.2.

2 Model problem

The main difficulty in a non-modal numerical simulation of infinite elastic wave-guides is the existence of wave-guide modes for which the signs of group and phase velocity differ. In order to mimic this essential difficulty in a simple, one dimensional setting, we are looking for solutions $u \in H^4_{\mathrm{loc}}(\mathbb{R}_+)$ to

$$\left(1 + \left(-\partial_x^2 - \zeta^2\right)^2\right) u(x) = \omega^2 u(x), \quad x > 0, \tag{2.1a}$$

$$\begin{pmatrix} \mathscr{B}_1 u \\ \mathscr{B}_2 u \end{pmatrix} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}, \quad x = 0, \tag{2.1b}$$

$$u \text{ satisfies a radiation condition.} \tag{2.1c}$$

$\omega > 0$ is the angular frequency, $\tilde{w}_1, \tilde{w}_2 \in \mathbb{C}$ given boundary data, and $\mathscr{B}_j$ for $j = 1, 2$ are linearly independent trace operators. We will specify the abstract radiation condition (2.1c) later in Definition 2.1 and equivalently in Definition 2.4. The model parameter $\zeta > 0$ has been introduced in order to be able to generalize the following ideas to other problems, see Sect. 2.4.

2.1 Dispersion relation

A function of the form $x \mapsto e^{ik(\omega)x}$ is a solution to (2.1a), if and only if $k(\omega)$ solves the dispersion relation

$$1 + \left(k^2 - \zeta^2\right)^2 = \omega^2. \tag{2.2}$$

We will always assume in the following that

$$\omega \notin \left\{1, \sqrt{\zeta^4 + 1}\right\}. \tag{2.3}$$

Thus, (2.1a) has the four fundamental solutions $x \mapsto e^{ik(\omega)x}$ with

$$k(\omega) = \pm\sqrt{\zeta^2 \pm \sqrt{\omega^2 - 1}}. \tag{2.4}$$

Since both signs occur, the choice of the branch cut in the definition of the complex square root function does not matter. The numbers $k(\omega) \in \mathbb{C}$ are called wave-numbers, real-valued wave-numbers are plotted in Fig. 2a for $\omega \in (0, 3)$. We can distinguish three cases:

1. $\omega \in (0, 1)$: The discriminant of the inner square root is negative and therefore all four wave-numbers have non-vanishing real and imaginary part.
2. $\omega \in (1, \sqrt{\zeta^4 + 1})$: The inner square root is real with absolute value smaller than ζ^2. Hence, in this case there exist four real wave-numbers.
3. $\omega > \sqrt{\zeta^4 + 1}$: The absolute value of the inner square root is larger than ζ^2 and therefore there exist two real wave-numbers and two wave-numbers with vanishing real and non-vanishing imaginary part.

To obtain unique solvability of (2.1) it is necessary that the radiation condition selects two of the four modes $e^{ik(\omega)\bullet}$. If $k(\omega)$ has a non-zero imaginary part, the choice is easy: If $\Im(k(\omega)) > 0$ the mode $e^{ik(\omega)\bullet}$ is called evanescent and will be regarded as 'physical', whereas if $\Im(k(\omega)) < 0$, the mode is exponentially increasing and will be regarded as 'unphysical'. Solutions $e^{ik(\omega)\bullet}$ with $\Im(k(\omega)) = 0$ are called guided modes, and here the choice is less obvious. One way to construct a physical decision criterion is to resort to the limiting absorption principle and replace ω by

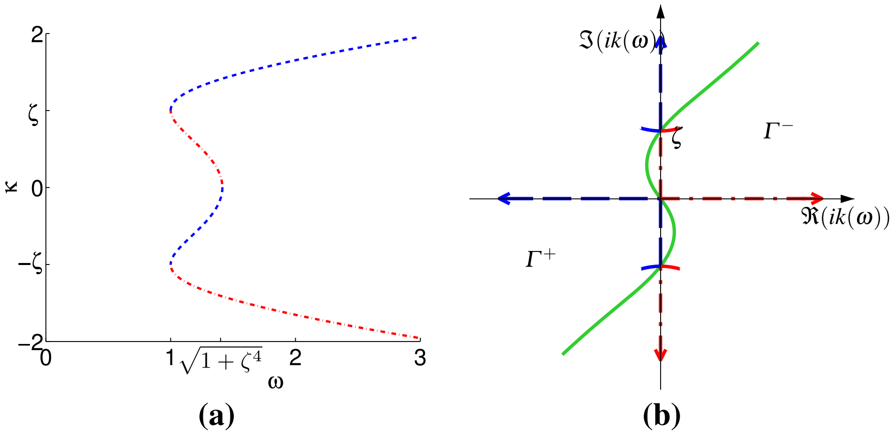


Fig. 2 **a** Dispersion curve of the model problem with $\zeta = 1$, *blue dashed* part has positive group velocity, *red dashed-dotted* part has negative group velocity; **b** $ik(\omega)$ for $\omega \in (0, 3)$. Outgoing wave-numbers (multiplied with i) are marked by *blue dashed lines*, and incoming wave-numbers by *red dashed-dotted lines*. The *green curve* Γ separates outgoing from incoming wave-numbers (color figure online)

$\omega + i\epsilon$ with a (small) absorption parameter $\epsilon > 0$. Under assumption (2.3) $k(\omega)$ is analytic at ω, $\partial_\omega k(\omega) \neq 0$, and

$$k(\omega + i\epsilon) \approx k(\omega) + i\partial_\omega k(\omega)\epsilon.$$

Therefore, $e^{ik(\omega+i\epsilon)\bullet}$ is exponentially decaying if $\partial_\omega k(\omega) > 0$ and exponentially increasing if $\partial_\omega k(\omega) < 0$. $\frac{\partial k}{\partial \omega}(\omega_0) = \frac{1}{(\partial\omega/\partial k)(k(\omega_0))}$ is the reciprocal of the group velocity $\partial\omega/\partial k$, and in particular both have the same sign. Hence, the limiting absorption principle selects guided modes if and only if they have positive group velocity $\partial\omega/\partial k$ (rather than positive phase velocity ω/k).

We summarize our discussion in the following definition:

Definition 2.1 (*modal radiation condition*) Suppose (2.3) holds true. We call the solution $x \mapsto e^{ik(\omega)x}$ to (2.1a)

$$\text{Outgoing mode}: \Leftrightarrow \begin{cases} \Im(k(\omega)) > 0, & \text{if } k(\omega) \notin \mathbb{R} \\ \partial_\omega k(\omega) > 0, & \text{if } k(\omega) \in \mathbb{R} \end{cases},$$

$$\text{Incoming mode}: \Leftrightarrow \begin{cases} \Im(k(\omega)) < 0, & \text{if } k(\omega) \notin \mathbb{R} \\ \partial_\omega k(\omega) < 0, & \text{if } k(\omega) \in \mathbb{R} \end{cases}.$$

A wave-number $k(\omega)$ is called outgoing, if $x \mapsto e^{ik(\omega)x}$ is an outgoing mode. A function is called outgoing if it is a linear combination of outgoing modes.

For the three cases from above we get

1. $\omega \in (0, 1)$: All wave-numbers have non-vanishing real and imaginary parts, and an outgoing solution is a superposition of two evanescent functions.
2. $\omega \in (1, \sqrt{\zeta^4 + 1})$: All wave-numbers are real. The two positive wave-numbers $k(\omega) = \sqrt{\zeta^2 \pm \sqrt{\omega^2 - 1}}$ satisfy

$$\partial_\omega k(\omega) = \pm \frac{\omega}{2\sqrt{\zeta^2 \pm \sqrt{\omega^2 - 1}}\sqrt{\omega^2 - 1}}.$$

   One positive wave-number has a positive and the other a negative group velocity. An outgoing solution may contain both a guided mode with $k > \zeta > 0$ and a guided mode with $k \in (-\zeta, 0)$.
3. $\omega > \sqrt{\zeta^4 + 1}$: Two wave-numbers are real and two wave-numbers are purely imaginary. An outgoing solution is a superposition of an evanescent mode and a guided mode with $k(\omega) > \sqrt{2}\zeta$.

The location of $ik(\omega)$ in $\mathbb{C}$ for $\omega \in (0, 3)$ is sketched in Fig. 2b (the green curve Γ and the labels $\Gamma^{\pm}$ will become clear in Assumption 2.3). For $\omega = 1$ there exist only two wave-numbers $k(1) = \pm\zeta$ and for $\omega = \sqrt{\zeta^4 + 1}$ there exist three wave-numbers at $\sqrt{2}\zeta$, $-\sqrt{2}\zeta$ and 0.

2.2 Complex scaling radiation condition

The modal radiation condition Definition 2.1 uses directly the wave-numbers $k(\omega)$ and is therefore frequency dependent. This is a severe drawback for resonance problems where the frequency is the sought resonance. Moreover, typically the dispersion relation is more complicated than (2.2). Hence, the computation of the wave-numbers may not be as easy as for this model problem.

One famous remedy is complex scaling: In the simplest case the real variable x is replaced by a complex scaled variable $\hat{x} := \sigma x$ with $\sigma \in \mathbb{C}$. This leads to modes of the form $x \mapsto e^{ik(\omega)\sigma x}$ and therefore for $|\sigma| = 1$ to a rotation of $ik(\omega)$ in the complex plane by $\arg(\sigma)$. The goal is to find σ such that for all outgoing wave-numbers k the real part of $ik(\omega)\sigma$ is negative, since then the complex scaled modes are exponentially decaying.

For our model problem this is possible for $\omega \in (0, 1)$ with $\sigma = 1$ and for $\omega > \sqrt{\zeta^4 + 1}$ with e.g. $\sigma = 1 + 1i$. For the second case $\omega \in (1, \sqrt{\zeta^4 + 1})$ such a rotation σ does not exist since outgoing wave-numbers are located on different sides of any straight line through the origin. Hence, a standard complex scaling method (frequently called perfectly matched layer method) does not work even for one fixed frequency $\omega \in (1, \sqrt{\zeta^4 + 1})$.

Remark 2.2 In [2], a modified PML is proposed for convected Helmholtz equations. It is more flexible than a standard linear PML since it allows to choose complex wave-numbers with negative and positive real part at the same time. Nevertheless, for this modified PML wanted (physical) wave-numbers still have to be separated by a straight line from the unwanted (unphysical) wave-numbers. For some intervals of frequencies this is neither the case for our model problem (see Fig. 2b) nor for an elastic wave-guide problem with dispersion curves given in Fig. 1.

2.3 Pole condition

Since our aim is a numerical simulation of (2.1) without using modes, we reformulate the modal radiation condition using the Laplace transform $\mathscr{L}u(s) := \int_{\mathbb{R}^+} u(x)e^{-sx}dx$ for the modes $x \mapsto e^{ik(\omega)x}$. The Laplace transform is originally defined only for $\Re(s)$ sufficiently large but the Laplace transformed modes can be holomorphically extended to $\mathbb{C}\backslash\{ik(\omega)\}$ as

$$\mathscr{L}\left\{e^{ik(\omega)\bullet}\right\}(s) = \frac{1}{s - ik(\omega)}, \qquad s \in \mathbb{C}\backslash\{ik(\omega)\}.$$

They are meromorphic functions with poles at $ik(\omega)$, see Fig. 2b for the location of the poles.

Assumption 2.3 Let Γ be an oriented curve fulfilling Assumption A.5 with domain Γ^- on the right and Γ^+ on the left as in Fig. 2b such that for all $\omega \in \mathbb{R}_{>0}\backslash\{1, \sqrt{\zeta^4 + 1}\}$ the poles $ik(\omega)$ of the Laplace transformed outgoing modes belong to Γ^+.

Due to Assumption A.5 Γ is point symmetric. Hence, Γ separates the poles of the Laplace transformed incoming modes from those of the Laplace transformed outgoing modes. As seen in the last subsection and in Fig. 2b, the use of a curved Γ is essential for this.

For our model problem, we need to assume that Γ is constructed such that

$$\left\{x \pm i\sqrt{x^2+\zeta^2}, x \in \left[-\sqrt{\tfrac{1}{2}(\sqrt{\zeta^4+1}-\zeta^2)}, 0\right)\right\} \subset \Gamma^+, \quad \text{1st case,} \quad (2.5a)$$

$$-i(0,\zeta) \cup i(\zeta, \sqrt{2}\zeta) \subset \Gamma^+, \quad \text{2nd case,} \quad (2.5b)$$

$$i(\sqrt{2}\zeta, \infty) \cup \mathbb{R}_{<0} \subset \Gamma^+, \quad \text{3rd case.} \quad (2.5c)$$

The existence of such a Γ for our model problem is obvious. In order to obtain a functional setting for the radiation condition we use Hardy spaces. Roughly speaking the Hardy space $H^-(\Gamma)$ on an unbounded curve $\Gamma \subset \mathbb{C}$ (see Fig. 2b) is the set of $L^2(\Gamma)$-boundary functions of functions, which are holomorphic below Γ. We refer to the Appendix A.1 for the exact definition and properties of such Hardy spaces.

Definition 2.4 (*Pole condition*) Let Γ be an oriented curve fulfilling Assumption A.5 with domain Γ^- on the right and Γ^+ on the left as in Fig. 2b such that (2.5) holds true. Moreover, let $H^-(\Gamma)$ be the Hardy space of Γ as defined in Definition A.7 and $\mathscr{L}u(s) := \int_{\mathbb{R}^+} u(x)e^{-sx}dx$ be the Laplace transform.

A function u satisfies the pole condition w.r.t. Γ if the Laplace transform $\mathscr{L}u$ of u exists for sufficiently large $\Re(s)$ and has a holomorphic extension to Γ^- such that

$$\mathscr{L}u \in H^-(\Gamma).$$

Since Γ^+ contains the outgoing poles, $(\mathscr{L}e^{ik\bullet})(s) = \frac{1}{s-ik}$ and due to Lemma A.8, the modal radiation condition 2.1 and the pole condition 2.4 are equivalent for solutions to (2.1a). However, as opposed to the modal radiation condition, the pole condition is independent of ω.

2.4 Pole condition for other dispersion relations

In this paper, the separating curves Γ constructed in Sect. 4 are always similar to the one in Fig. 2b: they start from ∞ in the third quadrant of the complex plane, cross the imaginary axis at $-i\zeta$, at the origin and at $i\zeta$ and go to ∞ in the first quadrant. The actual shape will depend on two complex parameters.

For the model problem this kind of curves is sufficient. In Fig. 1 the dispersion curves for an isotropic two-dimensional elastic wave-guide are given. In this example there exist two intervals of frequencies where exactly one mode has group and phase velocities with different signs. This is the same situation as in the second case of our model problem. Hence, the method presented here can be applied to the elastic wave-guide problem of Fig. 1 for a fixed interval of frequencies. The value of the parameter ζ has to be the wave-number with vanishing group velocity in this case, and the requirements on the curve Γ in (2.5) must be formulated accordingly. Numerical studies for two dimensional elastic wave-guides are reported in [10].

In principle, it seems possible to extend the method presented in the next sections to problems with finitely many modes with different signs of group and phase velocity. For frequencies ω, where an outgoing wave-number coincides with an incoming wave-number, e.g. for $\omega = 1$ and $\omega = \sqrt{1+\zeta^4}$ in our model problem, a separating curve Γ does not exist. In all our numerical experiments with isotropic two and three dimensional wave-guides this has happened only for a discrete number of frequencies.

3 Variational formulation

For numerical purposes it is convenient to reformulate (2.1) into a system of boundary value problems of order two. To this end we define $v := \frac{1}{\sqrt{\omega^2-1}}\left(-\partial_x^2 - \zeta^2\right) u$ and note that (2.1) is equivalent to finding $u, v \in H^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\begin{pmatrix} -\partial_x^2-\zeta^2 & -\sqrt{\omega^2-1} \\ -\sqrt{\omega^2-1} & -\partial_x^2-\zeta^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad x > 0, \tag{3.1a}$$

$$\mathscr{B}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.1b}$$

$$\mathscr{L}u, \mathscr{L}v \in H^-(\Gamma). \tag{3.1c}$$

The trace operator $\mathscr{B}$ is of no importance for the method, but we need to assume that it is chosen such that Problem (3.1) is well-posed. For simplicity we choose in the following Neumann boundary conditions $\mathscr{B}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$ and write Problem (3.1) in variational form

$$\int_0^\infty \left(\begin{pmatrix} f' \\ g' \end{pmatrix}^\top \begin{pmatrix} u' \\ v' \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix}^\top \begin{pmatrix} \zeta^2 & \sqrt{\omega^2-1} \\ \sqrt{\omega^2-1} & \zeta^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right) dx = - \begin{pmatrix} f(0) \\ g(0) \end{pmatrix}^\top \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{3.2}$$

for $(u, v)^\top \in \left(H^1_{\mathrm{loc}}(\mathbb{R}_+)\right)^2$ with $(\mathscr{L}u, \mathscr{L}v)^\top \in \left(H^-(\Gamma)\right)^2$ and test functions $(f, g)^\top \in \left(H^1_{\mathrm{comp}}(\mathbb{R}_+)\right)^2$. $H^1_{\mathrm{loc}}(\mathbb{R}_+)$ denotes the space of functions which belong to $H^1(I)$ for each compact subset $I \subset \mathbb{R}_+$, and $H^1_{\mathrm{comp}}(\mathbb{R}_+)$ denotes the set of $H^1(\mathbb{R}_+)$-functions with compact support in $[0, \infty)$. Due to the equivalence to (2.1a), (2.1b), (2.1c) the unique solution u_{ext}, v_{ext} to Problem (3.1) or equivalently Problem (3.2) is given by

$$\begin{pmatrix} u_{ext}(x) \\ v_{ext}(x) \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{k_+^2-\zeta^2}{\sqrt{\omega^2-1}} & \frac{k_-^2-\zeta^2}{\sqrt{\omega^2-1}} \end{pmatrix} \begin{pmatrix} C_1 e^{ik_+x} \\ C_2 e^{ik_-x} \end{pmatrix},$$

$$\begin{pmatrix} ik_+C_1 \\ ik_-C_2 \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{k_+^2-\zeta^2}{\sqrt{\omega^2-1}} & \frac{k_-^2-\zeta^2}{\sqrt{\omega^2-1}} \end{pmatrix}^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{3.3}$$

with the outgoing wave-numbers $k_+(\omega)$ and $k_-(\omega)$ according to Definition 2.1.

Remark 3.1 We could simplify Problem (3.1) by making a change of basis $\tilde{u} := u + v$ and $\tilde{v} := u - v$ since then (3.1a) becomes diagonal

$$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}-\partial_x^2-\zeta^2 & -\sqrt{\omega^2-1}\\ -\sqrt{\omega^2-1} & -\partial_x^2-\zeta^2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}-\partial_x^2-k_-(\omega)^2 & 0\\ 0 & -\partial_x^2-k_+(\omega)^2\end{pmatrix}$$

Here, we have used that $k_\pm(\omega)^2 = \zeta^2 \pm \sqrt{\omega^2 - 1}$ for the two outgoing wave-numbers $k_\pm(\omega)$ defined in Definition 2.1.

Hence, instead of a system of differential equations only two separated equations have to be solved. Moreover, in this case the radiation condition (3.1c) could be replaced by two different radiation conditions $\mathscr{L}\tilde{u} \in H^-((1-i)\mathbb{R})$ and $\mathscr{L}\tilde{v} \in H^+((1-i)\mathbb{R})$ with comparatively simple Hardy spaces of a complex half plane (see Definition A.3). Such problems can be solved with the standard Hardy space method as well as with a standard complex scaling method.

Nevertheless, in order to keep the difficulties arising e.g. in elastic wave-guide problems, we will use the more complicated formulation Problem (3.1) and in particular the radiation condition (3.1c) with a Hardy space of a curve $H^-(\Gamma)$. Only in the proofs of Theorem 3.4 and Lemma 5.1, which are only valid for this particular problem, we will make use of the decomposition into two separated problems.

We aim to transform (3.2) into a variational equation in $H^-(\Gamma)$. In [12, Sect. 2] this is done for the Helmholtz equation and implicitly for a Hardy space of a straight line. More precisely, the Hardy space $H^-(\mathbb{R})$ is mapped into the Hardy space $H^+(S^1)$ on the unit disk in [12, Sect. 2] using a Möbius transform, and the variational equation is formulated in $H^+(S^1)$. Since for the model problem a curved Γ is needed and in order to be self-consistent, we deduce here the variational formulation almost independent of [12, Sect. 2].

Let us assume that Γ fulfills (2.5) and Assumption A.5 and let $s_0 \in \Gamma^+$ be an arbitrary complex parameter. We define the Hilbert space $\mathscr{X} := \mathbb{C} \oplus H^-(\Gamma)$ with scalar product

$$\left\langle \begin{pmatrix} f_0 \\ F \end{pmatrix}, \begin{pmatrix} g_0 \\ G \end{pmatrix} \right\rangle_{\mathscr{X}} := f_0\overline{g_0} + \langle F, G\rangle_{L^2(\Gamma)}, \qquad \begin{pmatrix} f_0 \\ F \end{pmatrix}, \begin{pmatrix} g_0 \\ G \end{pmatrix} \in \mathscr{X},$$

and introduce two linear operators $\mathscr{S}_m, \mathscr{S}_s : \mathscr{X} \to H^-(\Gamma)$ by

$$\mathscr{S}_m \begin{pmatrix} f_0 \\ F \end{pmatrix}(s) := \frac{f_0 + F(s)}{s - s_0}, \; \mathscr{S}_s \begin{pmatrix} f_0 \\ F \end{pmatrix}(s) := \frac{f_0 s_0 + sF(s)}{s - s_0}, \quad \begin{pmatrix} f_0 \\ F \end{pmatrix} \in \mathscr{X}, s \in \Gamma. \tag{3.4}$$

These operators are bounded by

$$C_m := \sqrt{2}\max\left\{\|(\bullet - s_0)^{-1}\|_{L^2(\Gamma)}, \|(\bullet - s_0)^{-1}\|_{L^\infty(\Gamma)}\right\} \quad \text{and}$$
$$C_s := \sqrt{2}\max\left\{\|s_0(\bullet - s_0)^{-1}\|_{L^2(\Gamma)}, \|\bullet(\bullet - s_0)^{-1}\|_{L^\infty(\Gamma)}\right\},$$

respectively. Moreover, they are injective: e.g., if $\mathscr{S}_m \left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right) = 0$, then $F(s) = -f_0$ for all $s \in \Gamma$, and therefore $F \equiv 0$ and $f_0 = 0$ since F belongs to $L^2(\Gamma)$ and Γ is unbounded. Furthermore, for $u(x) = \sum_{\ell=1}^{n} c_\ell e^{ik_\ell x}$ with $ik_\ell \in \Gamma^+$ and $c_\ell \in \mathbb{C}$ for $\ell = 1, \ldots, n$, $\mathscr{L}u$ belongs to the range $\mathscr{S}_m(\mathscr{X})$ of $\mathscr{S}_m$ with

$$\mathscr{L}u = \mathscr{S}_m \left(\begin{smallmatrix} u_0 \\ U \end{smallmatrix}\right), \qquad u_0 = \sum_{\ell=1}^{n} c_\ell, \quad U(s) = \sum_{\ell=1}^{n} c_\ell \frac{ik_\ell - s_0}{s - ik_\ell}, \quad s \in \Gamma.$$

There are two reasons for introducing these operators:

1. By a limit theorem for the Laplace transform applied to $\mathscr{L}f = \mathscr{S}_m \left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right) \in \mathbb{C} \oplus H^-(\Gamma)$ we have

$$\lim_{x \searrow 0} f(x) = \lim_{s \to \infty} s\mathscr{L}f(s) = f_0. \tag{3.5}$$

   Thus, the scalar component of $\mathbb{C} \oplus H^-(\Gamma)$ represents the Dirichlet value $f(0)$ if f is continuous at the origin. This is essential for coupling the Hardy space variational formulation of the exterior problem to an interior problem.
2. For $f \in C^1(\mathbb{R}_+)$ with $\mathscr{L}f = \mathscr{S}_m \left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right) \in \mathscr{S}_m(\mathscr{X})$ we have

$$\mathscr{L}f' = \mathscr{S}_s \left(\begin{smallmatrix} f_0 \\ F \end{smallmatrix}\right). \tag{3.6}$$

   For this reason we call $\mathscr{S}_m$ the mass and $\mathscr{S}_s$ the stiffness operator.

Remark 3.2 In the subspace $\mathscr{S}_m(\mathscr{X}) \subset H^-(\Gamma)$ we can define a differential operator

$$\hat{\partial}_x := \mathscr{S}_s \mathscr{S}_m^{-1} \quad \text{such that} \quad \hat{\partial}_x \mathscr{L}f = \mathscr{L}\partial_x f$$

for all $f \in C^1(\mathbb{R}_+)$ with $\mathscr{L}f \in \mathscr{S}_m(\mathscr{X})$. For $\Gamma = \mathbb{R}$ it was shown in [12, Lemma 4.1] that there exists a norm isomorphism between $\mathscr{X}$ and $H^1(\mathbb{R}_+)$.

In the following we will use the integral identity

$$\int_0^\infty u(x)\, f(x)\, dx \;= q\,(\mathscr{L}u, \mathscr{L}f) \tag{3.7}$$

with the bilinear form

$$q\,(U, F) := \frac{-i}{2\pi} \int_\Gamma U(s)F(-s)\, ds, \quad U, F \in H^-(\Gamma), \tag{3.8}$$

which follows by contour deformations (see [12, Lemma A.1]) for the solutions u_{ext} and v_{ext} to Problem (3.1) as well as for their derivatives with suitable test functions f given later on in Lemma 3.3. In particular we have

$$\int_0^\infty u(x)\, f(x)\, dx \;= q_M \left(\mathscr{S}_m^{-1}\mathscr{L}u, \mathscr{S}_m^{-1}\mathscr{L}f\right), \tag{3.9a}$$

$$\int_0^\infty u'(x)\, f'(x)\, dx \ = q_S\left(\mathscr{S}_m^{-1}\mathscr{L}u, \mathscr{S}_m^{-1}\mathscr{L}f\right), \tag{3.9b}$$

with

$$q_M\left(\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right) := q\left(\mathscr{S}_m\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \mathscr{S}_m\left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right), \quad \left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right) \in \mathscr{X}, \tag{3.10a}$$

$$q_S\left(\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right) := q\left(\mathscr{S}_s\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \mathscr{S}_s\left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right), \quad \left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right) \in \mathscr{X}. \tag{3.10b}$$

Thus, Eq. (3.2) leads to the bilinear form

$$\begin{aligned} Q\left(\begin{pmatrix}\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right)\\ \left(\begin{smallmatrix}v_0\\V\end{smallmatrix}\right)\end{pmatrix}, \begin{pmatrix}\left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\\ \left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right)\end{pmatrix}\right) := {} & q_S\left(\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right) + q_S\left(\left(\begin{smallmatrix}v_0\\V\end{smallmatrix}\right), \left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right)\right) \\ & - \zeta^2\left(q_M\left(\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right) + q_M\left(\left(\begin{smallmatrix}v_0\\V\end{smallmatrix}\right), \left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right)\right)\right) \\ & - \sqrt{\omega^2 - 1}\left(q_M\left(\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right)\right)\right. \\ & \left. + q_M\left(\left(\begin{smallmatrix}v_0\\V\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right)\right)\right) \end{aligned} \tag{3.11}$$

for $\left(\begin{smallmatrix}u_0\\U\end{smallmatrix}\right), \left(\begin{smallmatrix}v_0\\V\end{smallmatrix}\right), \left(\begin{smallmatrix}f_0\\F\end{smallmatrix}\right), \left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right) \in \mathscr{X}$.

In the following lemma we give test functions for which (3.7) holds true. In particular, the integrand on the left hand side of (3.7) will be exponentially decaying, since u_{ext} and v_{ext} as well as their derivatives are bounded due to (3.3), and the test functions are exponentially decaying. Therefore, the integral on the left hand side of (3.7) is well defined.

Lemma 3.3 *Let $\Lambda^+ \subset \left(\Gamma^+ \backslash \{s_0\}\right) \cap \{s \in \mathbb{C} \colon \Re s < 0\}$ be a set with some cluster point. Moreover, let $p_0 \in \Lambda^+$ and define*

$$f_\lambda(x) \ := (e^{\lambda x} - e^{p_0 x})/(\lambda - p_0), \qquad \tilde{f}_{p_0}(x) := e^{p_0 x} \tag{3.12}$$

for $x \geq 0$ and $\lambda \in \Lambda^+ \backslash \{p_0\}$. Then

$$\begin{aligned} \mathscr{L} f_\lambda \ &= \mathscr{S}_m\left(\begin{smallmatrix}0\\F_\lambda\end{smallmatrix}\right) \quad \text{with} \quad F_\lambda(s) := \frac{s - s_0}{s - p_0}\frac{1}{s - \lambda}, \\ \mathscr{L}\tilde{f}_{p_0} \ &= \mathscr{S}_m\begin{pmatrix}1\\ \tilde{F}_{p_0}\end{pmatrix} \quad \text{with} \quad \tilde{F}_{p_0}(s) := \frac{p_0 - s_0}{s - p_0} \end{aligned} \tag{3.13}$$

for $s \in \Gamma$ and $\lambda \in \Lambda^+ \backslash \{p_0\}$.
Moreover, span$\{\mathscr{S}_m^{-1}\mathscr{L}\tilde{f}_{p_0}\} \cup \{\mathscr{S}_m^{-1}\mathscr{L}f_\lambda : \lambda \in \Lambda^+\backslash\{p_0\}\}$ *is dense in $\mathscr{X}$.*

Proof Straightforward computations yield the identities in (3.13). To show density, we will demonstrate that any given $\left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right) \in \mathscr{X}$ can be approximated arbitrarily accurately. Note that $\left(\begin{smallmatrix}g_0\\G\end{smallmatrix}\right) - g_0\mathscr{S}_m^{-1}\mathscr{L}\tilde{f}_{p_0} = \begin{pmatrix}0\\ G - g_0\tilde{F}_{p_0}\end{pmatrix}$. Since span $\{(\bullet - \lambda)^{-1} : \lambda \in \Lambda^+\backslash\{p_0\}\}$ is dense in $H^-(\Gamma)$ by Lemma A.10, $\frac{s-p_0}{s-s_0}(G(s) - g_0\tilde{F}_{p_0}(s))$ can be approximated in $H^-(\Gamma)$ to arbitrary accuracy $\epsilon > 0$ by linear combinations of the form

$\sum_{j=1}^n \frac{c_j}{s-\lambda_j}$, $c_j \in \mathbb{C}$, $\lambda_j \in \Lambda^+\backslash\{p_0\}$. Thus

$$\begin{aligned}
&\left\| G - g_0\tilde{F}_{p_0} - \sum_{j=1}^n c_j F_{\lambda_j} \right\|_{L^2(\Gamma)} \\
&= \left\| \frac{\bullet - s_0}{\bullet - p_0} \left(\frac{\bullet - p_0}{\bullet - s_0}(G - g_0\tilde{F}_{p_0}) - \sum_{j=1}^n \frac{c_j}{\bullet - \lambda_j} \right) \right\|_{L^2(\Gamma)} \\
&\leq \epsilon \left\| \frac{\bullet - s_0}{\bullet - p_0} \right\|_{L^\infty(\Gamma)},
\end{aligned}$$

and the claim is proven. □

Theorem 3.4 (Variational formulation) *Let $\omega \in \mathbb{R}^+\backslash\{1, \sqrt{\zeta^4+1}\}$, Γ fulfill (2.5) and Assumption A.5, let $s_0 \in \Gamma^+$ and $\mathscr{X} = \mathbb{C} \oplus H^-(\Gamma)$. Let the operators $\mathscr{S}_m, \mathscr{S}_s : \mathscr{X} \to H^-(\Gamma)$ and the bilinear form $Q : \mathscr{X}^2 \times \mathscr{X}^2 \to \mathbb{C}$ be as defined in (3.4) and (3.11).*

If $\binom{u}{v} \in H^1_{\text{loc}}(\mathbb{R}_+)$ is the unique solution of (3.2) (respectively Problem (3.1), then $\binom{u_0}{U} := \mathscr{S}_m^{-1}\mathscr{L}u$, $\binom{v_0}{V} := \mathscr{S}_m^{-1}\mathscr{L}v \in \mathscr{X}$ solve

$$Q\left(\begin{pmatrix}\binom{u_0}{U}\\ \binom{v_0}{V}\end{pmatrix}, \begin{pmatrix}\binom{f_0}{F}\\ \binom{g_0}{G}\end{pmatrix}\right) = -(f_0 w_1 + g_0 w_2) \quad \textit{for all } \binom{f_0}{F}, \binom{g_0}{G} \in \mathscr{X}. \tag{3.14}$$

Vice versa, (3.14) is uniquely solvable with solution $\binom{u_0}{U}, \binom{v_0}{V} \in \mathscr{X}$ and there exists a solution $\binom{u}{v} \in \left(H^2_{\text{loc}}(\mathbb{R}_+)\right)^2$ to Problem (3.1) such that $u(0) = u_0$ and $v(0) = v_0$.

Proof If in (3.2) test functions f, g of form (3.12) are chosen, (3.7) can be applied and (3.14) holds true for all $\binom{f_0}{F}, \binom{g_0}{G}$ of form (3.13), which form a dense subspace of $\mathscr{X}^2$ due to Lemma 3.3. Since the operators $\mathscr{S}_m, \mathscr{S}_s$ and the bilinear form q are bounded, so is Q and (3.14) holds true for all $\binom{f_0}{F}, \binom{g_0}{G} \in \mathscr{X}$.

We already know that Problem (3.1) is uniquely solvable for arbitrary boundary values w_1, w_2. Moreover, due to the first part of the theorem, the transformed solution solves the Hardy space variational formulation. Hence, there exists a solution to (3.14), so the second part of the theorem is proven if this solution is unique.

As in Remark 3.1 a change of basis $\binom{\tilde{u}_0}{\tilde{U}} := \binom{u_0+v_0}{U+V}$, $\binom{\tilde{v}_0}{\tilde{v}} := \binom{u_0-v_0}{U-V}$, $\binom{\tilde{f}_0}{\tilde{f}} := \binom{f_0+g_0}{F+G}$, $\binom{\tilde{g}_0}{\tilde{g}} := \binom{f_0-g_0}{F-G}$ leads to the two separate problems

$$q_S\left(\binom{\tilde{u}_0}{\tilde{u}}, \binom{\tilde{f}_0}{\tilde{f}}\right) - k_-(\omega)^2 q_M\left(\binom{\tilde{u}_0}{\tilde{u}}, \binom{\tilde{f}_0}{\tilde{f}}\right) = -(w_1+w_2)\tilde{f}_0, \tag{3.15a}$$

$$q_S\left(\binom{\tilde{v}_0}{\tilde{v}}, \binom{\tilde{g}_0}{\tilde{g}}\right) - k_+(\omega)^2 q_M\left(\binom{\tilde{v}_0}{\tilde{v}}, \binom{\tilde{g}_0}{\tilde{g}}\right) = -(w_1-w_2)\tilde{g}_0, \tag{3.15b}$$

in $\mathscr{X}$. Note that we are using the same Hardy space $H^-(\Gamma)$ for both problems. Equations (3.15a), (3.15b) are equivalent to operator equations of the form $\begin{pmatrix} a^\pm & b^\pm \\ c^\pm & D^\pm \end{pmatrix}\begin{pmatrix} u_0 \\ U \end{pmatrix} = \begin{pmatrix} -(w_1 \pm w_2) \\ 0 \end{pmatrix}$ with $D^\pm : H^-(\Gamma) \to H^-(\Gamma)$. We show in the following implicitly that there exist unique inverses $\left(D^\pm\right)^{-1} : D^\pm(H^-(\Gamma)) \to H^-(\Gamma)$. Assuming this, the proof is complete since then we can use the Schur complement to reduce (3.15a), (3.15b) to scalar equations for u_0. Solvability ensures uniqueness of these equations for u_0, and therefore $(u_0, U)^\top$ is unique.

In order to construct $D^\pm$ we use the Definitions (3.4) and (3.8) such that (3.15a), (3.15b) lead to

$$\int_\Gamma F(-s)\, \frac{\left(-s_0 s - k_\pm(\omega)^2\right) u_0 + \left(-s^2 - k_\pm(\omega)^2\right) U(s)}{s^2 - s_0^2}\, ds + \int_\Gamma f_0\, \frac{\left(s_0^2 - k_\pm(\omega)^2\right) u_0 + \left(s_0 s - k_\pm(\omega)^2\right) U(s)}{s^2 - s_0^2}\, ds = -\frac{2\pi}{i}(w_1 \pm w_2) f_0. \tag{3.16}$$

Let $\begin{pmatrix} u_0^\pm \\ U_\pm \end{pmatrix} \in \mathscr{X}$ be solutions to (3.16) and $\Lambda \subset \Gamma^- \backslash \{-s_0, -ik_\pm(\omega)\}$ be an infinite set with cluster point. Since $F_\lambda(s) := (s+\lambda)^{-1}$, $s \in \Gamma$, $\lambda \in \Lambda$, belongs to $H^-(\Gamma)$ due to A.8(4), we choose the test functions $\begin{pmatrix} 0 \\ F_\lambda \end{pmatrix} \in \mathscr{X}$. Since for these test functions $f_0 = 0$, the second integral and the right hand side in (3.16) vanish and therefore lead to the identity

$$-\frac{1}{2\pi i}\int_\Gamma \frac{s^2 + k_\pm(\omega)^2}{(s-\lambda)(s^2 - s_0^2)} U_\pm(s) ds = \left(\frac{1}{2\pi i}\int_\Gamma \frac{s_0 s + k_\pm(\omega)^2}{(s-\lambda)(s^2 - s_0^2)} ds\right) u_0^\pm.$$

Partial fraction decomposition for the left hand side together with A.8(5) applied to $z = \lambda$, s_0, and $-s_0$ leads to

$$\frac{\lambda^2 + k_\pm^2(\omega)}{\lambda^2 - s_0^2} U_\pm^{\text{vol}}(\lambda) + \frac{s_0^2 + k_\pm(\omega)^2}{2s_0(\lambda + s_0)} U_\pm^{\text{vol}}(-s_0) = \text{rhs}_\pm(\lambda) u_0^\pm, \quad \lambda \in \Lambda.$$

Note that by Lemma A.8 there is a one-to-one correspondence between $U_\pm$ and the volume functions $U_\pm^{\text{vol}}$ which are holomorphic in Γ^-. The equations for $U_\pm^{\text{vol}}$ are solvable since there exist solutions to (3.16), and the solutions $U_\pm^{\text{vol}}$ are unique since for $\text{rhs}_\pm(\lambda) u_0^\pm = 0$ we have $U_\pm^{\text{vol}}(\lambda) = -\frac{(s_0^2 + k_\pm(\omega)^2) U_\pm^{\text{vol}}(-s_0)}{2s_0(\lambda - ik_\pm(\omega))} \frac{\lambda - s_0}{\lambda + ik_\pm(\omega)}$. Due to $-ik_\pm(\omega) \in \Gamma^-$, $s_0 \in \Gamma^+$ and $\lambda \in \Lambda$ this function is holomorphic in Γ^- if and only if $U_\pm^{\text{vol}} \equiv 0$. Hence, there exists a unique linear mapping $u_0^\pm \mapsto U_\pm$ and the proof is complete. □

The special test functions of Lemma 3.3 will not be used again in the remainder of this paper. They are only needed to derive the variational formulation. Due to the density of the Laplace transformed test functions in $\mathscr{X}$, we are able to choose more convenient trial and test functions later on.

4 Choice of Γ and discretization of $H^-(\Gamma)$

Recall that we have formulated some conditions on Γ in Assumption A.5 and Definition 2.4 depending on the parameter ζ of the differential equation. They leave a lot of freedom in the choice of Γ. Let $H_N^-(\Gamma) \subset H^-(\Gamma)$ be a finite dimensional subset of $H^-(\Gamma)$ and $\mathscr{X}_N := \mathbb{C} \oplus H_N^-(\Gamma) \subset \mathscr{X}$. Then, we are looking for solutions $\binom{u_0}{U_N}, \binom{v_0}{V_N} \in \mathscr{X}_N$ such that

$$Q\left(\begin{pmatrix} \binom{u_0}{U_N} \\ \binom{v_0}{V_N} \end{pmatrix}, \begin{pmatrix} \binom{f_0}{F_N} \\ \binom{g_0}{G_N} \end{pmatrix}\right) = -(f_0 w_1 + g_0 w_2) \text{ for all } \binom{f_0}{F_N}, \binom{g_0}{G_N} \in \mathscr{X}_N. \tag{4.1}$$

Usually, we should first specify the correct space $H^-(\Gamma)$ and then define the finite dimensional subspace $H_N^-(\Gamma) \subset H^-(\Gamma)$. Here, we go the other way around and first define some nice basis functions. In order to get stability, they will automatically lead us to a family of curves Γ depending on two complex parameters, which satisfy the conditions for the parameter ζ. This also specifies the space $H^-(\Gamma)$.

Let us mention that we use a parameter s_0 in this section. This parameter could be chosen independently to s_0 used in Sect. 3. However, we will see in Lemma 4.11 that it is natural to choose these two parameters equal. Therefore, we will not distinguish between them in the rest of the paper.

4.1 Choice of the basis functions

Our first step is to consider the functions

$$\Psi_n^{s_0}(s) := \frac{2s_0}{s - s_0}\left(\frac{s + s_0}{s - s_0}\right)^n, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{4.2}$$

The restrictions $\Psi_n^{s_0}|_{-is_0\mathbb{R}}$ form an orthogonal basis of $H^-(-is_0\mathbb{R})$ due to A.4(3). For general Γ this is of course no longer true. However, at least the following holds true for general Γ:

Lemma 4.1 *Suppose Assumption A.5 is satisfied and let $s_0 \in \Gamma^+$. Then $\Psi_n^{s_0} \in H^-(\Gamma)$ for all $n \in \mathbb{N}_0$, every finite subset of $\bigcup_{n\in\mathbb{N}_0}\{\Psi_n^{s_0}\}$ is linearly independent, and the* span *of $\bigcup_{n\in\mathbb{N}_0}\{\Psi_n^{s_0}\}$ is dense in $H^-(\Gamma)$.*

Proof To show $\Psi_n^{s_0} \in H^-(\Gamma)$, recall from Lemma A.8(4) that $(\bullet - s_0)^{-1} \in H^-(\Gamma)$. Since $\frac{s+s_0}{s-s_0}$ is analytic in $s \in \Gamma^-$ and bounded in Γ^-, we obtain $\Psi_n^{s_0} \in H^-(\Gamma)$.

To show linear independence, we assume that $\sum_{n=0}^N \alpha_n \Psi_n^{s_0} = 0$ for some $N \in \mathbb{N}_0$ and $\alpha_n \in \mathbb{C}$. Like in the proof of Theorem 3.4 we use the one-to-one correspondence (Lemma A.8(5)) between $\sum_{n=0}^N \alpha_n \Psi_n^{s_0}(s) \in H^-(\Gamma)$ and the volumetric function $(\sum_{n=0}^N \alpha_n \Psi_n^{s_0}(s))^{\mathrm{vol}}$. For the sake of readability we drop the superindex in the following. Thus $\sum_{n=0}^N \alpha_n \Psi_n^{s_0}(s) = 0$ for all $s \in \Gamma^-$. Choosing $s = -s_0$ shows $\alpha_0 = 0$. To show $\alpha_1 = \alpha_2 = \cdots = 0$, we repeat this argument for $(s+s_0)^{-j}\sum_{n=j}^{N}\alpha_n\Psi_n^{s_0}$. This shows linear independence.

We denote the distance of a point $s \in \mathbb{C}$ to a set $M \subset \mathbb{C}$ by $d(s, M) := \inf_{z\in M}|s-z|$. We show in the following that

$$\frac{1}{s-\lambda} \in \overline{\operatorname{span}\bigcup_{n\in\mathbb{N}_0}\{\Psi_n^{s_0}\}}^{H^-(\Gamma)}$$

for all λ with $|\lambda - s_0| < d(s_0, \Gamma)$. Lemma A.10 then yields the claim.

For $N \in \mathbb{N}$ we can write $\sum_{n=0}^{N}\alpha_n\Psi_n^{s_0}(s) = \frac{\sum_{n=0}^{N}\alpha_n 2s_0(s-s_0)^{N-n}(s+s_0)^n}{(s-s_0)^{N+1}}$. Since $\Psi_n^{s_0}, n = 0\ldots N$ are linearly independent, so are the polynomials $2s_0(s-s_0)^{N-n}(s+s_0)^n = (s-s_0)^{N+1}\Psi_n^{s_0}(s), n = 0\ldots N$. Thus span $\bigcup_{n=0\ldots N}\{\Psi_n^{s_0}\} = \{\frac{\sum_{n=0}^{N}\tilde{\alpha}_n s^n}{(s-s_0)^{N+1}} : \tilde{\alpha}_n \in \mathbb{C}, n = 0\ldots N\}$ holds. Let us define $\tilde{\alpha}_n$ such that the numerator of

$$\frac{1}{s-\lambda} - \frac{\sum_{n=0}^{N}\tilde{\alpha}_n^N s^n}{(s-s_0)^{N+1}} = \frac{(s-s_0)^{N+1} - (s-\lambda)\sum_{n=0}^{N}\tilde{\alpha}_n^N s^n}{(s-\lambda)(s-s_0)^{N+1}}$$

becomes constant in s. Comparing coefficients shows that this is the case for the choice

$$\tilde{\alpha}_n^N := \sum_{k=n+1}^{N+1}\binom{N+1}{k}(-s_0)^{N+1-k}\lambda^{k-n-1}$$

and hence

$$\frac{1}{s-\lambda} - \frac{\sum_{n=0}^{N}\tilde{\alpha}_n s^n}{(s-s_0)^{N+1}} = \frac{1}{s-\lambda}\left(\frac{\lambda - s_0}{s-s_0}\right)^{N+1}.$$

Due to $|\lambda - s_0| < d(s_0, \Gamma)$, the density of $\{\Psi_n^{s_0}, n \in \mathbb{N}_0\}$ in $H^-(\Gamma)$ follows with $\left\|\frac{1}{\bullet-\lambda} - \sum_{n=0}^{N}\alpha_{n,N}\Psi_n^{s_0}\right\|_{H^-(\Gamma)} \to 0$ for $N \to \infty$. □

We know that in general the solution of Problem (3.14) has two poles in Γ_ζ^+. Thus in order to mimic the nature of the solution in our discrete space, it seems appropriate to choose s_0, s_1 with $\Im s_0 > \zeta$ and $\Im s_1 \in (-\zeta, 0)$ and use discrete spaces $H_N^-(\Gamma) = \operatorname{span}\bigcup_{n=0}^{N}\{\Psi_n^{s_0}, \Psi_n^{s_1}\}$. In the standard Hardy space method, we obtain tridiagonal matrices for the mass and stiffness terms. The aim to keep this property (see Sect. 4.4), leads us to a set of "mixed" basis functions $\Psi_n^{s_0,s_1}$. With the same techniques as in the previous theorem, it is easy to show the following:

Lemma 4.2 *Let Assumption A.5 hold true, let $s_0, s_1 \in \Gamma^+$, let $\lfloor x\rfloor := \max\{m \in \mathbb{Z} : m \le x\}$ and*

$$\Psi_n^{s_0,s_1} := \frac{s_0+s_1}{s-s_1}\left(\frac{s+s_0}{s-s_0}\right)^{\lfloor (n+1)/2\rfloor}\left(\frac{s+s_1}{s-s_1}\right)^{\lfloor n/2\rfloor}, \quad n \in \mathbb{N}_0. \tag{4.3}$$

Then $\Psi_n^{s_0,s_1} \in H^-(\Gamma)$, $n \in \mathbb{N}_0$, *every finite subset of* $\bigcup_{n\in\mathbb{N}_0}\{\Psi_n^{s_0,s_1}\}$ *is linearly independent, and the* span *of* $\bigcup_{n\in\mathbb{N}_0}\{\Psi_n^{s_0,s_1}\}$ *is dense in* $H^-(\Gamma)$.

In previous versions of the method (see [8,9]), other kinds of basis functions were used. Studying the reasons for the instability of the corresponding discrete problems led to the current basis functions $\Psi_n^{s_0,s_1}$.

4.2 Definition and properties of the curves Γ_{s_0,s_1}

Our aim is to find Γ such that $\Psi_n^{s_0,s_1}$, $n \in \mathbb{N}_0$, form a Riesz basis of the space $H^-(\Gamma)$. We denote the entries of the Gram matrix by

$$T_{m,n} := \left\langle \Psi_n^{s_0,s_1}, \Psi_m^{s_0,s_1}\right\rangle_{H^-(\Gamma)}, \quad n, m \in \mathbb{N}_0, \tag{4.4a}$$

and compute for $l_1, l_2 \in \mathbb{N}_0$

$$\begin{pmatrix} T_{2l_1,2l_2} & T_{2l_1,2l_2+1} \\ T_{2l_1+1,2l_2} & T_{2l_1+1,2l_2+1} \end{pmatrix} = \int_\Gamma g(s)^{l_2-l_1}|g(s)|^{l_1} f(s)\,|ds| \tag{4.4b}$$

with

$$f(s) := \frac{|s_0+s_1|^2}{|s-s_1|^2}\left(\begin{matrix} 1 & (\frac{s+s_0}{s-s_0}) \\ \overline{(\frac{s+s_0}{s-s_0})} & \left|\frac{s+s_0}{s-s_0}\right|^2 \end{matrix}\right), \quad g(s) := \frac{s+s_0}{s-s_0}\,\frac{s+s_1}{s-s_1}, \quad s \in \Gamma. \tag{4.4c}$$

A necessary criterion for a Riesz basis is $\sup_{m,n}|T_{m,n}| < \infty$. Since this is the case if $|g(s)| = 1$ for $s \in \Gamma$, we are led to choose Γ as the algebraic variety

$$\Gamma = \Gamma_{s_0,s_1} := \left\{ s \in \mathbb{C} \colon \left| \frac{s+s_0}{s-s_0}\,\frac{s+s_1}{s-s_1}\right| = 1 \right\}. \tag{4.5}$$

Figure 3 shows an example of the set Γ_{s_0,s_1}, which turns out to be a curve, for a typical choice of s_0 and s_1 in our context. These curves have a number of interesting geometrical properties, which we are going to explore in this subsection. In particular, they satisfy Assumption A.5 under certain conditions on s_0 and s_1:

Lemma 4.3 *Assume that* $|s_1| + |s_0| \neq 0$ *and* $s_1 \notin s_0\mathbb{R}_{<0}$.

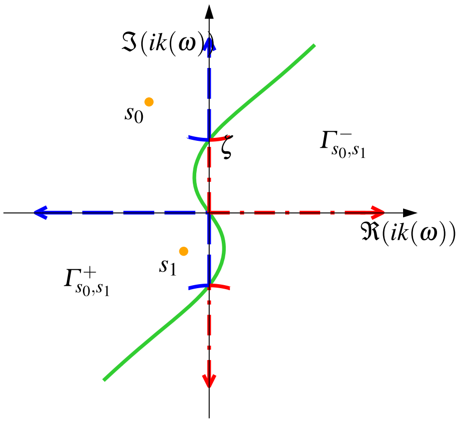
(1) $\Gamma_{s_0,s_1} = \gamma_{s_0,s_1}(\mathbb{R})$ *satisfies Assumption* A.5 *with*

$$\gamma_{s_0,s_1}(\rho) := -i\rho\frac{\rho^2(s_0+s_1) + |s_0|^2 s_1 + |s_1|^2 s_0}{|\rho^2(s_0+s_1) + |s_0|^2 s_1 + |s_1|^2 s_0|}, \qquad \rho \in \mathbb{R}.$$

It is actually an oriented C^∞*-curve with*

$$\gamma'_{s_0,s_1}(0) = -i\frac{|s_0|^2 s_1 + |s_1|^2 s_0}{||s_0|^2 s_1 + |s_1|^2 s_0|} \quad \textit{and} \quad \sigma_\infty = \lim_{|\rho|\to\infty}\frac{\gamma(\rho)}{\rho} = -i\frac{s_0+s_1}{|s_0+s_1|}.$$

Fig. 3 Algebraic variety Γ_{s_0,s_1} for the points s_0, s_1 indicated by *yellow dots* (color figure online)



(2) *If* $s_0 = 0$ *or* $s_1 \in s_0\mathbb{R}_{\geq 0}$, *then* Γ_{s_0,s_1} *is a straight line.*
(3) $\Gamma_{s_0,s_1} \cap \mathbb{R} = \{0\}$ *if* $\Re s_0 < 0$ *and* $\Re s_1 < 0$.
(4) *If* $\frac{|s_0|^2\Im(s_1)+|s_1|^2\Im(s_0)}{\Im(s_0+s_1)} < 0$, *then* $\Gamma_{s_0,s_1} \cap i\mathbb{R} = \{0, i\xi, -i\xi\}$ *with*

$$\xi := \sqrt{-\frac{|s_0|^2\Im(s_1)+|s_1|^2\Im(s_0)}{\Im(s_0+s_1)}}. \tag{4.6}$$

(5) *Using Equation* (A.3) *as definition of* $\Gamma^{\pm}_{s_0,s_1}$, *we have* $s_0, s_1 \in \Gamma^+_{s_0,s_1}$.
(6) *If* $\Re s_0 < 0$, $\Re s_1 < 0$, $\frac{|s_0|^2\Im(s_1)+|s_1|^2\Im(s_0)}{\Im(s_0+s_1)} < 0$, *and* $\Im(s_0 + s_1) > 0$, *then*

$$i(-\infty, -\xi) \cup i(0, \xi) \cup \mathbb{R}_{>0} \subset \Gamma^-_{s_0,s_1}.$$

Proof (1) A computation shows that

$$\begin{aligned}\{s \in \Gamma_{s_0,s_1} : |s| = \rho\} &= \left\{s \in \mathbb{C}\colon |s+s_0|^2|s+s_1|^2 = |s-s_0|^2|s-s_1|^2, |s| = \rho\right\} \\ &= \left\{s \in \mathbb{C}\colon \Re\left(s\left[\rho^2(\overline{s_0+s_1}) + |s_0|^2\overline{s_1} + |s_1|^2\overline{s_0}\right]\right) = 0,\right. \\ &\qquad \left. |s| = \rho\right\}.\end{aligned}$$

As $\{s \in \mathbb{C} : \Re(s\overline{z}) = 0, |s| = \rho\} = \{i\rho z/|z|, -i\rho z/|z|\}$ for all $z \in \mathbb{C}\backslash\{0\}$ we obtain the parameterization of Γ_{s_0,s_1}. The conditions on s_0, s_1 ensure that the expression in brackets does not vanish for $\rho \neq 0$. Smoothness of Γ_{s_0,s_1} at the origin and elsewhere, the behavior at infinity, and $\sup_{\rho\in\mathbb{R}} |\gamma_{s_0,s_1}| < \infty$ are easy to see.

(2) This is straightforward!

(3) This follows from the equivalence of $\Im(\gamma_{s_0,s_1}(\rho)) = 0$ and $\rho^2\Re(s_0 + s_1) + |s_0|^2\Re(s_1) + |s_1|^2\Re(s_0) = 0$ for $\rho \neq 0$.

(4) This follows from the equivalence of $\Re(\gamma_{s_0,s_1}(\rho)) = 0$ and $\rho^2\Im(s_0 + s_1) + |s_0|^2\Im(s_1) + |s_1|^2\Im(s_0) = 0$ for $\rho \neq 0$.

(5) For $s_0 = s_1$ we have $s_0 = \gamma_{s_0,s_1}(|s_0|)\exp(i\pi/2) \in \Gamma^+_{s_0,s_1}$. Since the mapping $(s_0, s_1, \rho) \mapsto \gamma_{s_0,s_1}$ is continuous, since $s_0, s_1 \notin \Gamma_{s_0,s_1}$ due to (4.5) and since the set of admissible parameters s_0, s_1 is connected, it follows that $s_0, s_1 \in \Gamma^+_{s_0,s_1}$ for all admissible s_0, s_1.

(6) Under the given assumptions we have $\Re\sigma_\infty > 0$, and $\Im\sigma_\infty > 0$. These inequalities imply together with part 4 that $\mathbb{R}_{>0} \subset \Gamma^-_{s_0,s_1}$, and together with part 5 that $i(-\infty, -\xi) \subset \Gamma^-_{s_0,s_1}$ (see (A.3)). To show that $i(0, \xi) \subset \Gamma^-_{s_0,s_1}$ note that $\Re\gamma'_{s_0,s_1}(0) < 0$, $\lim_{\rho\to\infty} \Re\gamma_{s_0,s_1}(\rho) = \infty$, and for $\rho > 0$ the path $\gamma_{s_0,s_1}(\rho)$ never crosses the real axis, and it crosses the imaginary axis only at $i\xi$. □

Remark 4.4 For the model problem due to (2.5) the parameters s_0, s_1 with negative real part have to be chosen such that $\xi = \zeta$. Since $r\Gamma_{s_0,s_1} = \Gamma_{rs_0,rs_1}$ for $r > 0$ this can be achieved if one pair of parameters is known for e.g. $\xi = 1$.

Besides γ_{s_0,s_1} in Lemma 4.3 there exists another useful parameterization of Γ_{s_0,s_1} needed in Sect. 4.3: note from (4.4c) and (4.5) that $g(\Gamma_{s_0,s_1}) \subset S^1$. Solving $z = g(s)$ for s we obtain

$$g_\pm^{-1}(z) := \frac{1}{2}\left(\frac{z+1}{z-1}(s_0+s_1) \pm \sqrt{\frac{(z+1)^2}{(z-1)^2}(s_0+s_1)^2 - 4s_0s_1}\right), \quad z \in S^1\backslash\{1\}. \tag{4.7}$$

Lemma 4.5 *Let $s_0 + s_1 \neq 0$ and $s_0s_1 \neq 0$.*

(1) $g(g_\pm^{-1}(z)) = z$ *for all* $z \in S^1\backslash\{1\}$.

(2) *The sets* $\Gamma^1_{s_0,s_1} := g_+^{-1}(S^1\backslash\{1\})$, $\Gamma^2_{s_0,s_1} := g_-^{-1}(S^1\backslash\{1\})$ *and* $\{0\}$ *are pairwise disjoint, and*

$$\Gamma_{s_0,s_1} = \{0\} \cup \Gamma^1_{s_0,s_1} \cup \Gamma^2_{s_0,s_1}. \tag{4.8}$$

(3) *For* $s \in \Gamma^1_{s_0,s_1}$ *we have* $\frac{s_0s_1}{s} \in \Gamma^2_{s_0,s_1}$, $g_+^{-1}(g(s)) = s$ *and* $g_-^{-1}(g(s)) = \frac{s_0s_1}{s}$.

(4) *We have*

$$\lim_{\theta\searrow 0} g_+^{-1}(e^{i\theta}) = 0, \quad \lim_{\theta\nearrow 0} g_+^{-1}(e^{i\theta}) = \infty, \quad \lim_{\theta\searrow 0} g_-^{-1}(e^{i\theta}) = -\infty, \quad \lim_{\theta\nearrow 0} g_-^{-1}(e^{i\theta}) = 0.$$

(5) $\Gamma_{s_0,s_1} = \{s \in \mathbb{C} : |g(s)| = 1\}$, $\Gamma^+_{s_0,s_1} = \{s \in \mathbb{C} : |g(s)| > 1\}$, $\Gamma^-_{s_0,s_1} = \{s \in \mathbb{C} : |g(s)| < 1\}$.

Proof (1) This follows by construction as $z = g(s)$ is equivalent to $s = g_+^{-1}(z)$ or $s = g_-^{-1}(z)$.

(2) Since $\sqrt{s_0s_1} \notin \Gamma_{s_0,s_1}$, the discriminant $(\frac{z+1}{z-1})^2(s_0+s_1)^2 - 4s_0s_1$ never vanishes for $z \in S^1\backslash\{1\}$, so $g_+^{-1}(z) \neq g_-^{-1}(z)$. Together with part 1 this shows $\Gamma^1_{s_0,s_1} \cap \Gamma^2_{s_0,s_1} = \emptyset$. Moreover, $0 \notin \Gamma^1_{s_0,s_1} \cup \Gamma^2_{s_0,s_1}$ as $s_0s_1 \neq 0$. Finally, $\Gamma_{s_0,s_1} \subset \{0\} \cup \Gamma^1_{s_0,s_1} \cup \Gamma^2_{s_0,s_1}$ since $g(s) = 1$ is equivalent to $s(s_0 + s_1) = 0$, and as $s_0 + s_1 \neq 0$ it is also equivalent to $s = 0$.

(3) Let $s \in \Gamma^1_{s_0,s_1}$. The identity $g_+^{-1}(g(s)) = s$ is obvious. By the definition of g we have $g(s) = g(\frac{s_0 s_1}{s})$. As $s \neq \frac{s_0 s_1}{s}$ for $s \in \Gamma_{s_0,s_1}\backslash\{0\}$, applying g_+^{-1} to both sides of the last equation shows that $\frac{s_0 s_1}{s} \in \Gamma^2_{s_0,s_1}$. Hence, $g_-^{-1}(g(s)) = g_-^{-1}(g(\frac{s_0 s_1}{s})) = \frac{s_0 s_1}{s}$.
(4) This is straightforward.
(5) The first equality is obvious from the definition. By continuity of g, we must have $\{s \in \mathbb{C} : |g(s)| > 1\} = \Gamma^+_{s_0,s_1}$ or $\{s \in \mathbb{C} : |g(s)| > 1\} = \Gamma^-_{s_0,s_1}$. As $\lim_{s\to s_0} |g(s)| = \infty$ and since by Lemma 4.3(5) $s_0 \in \Gamma^+_{s_0,s_1}$, the first alternative holds true. □

4.3 Stability of the basis $\Psi_n^{s_0,s_1}$ in $H^-(\Gamma_{s_0,s_1})$

In the standard Hardy space method for $H^-(\kappa_0\mathbb{R})$ an orthogonal basis is used for discretization. Thus, stability of the discrete problems ensures the boundedness of the condition numbers of the system matrices. As our Hardy space is more complicated, we do not use an orthogonal basis. However, the basis is stable as the next lemmata show.

Lemma 4.6 *If* $s_0, s_1 \in \mathbb{C}\backslash\{0\}$ *and* $s_1 \notin s_0\mathbb{R}_{<0}$, *the infinite matrix* $T : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ *with matrix entries* $T_{m,n}$ *defined by* (4.4a) *for* $\Gamma = \Gamma_{s_0,s_1}$ *is associated to the block Toeplitz operator* $\mathscr{T}(a) : [H^+(S^1)]^2 \to [H^+(S^1)]^2$ *with continuous symbol*

$$a(z) := \begin{cases} \frac{f}{|g'|} \circ g_+^{-1}(z) + \frac{f}{|g'|} \circ g_-^{-1}(z), & z \in S^1\backslash\{1\}, \\ \frac{|s_0+s_1|}{2} \begin{pmatrix} 1+\frac{|s_0|}{|s_1|} & 1-\frac{|s_0|}{|s_1|} \\ 1-\frac{|s_0|}{|s_1|} & 1+\frac{|s_0|}{|s_1|} \end{pmatrix}, & z = 1 \end{cases} \tag{4.9}$$

(*see Definition* A.15). *Here* $f : \Gamma_{s_0,s_1} \to \mathbb{C}^{2\times 2}$ *and* $g : \Gamma_{s_0,s_1} \to S^1$ *are defined in* (4.4c), *and* $g_\pm^{-1}$ *in* (4.7).

Proof It follows immediately from (4.4b) that T has a 2×2 block Toeplitz structure, and is associated to the block Toeplitz operator $\mathscr{T}(a)$ with symbol $a(z) = \sum_{k\in Z} a_k z^k$,

$$a_k := \int_{\Gamma_{s_0,s_1}} g(s)^{-k} f(s)|ds|, \qquad k \in \mathbb{Z}.$$

provided that $a \in L^\infty(S^1)^{2\times 2}$. Since S^1 is compact, it suffices to show that a is continuous. With the substitutions of variables $s = g_+^{-1}(z)$ on $\Gamma^1_{s_0,s_1}$ and $s = g_-^{-1}(z)$ on $\Gamma^2_{s_0,s_1}$ it follows from Lemma 4.5 that

$$a_k = \int_{S^1} z^{-k} \left(\frac{f}{|g'|} \circ g_+^{-1}(z) + \frac{f}{|g'|} \circ g_-^{-1}(z) \right) |dz|,$$

i.e. a has the form (4.9) at least for $z \neq 1$. It remains to show that a is continuous. As $\sqrt{s_0 s_1} \notin \Gamma_{s_0,s_1}$ the derivative $g'(s) = \frac{-2(s_0+s_1)(s^2-s_0 s_1)}{(s-s_0)^2(s-s_1)^2}$ never vanishes on Γ_{s_0,s_1}, showing continuity of a on $S^1\backslash\{1\}$. Moreover, due to Lemma 4.5(5) we have

$$\lim_{s\to 0} \frac{f(s)}{|g'(s)|} = \frac{|s_0+s_1|}{2} \frac{|s_0|}{|s_1|} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \lim_{s\to\pm\infty} \frac{f(s)}{|g'(s)|} = \frac{|s_0+s_1|}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which yields continuity of $a(z)$ at $z = 1$. □

Lemma 4.7 *The operator $\mathscr{T}(a): [H^+(S^1)]^2 \to [H^+(S^1)]^2$ defined in the previous lemma is continuously invertible.*

Proof Due to Theorem A.16 it suffices to show that $\inf_{z\in S^1} \min(speca(z)) > 0$. As a is continuous, we only have to show that all the Hermitian 2×2 matrices $a(z)$ are strictly positive definite. By Lemma 4.5(3), $a \circ g$ is the sum of two matrices

$$(a \circ g)(s) = \frac{|s_0 + s_1|}{2} \left(\frac{|s - s_0|^2}{|s^2 - s_0 s_1|} \begin{pmatrix} 1 & \frac{s+s_0}{s-s_0} \\ \frac{s+s_0}{s-s_0} & \left|\frac{s+s_0}{s-s_0}\right|^2 \end{pmatrix} + \frac{|s_0|}{|s_1|} \frac{|s - s_1|^2}{|s^2 - s_0 s_1|} \begin{pmatrix} 1 & -\frac{s+s_1}{s-s_1} \\ -\frac{s+s_1}{s-s_1} & \left|\frac{s+s_1}{s-s_1}\right|^2 \end{pmatrix} \right)$$

for $s \in \Gamma^1_{s_0,s_1}$. Both are Hermitian, positive semi-definite, and the kernels are spanned by the vectors $\begin{pmatrix} \frac{s+s_0}{s-s_0} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -\frac{s+s_1}{s-s_1} \\ -1 \end{pmatrix}$, respectively. Since $\frac{s+s_0}{s-s_0} \neq -\frac{s+s_1}{s-s_1}$ for $s \in \Gamma^1_{s_0,s_1}$, the intersection of the kernels is empty and thus $a(z)$ is positive definite for every $z \in S^1 \backslash \{1\}$. Since $a(1)$ is also positive definite, the proof is complete. □

Theorem 4.8 (Riesz basis of $H^-(\Gamma_{s_0,s_1})$) *If $s_0, s_1 \in \mathbb{C}\backslash\{0\}$ and $s_1 \notin s_0\mathbb{R}_{<0}$, then $\{\Psi_n^{s_0,s_1}, n \in \mathbb{N}_0\}$ is a Riesz basis of $H^-(\Gamma_{s_0,s_1})$. More precisely,*

$$\|\mathscr{T}(a)^{-1}\|^{-1} \sum_{n\in\mathbb{N}_0} |\beta_n|^2 \le \left\| \sum_{n\in\mathbb{N}_0} \beta_n \Psi_n^{s_0,s_1} \right\|^2_{H^-(\Gamma_{s_0,s_1})} \le \|\mathscr{T}(a)\| \sum_{n\in\mathbb{N}_0} |\beta_n|^2 \tag{4.10}$$

for all $(\beta_n) \in l^2(\mathbb{N}_0)$ with $\mathscr{T}(a)$ defined in Lemma 4.6, and for every $U \in H^-(\Gamma_{s_0,s_1})$ there exists a unique sequence $(\beta_n) \in l^2(\mathbb{N}_0)$ such that

$$U(s) = \sum_{n\in\mathbb{N}_0} \beta_n \Psi_n^{s_0,s_1}(s), \qquad s \in \Gamma_{s_0,s_1} \cup \Gamma^-_{s_0,s_1}. \tag{4.11}$$

Proof (4.10) is a consequence of the two preceding lemmata since

$$\left\| \sum_{n\in\mathbb{N}_0} \beta_n \Psi_n^{s_0,s_1} \right\|^2_{H^-(\Gamma_{s_0,s_1})} = \langle (\beta_n), T(a)\,(\beta_n)\rangle_{l^2(\mathbb{N}_0)}$$

and since by the Rayleigh-Ritz principle

$$\sup_{\|(\beta_n)\|=1} \langle (\beta_n), T(a)\,(\beta_n)\rangle_{l^2(\mathbb{N}_0)} = \sup \operatorname{spec}(T(a)) = \|T(a)\| = \|\mathscr{T}(a)\|,$$

$$\inf_{\|(\beta_n)\|=1} \langle(\beta_n), T(a)\,(\beta_n)\rangle_{l^2(\mathbb{N}_0)} = \inf \operatorname{spec}(T(a)) = \|T(a)^{-1}\|^{-1} = \|\mathscr{T}(a)^{-1}\|^{-1}.$$

(4.11) then follows from the density Lemma 4.2. □

As the solution to Problem (3.14) consists of terms $\frac{1}{s-\lambda}$, the next corollary is of importance.

Corollary 4.9 *For $\lambda \in \Gamma^{+}_{s_0,s_1}$ the coefficients of the expansion $\frac{1}{s-\lambda} = \sum_{n\in\mathbb{N}_0} \beta_n \Psi_n^{s_0,s_1}(s)$ with $s \in \Gamma_{s_0,s_1} \cup \Gamma^{-}_{s_0,s_1}$ are given by*

$$\beta_n := \frac{1}{\lambda + s_0}\left(\frac{\lambda - s_0}{\lambda + s_0}\right)^{\lfloor n/2\rfloor}\left(\frac{\lambda - s_1}{\lambda + s_1}\right)^{\lfloor (n+1)/2\rfloor} \tag{4.12}$$

and $|\beta_n| \leq q^n$ with $q < 1$ and $n \in \mathbb{N}_0$.

Proof The existence and uniqueness of the expansion follows from Theorem 4.8 and Lemma A.8(4). Evaluating $(s-\lambda)^{-1} = \sum_{n\in\mathbb{N}_0} \beta_n \Psi_n^{s_0,s_1}$ at $s = -s_0$ yields $\beta_0 = \frac{1}{\lambda+s_0}$. Evaluating further at $s = -s_1$ yields $\beta_1 = \frac{1}{\lambda+s_0} \cdot \frac{\lambda-s_1}{\lambda+s_1}$. Since $\frac{1}{s-\lambda} - \beta_0 \Psi_0^{s_0,s_1} - \beta_1 \Psi_1^{s_0,s_1} = \frac{\lambda-s_0}{\lambda+s_0} \cdot \frac{\lambda-s_1}{\lambda+s_1} \cdot \frac{s+s_0}{s-s_1} \cdot \frac{s+s_1}{s-s_0} \cdot \frac{1}{s-\lambda}$ an induction argument shows that $(\beta_n)_{n\in\mathbb{N}_0}$ has to be given by (4.12). The geometric decay of $|\beta_n|$ follows from $\left|\frac{\lambda-s_0}{\lambda+s_0}\frac{\lambda-s_1}{\lambda+s_1}\right| = |g(\lambda)|^{-1} < 1$ (see Lemma 4.5(5)). □

The following corollary is concerned with the dependence of the condition number of $\mathscr{T}(a)$ and hence the condition number of the basis $\{\Psi_n^{s_0,s_1}\}$ on the scaling of the poles s_0 and s_1.

Corollary 4.10 *Let s_0 and s_1 be such that $s_1, s_0 \neq 0$ and $s_1 \notin s_0\mathbb{R}_{\leq 0}$ and $\sigma \in \mathbb{C}\backslash\{0\}$. Moreover, in this corollary we indicate the dependence of the function a in (4.9) on s_0 and s_1 by a_{s_0,s_1}. Then*

$$\operatorname{cond}(\mathscr{T}(a_{s_0,s_1})) = \operatorname{cond}(\mathscr{T}(a_{\sigma s_0,\sigma s_1})).$$

Proof We have $\Gamma_{\sigma s_0,\sigma s_1} = \sigma\Gamma_{s_0,s_1} := \left\{\sigma s \in \mathbb{C} \mid s \in \Gamma_{s_0,s_1}\right\}$, and $a_{\sigma s_0,\sigma s_1}(g(\sigma s)) = |\sigma| a_{s_0,s_1}(g(s))$ (see the proof of Lemma 4.7). Therefore, $\mathscr{T}(a_{\sigma s_0,\sigma s_1}) = |\sigma|\mathscr{T}(a_{s_0,s_1})$. This proves the claim. □

Due to Theorem A.16 the formula

$$\operatorname{cond}(\mathscr{T}(a_{s_0,s_1})) = \frac{\sup\{\lambda \in \bigcup_{z\in S^1} spec\, a_{s_0,s_1}(z)\}}{\inf\{\lambda \in \bigcup_{z\in S^1} spec\, a_{s_0,s_1}(z)\}} \tag{4.13}$$

can be used to compute $\operatorname{cond}(\mathscr{T}(a_{s_0,s_1}))$ numerically.

4.4 Matrix representations

In this subsection we provide the building blocks for assembling the system matrix for wave equations with Hardy space infinite elements. For our model problem we will discuss in Sect. 5 how these building blocks are put together.

For $N \in \mathbb{N}_0$ define

$$H_N^-(\Gamma_{s_0,s_1}) := \operatorname{span} \bigcup_{n=0}^{N} \{\Psi_n^{s_0,s_1}\}, \quad \mathscr{X}_N := \mathbb{C} \oplus H_N^-(\Gamma_{s_0,s_1}), \tag{4.14}$$

and as basis functions of $\mathscr{X}_N$

$$\Phi_{-1}^{s_0,s_1} := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \quad \Phi_n^{s_0,s_1} := \left(\begin{smallmatrix} 0 \\ \Psi_n^{s_0,s_1} \end{smallmatrix}\right), \quad n = 0, ..., N. \tag{4.15}$$

Since the operators $\mathscr{S}_m, \mathscr{S}_s : \mathscr{X} \to H^-(\Gamma_{s_0,s_1})$ defined in (3.4) are used in the bilinear form (3.11) of the variational formulation (3.14), the following lemma simplifies the calculation of the matrices representing the bilinear form with respect to the basis $\{\Phi_n^{s_0,s_1} : n = -1, 0, 1, \ldots\}$. This lemma holds true if the parameter s_0 in the definition of $\mathscr{S}_m$ and $\mathscr{S}_s$ coincides with the parameter s_0 of the basis functions $\Phi_n^{s_0,s_1}$. For this reason we have refrained from introducing different symbols for these parameters in the beginning of this section.

Lemma 4.11 *We have*

$$\mathscr{S}_m(\mathscr{X}_N) = \mathscr{S}_s(\mathscr{X}_N) = \operatorname{span} \left\{\Psi_n^{s_1,s_0} \; : \; n = 0, ..., N+1\right\}$$

(note the reversed order of s_0 and s_1 in the basis elements!), and the matrix representation of $s_1\mathscr{S}_m, \mathscr{S}_s : \mathscr{X} \to H^-(\Gamma_{s_0,s_1})$ ($\mathscr{S}_m$ is multiplied with s_1) with respect to the bases $\{\Phi_{-1}^{s_0,s_1}, \Phi_0^{s_0,s_1}, \Phi_1^{s_0,s_1}, \cdots\}$ and $\{\Psi_0^{s_1,s_0}, \Psi_1^{s_1,s_0}, \cdots\}$ is given by the infinite 2×2 bidiagonal block Toeplitz matrices

$$S_\pm := \frac{1}{2}\mathrm{id} \pm \frac{1}{2} \begin{pmatrix} \boxed{\begin{matrix} \frac{s_0-s_1}{s_0+s_1} & 1 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} \frac{s_1-s_0}{s_0+s_1} & 0 \\ 1 & 0 \end{matrix}} & & \\ & \boxed{\begin{matrix} \frac{s_0-s_1}{s_0+s_1} & 1 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} \frac{s_1-s_0}{s_0+s_1} & 0 \\ 1 & 0 \end{matrix}} & \\ & & \boxed{\begin{matrix} \frac{s_0-s_1}{s_0+s_1} & 1 \\ 0 & 0 \end{matrix}} & \ddots \\ & & & \ddots \end{pmatrix}. \tag{4.16}$$

Correspondingly $s_1\mathscr{S}_m, \mathscr{S}_s : \mathscr{X}_N \to \{\Psi_0^{s_1,s_0}, \ldots, \Psi_{N+1}^{s_1,s_0}\}$ are represented by the upper left $2(N+2) \times 2(N+2)$ blocks $S_{\pm,N+2}$ of $S_\pm$.

Since the proof is a straightforward computation, we omit it here.

Lemma 4.12 *$D := \left(q(\Psi_n^{s_1,s_0}, \Psi_m^{s_1,s_0})\right)_{n,m\in\mathbb{N}_0}$ with the bilinear form q defined in (3.8) is a block diagonal matrix*

$$D = \frac{-(s_0+s_1)^2}{2s_0}\begin{pmatrix} \boxed{\begin{matrix} 1 & \frac{s_0-s_1}{s_0+s_1} \\ \frac{s_0-s_1}{s_0+s_1} & 1 \end{matrix}} & & \\ & \boxed{\begin{matrix} 1 & \frac{s_0-s_1}{s_0+s_1} \\ \frac{s_0-s_1}{s_0+s_1} & 1 \end{matrix}} & \\ & & \ddots \end{pmatrix}. \tag{4.17}$$

Proof Since $\Psi_n^{s_1,s_0}$ are meromorphic functions, which decay like $1/s$ for $s \in \Gamma_{s_0,s_1} \to \pm\infty$, the integrals over Γ_{s_0,s_1} can be computed by the residue theorem. E.g. for $n = 0$ and $m = 1$ we compute

$$\begin{aligned} q(\Psi_0^{s_1,s_0}, \Psi_1^{s_1,s_0}) &= \frac{1}{2\pi i}\int_{\Gamma_{s_0,s_1}} \frac{s_0+s_1}{s-s_0}\,\frac{s_0+s_1}{-s-s_0}\,\frac{-s+s_1}{-s-s_1}ds \\ &= -\mathrm{Res}_{s_0}\left(\frac{1}{s-s_0}\,\frac{(s_0+s_1)^2(s-s_1)}{(s+s_0)(s+s_1)}\right) \\ &= -\lim_{s\to s_0}\frac{(s_0+s_1)^2(s-s_1)}{(s+s_0)(s+s_1)} = -\frac{(s_0+s_1)^2}{2s_0}\,\frac{s_0-s_1}{s_0+s_1}. \end{aligned}$$

□

Let us summarize the key results of this section, which are independent of our model problem: There exists an implicitly given curve Γ_{s_0,s_1}, for which the basis functions $\Psi_n^{s_0,s_1}$ form a Riesz basis of $H^-(\Gamma_{s_0,s_1})$. But as the last lemma shows, this curve is not needed for an implementation of the method since the integrals can be worked out analytically. Hence, for an implementation of the method no quadrature formula, but only the three preceding matrices $S_{\pm,N+2}$ and D_{N+2} are needed.

Note that for $s_0 = s_1 = i\kappa_0$ the matrices are identical to those of the standard Hardy space method (see e.g. [13, Sect. 6.4]).

5 Convergence of the Galerkin method

After the general studies of the curved Hardy spaces $H^-(\Gamma_{s_0,s_1})$ and their discretization we now return to our model problem (2.1). Due to Theorem 4.8 the variational formulation (3.14) is equivalent to find the solutions $(x, y)^\top \in (l^2(\{-1, 0, 1\ldots\})^2$ of the infinite dimensional linear system

$$A\begin{pmatrix} x \\ y \end{pmatrix} = b \tag{5.1}$$

with system matrix $A := \begin{pmatrix} S-\zeta^2 M & -\sqrt{\omega^2-1}M \\ -\sqrt{\omega^2-1}M & S-\zeta^2 M \end{pmatrix}$, right hand side $b := \begin{pmatrix} (-w_1,0,\ldots)^\top \\ (-w_2,0,\ldots)^\top \end{pmatrix}$ and

$$\begin{aligned} M &:= (q(\mathscr{S}_m(\Phi_l), \mathscr{S}_m(\Phi_k)))_{l,k\geq -1} = s_1^{-2} S_-^\top D S_-, \\ S &:= (q(\mathscr{S}_s(\Phi_l), \mathscr{S}_s(\Phi_k)))_{l,k\geq -1} = S_+^\top D S_+. \end{aligned}$$

Similarly, the discrete variational formulation (4.1) is equivalent to find solutions $(x_N, y_N)^\top \in (\mathbb{C}^{N+2})^2$ of

$$A_N \begin{pmatrix} x_N \\ y_N \end{pmatrix} = b_N \tag{5.2}$$

with $A_N := \begin{pmatrix} S_N-\zeta^2 M_N & -\sqrt{\omega^2-1}M_N \\ -\sqrt{\omega^2-1}M_N & S_N-\zeta^2 M_N \end{pmatrix}$, M_N and S_N the $(N+2)\times(N+2)$ left upper blocks of M and S, and right hand side $b_N := \begin{pmatrix} (-w_1,0,\ldots,0)^\top \\ (-w_2,0,\ldots,0)^\top \end{pmatrix} \in (\mathbb{C}^{N+2})^2$.

We have to choose s_0 and s_1 such that the blue parts in Fig. 2b are contained in Γ^+, i.e. condition (2.5) holds true. By point symmetry the red parts are then contained in Γ^-. If s_0 and s_1 satisfy the conditions of Lemma 4.3, we already know that three of the five parts of the blue set are contained in $\Gamma^+_{s_0,s_1}$. Due to Lemma 4.5 the remaining parts of the blue set are contained in $\Gamma^+_{s_0,s_1}$ if

$$g\left(x \pm \sqrt{x^2+\zeta^2 i}\right) > 1 \qquad \text{for all } x \in [x_0, 0). \tag{5.3}$$

where $x_0 := -\frac{1}{\sqrt{2}}\left(\sqrt{\zeta^4+1}-\zeta^2\right)^{1/2}$.

In practice one might first choose arbitrary $\tilde{s_0}$ and $\tilde{s_1}$ with negative real parts and $\Im(\tilde{s_0}+\tilde{s_1}) > 0$ and use them as starting points for a minimization of $|\zeta^2 + \frac{|s_0|^2\Im(s_1)+|s_1|^2\Im(s_0)}{\Im(s_0+s_1)}|$. The condition (5.3) can easily be checked numerically.

Lemma 5.1 *(Stability of the Hardy space method) Suppose $s_0, s_1 \in \mathbb{C}$ satisfy the assumption of Lemma 4.3 and Eq. (5.3) and $\omega \in \mathbb{R}^+\backslash\{1, \sqrt{\zeta^4+1}\}$. Then there exist constants $c_1, c_2 > 0$ such that for all $N \in 2\mathbb{N}_0$ A_N is invertible, $||A_N||_2 \leq c_1$, and $||A_N^{-1}||_2 \leq c_2$.*

Proof Due to Remark 3.1 it suffices to show the assertion with A_N replaced by $R_{\pm,N} := S_N - k_\pm(\omega)^2 M_N$. Since $R_{\pm,N}$ are the truncations of the block tridiagonal Toeplitz matrices $R_\pm := S - k_\pm(\omega)^2 M$, $\|R_{\pm,N}\|_2$ are uniformly bounded in N. Straightforward calculations yield

$$-2R_\pm = \begin{pmatrix} \boxed{\begin{matrix} s_0 & s_0 \\ s_0 & s_0+s_1 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ s_1 & 0 \end{matrix}} & \\ \boxed{\begin{matrix} 0 & s_1 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} s_0+s_1 & s_0 \\ s_0 & s_0+s_1 \end{matrix}} & \ddots \\ & \ddots & \ddots \end{pmatrix} - \frac{k_\pm(\omega)^2}{s_0 s_1} \begin{pmatrix} \boxed{\begin{matrix} s_1 & -s_1 \\ -s_1 & s_0+s_1 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ -s_0 & 0 \end{matrix}} & \\ \boxed{\begin{matrix} 0 & -s_0 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} s_0+s_1 & -s_1 \\ -s_1 & s_0+s_1 \end{matrix}} & \ddots \\ & \ddots & \ddots \end{pmatrix}.$$

The submatrix $G \in \mathbb{C}^{N/2\times N/2}$ for the even degrees of freedom is a diagonal matrix with constant diagonal entries $(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)$, which are non-vanishing due to $s_0+s_1\neq 0$, $\pm i\sqrt{s_0s_1}\in \Gamma_{s_0,s_1}$, $ik_\pm \in \Gamma^+_{s_0,s_1}$. Hence, we build the Schur complement with respect to the even degrees of freedom and get a symmetric, tridiagonal matrix $F\in\mathbb{C}^{N/2\times N/2}$ for the odd degrees of freedom with entries

$$
\begin{aligned}
F_{1,1} &= s_0 - \frac{k_\pm(\omega)^2}{s_0} - \frac{\left(s_0+\frac{k_\pm(\omega)^2}{s_0}\right)^2}{(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)},\\
F_{j,j} &= (s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right) - \frac{\left(s_0+\frac{k_\pm(\omega)^2}{s_0}\right)^2+\left(s_1+\frac{k_\pm(\omega)^2}{s_1}\right)^2}{(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)},\\
&\quad j=2,\dots,N/2,\\
F_{j,j-1} &= F_{j-1,j} = -\frac{\left(s_0+\frac{k_\pm(\omega)^2}{s_0}\right)\left(s_1+\frac{k_\pm(\omega)^2}{s_1}\right)}{(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)},\quad j=2,\dots,N/2.
\end{aligned}
$$

At first we neglect the difference $F_{1,1}\neq F_{j,j}$, $j\geq 2$ in the first diagonal entry and define a symmetric, tridiagonal Toeplitz operator $\mathscr{T}(\sigma): H^+(S^1)\to H^+(S^1)$ with symbol $\sigma(z)=F_{2,2}+2F_{1,2}\Re(z)$, $z\in S^1$. The range $\sigma(S^1)\subset\mathbb{C}$ of the symbol is a bounded straight line. It is straightforward to check that

$$
\sigma(z)=\begin{cases}\frac{2(1-\Re z)(ik_\pm-g_+^{-1}(z))(ik_\pm-g_-^{-1}(z))(ik_\pm-g_+^{-1}(\bar z))(ik_\pm-g_-^{-1}(\bar z))}{s_0s_1(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)}, & z\neq 1,\\ \frac{-4k_\pm^2(s_0+s_1)^2}{s_0s_1(s_0+s_1)\left(1-\frac{k_\pm(\omega)^2}{s_0s_1}\right)}, & z=1\end{cases}
\tag{5.4}
$$

with the mappings $g_\pm^{-1}: S^1\to\Gamma_{s_0,s_1}$ given in (4.7). Since $ik_\pm\notin\Gamma_{s_0,s_1}$ and due to Proposition A.13, $0\notin\sigma(S^1)=\operatorname{spec}\mathscr{T}(\sigma)$ and therefore $\mathscr{T}(\sigma)$ is positive definite, e.g. there exists a constant $\alpha:=\inf_{s\in\sigma(S^1)}|s|>0$ such that $|(\mathscr{T}(\sigma)V,V)_{L^2(S^1)}|\geq \alpha\|V\|^2_{L^2(S^1)}$ for all $V\in H^+(S^1)$. Since $F_{2,2}-F_{1,1}=\frac{\sigma(i)}{2}$ it follows by the Lemma of Lax-Milgram that the matrix $F=\left(T(\sigma)-\frac{1}{2}\begin{pmatrix}\sigma(i)&0\\0&0\end{pmatrix}\right)_{N/2}$ is invertible for all $N\in 2\mathbb{N}$ and the inverse is uniformly bounded by $2/\alpha$. □

Since A_N is the discretization matrix with respect to a Riesz basis and since A_N is stable, we obtain exponential convergence of the Hardy space method:

Theorem 5.2 *Suppose $s_0,s_1\in\mathbb{C}$ satisfy the assumptions of Lemma 4.3 and Eq. (5.3), $\omega\in\mathbb{R}^+\setminus\{1,\sqrt{\zeta^4+1}\}$, and $\binom{u_0}{U},\binom{v_0}{V}\in\mathscr{X}$ is the unique solution of (3.14). Then the discrete problems (4.1) with $\mathscr{X}_N$ as defined in (4.14) are uniquely solvable for all $N\in 2\mathbb{N}_0$ with solution $\binom{u_{0,N}}{U_N},\binom{v_{0,N}}{V_N}\in\mathscr{X}_N$ and there exist con-*

stants $C_1, C_2, C_3 > 0$ independent of N such that

$$\left\| \binom{u_0}{U} - \binom{u_{0,N}}{U_N} \right\|_{\mathscr{X}} \le C_1 \inf_{(f_{0,N}, F_N)^\top \in \mathscr{X}_N} \left\| \binom{u_0}{U} - \binom{f_{0,N}}{F_N} \right\|_{\mathscr{X}} \le C_2 e^{-C_3 N},$$
$$\left\| \binom{v_0}{V} - \binom{v_{0,N}}{V_N} \right\|_{\mathscr{X}} \le C_1 \inf_{(f_{0,N}, F_N)^\top \in \mathscr{X}_N} \left\| \binom{v_0}{V} - \binom{f_{0,N}}{F_N} \right\|_{\mathscr{X}} \le C_2 e^{-C_3 N}.$$

In particular,

$$\sqrt{|u_0 - u_{0,N}|^2 + |v_0 - v_{0,N}|^2} \le C_2 e^{-C_3 N}. \tag{5.5}$$

Proof The last lemma together with Theorem 4.8 and [14, Theorem 13.6] prove the convergence of the method and the first error estimates. Since by construction U, V are linear combinations of $1/(s - ik_+(\omega))$ and $1/(s - ik_-(\omega))$ with $ik_\pm(\omega) \in \Gamma^+_{s_0,s_1}$ (see Fig. 3), Corollary 4.9 gives the exponential convergence. □

The convergence rate will depend on the size of $\left|\frac{ik_\pm(\omega)-s_0}{ik_\pm(\omega)+s_0}\frac{ik_\pm(\omega)-s_1}{ik_\pm(\omega)+s_1}\right| < 1$. In particular, if the distance of $ik_\pm(\omega)$ to $\Gamma_{s_0,s_1} = \{s \in \mathbb{C}, \left|\frac{s-s_0}{s+s_0}\frac{s-s_1}{s+s_1}\right| = 1\}$ is small, then the convergence will be slow. In the numerical section in Fig. 5 we have studied the convergence rates for a typical choice of the parameters s_0 and s_1.

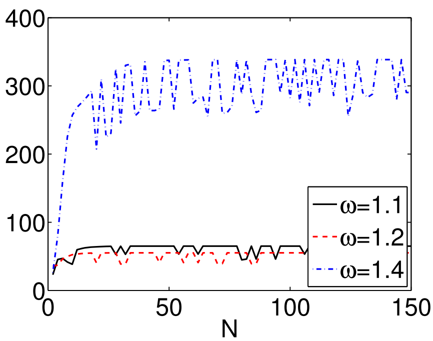
6 Numerical results

Since the aim of this paper is not to solve the specific model problem, but to introduce a new method suitable e.g. for elastic wave-guide problems, we report in the first part of this section only shortly on some numerical results for the model problem. In the second part we will investigate the dependence of the performance on the parameters of the method. This part should help if the method is applied to more complicated problems.

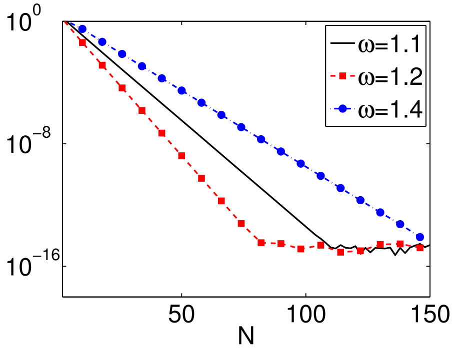
6.1 Model problem

Figure 4a shows numerical verifications of our theoretical results. In particular, in Fig. 4a we see that the condition numbers of the discretization matrices A_N remain bounded for $N \to \infty$ (cf. Lemma 5.1). Recall that we excluded $\omega = 1$ and $\omega = \sqrt{2}$ in assumption (2.3) for $\zeta = 1$. For these frequencies even the continuous problem is not uniquely solvable. Hence, we have to expect that the condition numbers will grow for $\omega \to \{1, \sqrt{2}\}$. Therefore, for $\omega = 1.4$ the condition numbers are larger than for other frequencies, but still moderate.

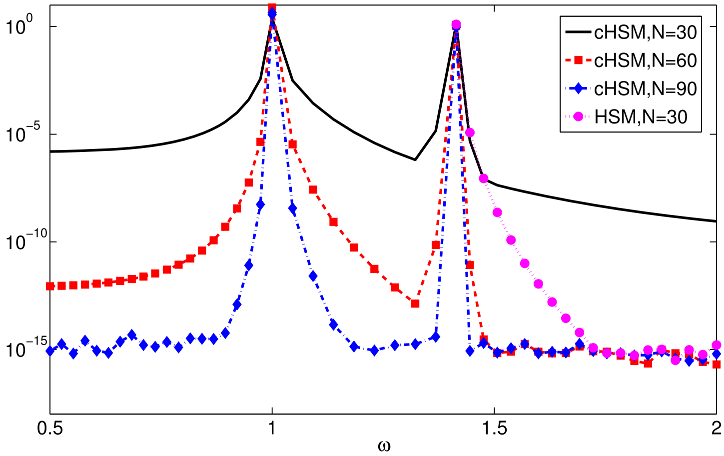
Figure 4b illustrates the exponential convergence of the Hardy space infinite element method proven in Theorem 5.2. As mentioned after Theorem 5.2, the convergence rate becomes worse if the poles $ik_\pm(\omega)$ are in the neighborhood of the curve Γ_{s_0,s_1}. This is the case e.g. for ω in the neighborhood of the degenerated cases $\omega = 1$



(a) $\mathrm{cond}_2(A_N)$



(b) $\sqrt{|u_0 - u_{0,N}|^2 + |v_0 - v_{0,N}|^2}$



(c) $\sqrt{|u_0 - u_{0,N}|^2 + |v_0 - v_{0,N}|^2}$

Fig. 4 *Upper panels* spectral condition number of A_N and Dirichlet error vs. number of dofs N for different frequencies ω. *Lower panel* Dirichlet error vs. frequency ω for the new Hardy space method with different numbers of dofs N and the standard Hardy space method (only for $\omega > \sqrt{2}$) with parameter $\kappa_0 = 0.1 + 1i$. For all computations $\zeta = 1$, $s_0 = -0.8197 + 1.5152i$, and $s_1 = -0.3437 - 0.5372i$

and $\omega = \sqrt{2}$. Therefore, the convergence rate for $\omega = 1.4 \approx \sqrt{2}$ is the worst in Fig. 4b.

For these two figures we have used three different frequencies for which we have two propagating modes with different signs of the phase velocities (see Sect. 2). For Fig. 4c we vary the frequencies $\omega \in (0.5, 2)\setminus\{1, \sqrt{2}\}$ in order to illustrate that the method converges in all three cases (two evanescent modes, two propagating mode, and one propagating and one evanescent mode). The standard Hardy space method is only used for $\omega > \sqrt{2}$ since for the other two cases it yields wrong results. Nevertheless, we see from Fig. 4c that the error in the neighborhood of the critical frequencies $\omega = \{1, \sqrt{2}\}$ is comparatively large, as already seen in Fig. 4b. Modifications to the

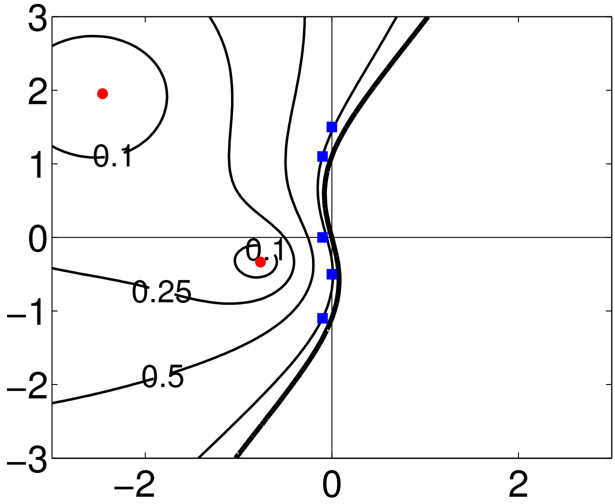


Fig. 5 Optimal parameters $s_0 = -2.4574 + 1.9520i$ and $s_1 = -0.7636 - 0.3362i$ (*dots*) with corresponding Γ_{s_0,s_1} (*thick solid line*) for the points λ_n, $n = 1, \ldots, 5$ (*squares*) and the level sets $\{s \in \mathbb{C},\ F_{s_0,s_1}(s) = c\}$ (*thin lines*) for $c = 0.8638$, 0.5, 0.25 and 0.1

method in order to achieve better approximation properties for these frequencies are currently under investigation.

6.2 Dependence of the method on the parameters s_0 and s_1

In a practical implementation the question arises how to choose the parameters s_0 and s_1. Therefore, we have chosen a reference function $u(x) = \sum_{n=1}^{5} e^{\lambda_n x}$, $x \geq 0$, with Laplace transform $U := (\mathscr{L}u) \in H^-(\Gamma)$ given by $U(s) = \sum_{n=1}^{5}(s-\lambda_n)^{-1}$, $s \in \Gamma$, with $\zeta = 1$ and the five characteristic poles $\lambda_1 = 1.5i$ (corresponds to a propagating mode with positive phase velocity), $\lambda_2 = -0.5i$ (propagating with negative phase velocity), $\lambda_3 = -0.1$ (purely evanescent), $\lambda_4 = -0.1 + 1.1i$, and $\lambda_5 = -0.1 - 1.1i$ (both evanescent and oscillating). In Fig. 5 these poles are denoted by squares in the complex plane.

Moreover, in order to find optimal parameters s_0 and s_1, we define the functional $F_{s_0,s_1}(s) := \left|\frac{s-s_0}{s+s_0}\frac{s-s_1}{s+s_1}\right|$, $s \in \Gamma$ and search for s_0 and s_1 as the minimizer of $E(s_0, s_1) := \max_{n=1}^{5} F_{s_0,s_1}(\lambda_n)$ under the constraint $\sqrt{-\frac{|s_0|^2\Im(s_1)+|s_1|^2\Im(s_0)}{\Im(s_0+s_1)}} = \zeta = 1$. Due to Corollary 4.9 this corresponds to minimizing the approximation error, which can be bounded by $5E(s_0, s_1)^{(N/2)}$ if we use N basis functions in $H^-(\Gamma)$. The optimal parameters s_0 and s_1 are given as dots and the corresponding curve Γ_{s_0,s_1} with a solid thick line. Additionally, some level sets of $F_{s_0,s_1}(s)$ are given in Fig. 5 in order to illustrate which approximation error can be expected for arbitrary poles $s \in \Gamma^+_{s_0,s_1}$.

The parameters s_0 and s_1 also influence the condition number $\mathrm{cond}(\mathscr{T}(a_{s_0,s_1}))$ and therefore the stability constant of Theorem 5.2. For Fig. 6 we have used (4.13) with 100 uniformly distributed sample points $z \in S^1$ to compute $\mathrm{cond}(\mathscr{T}(a_{s_0,s_1}))$ numerically. Note that due to Corollary 4.10 it is sufficient to set $\widetilde{s_0} := 1$ and vary

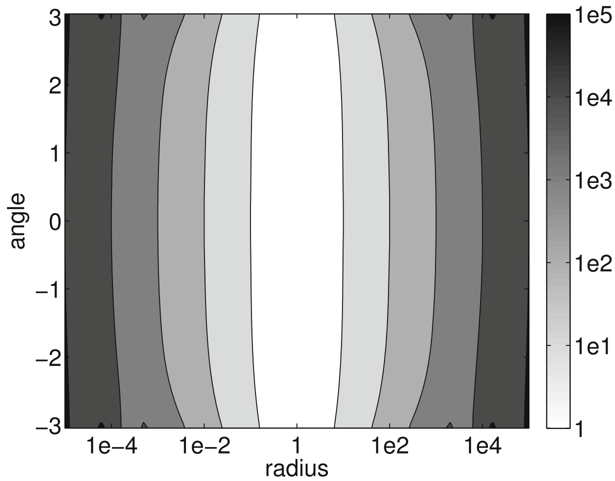


Fig. 6 cond($\mathscr{T}(a_{1,\widetilde{s_1}})$) for $\widetilde{s_1} = re^{i\phi}$ with radius $r \in (10^{-5}, 10^5)$ and angle $\phi \in (-\pi + 0.1, \pi - 0.1)$. The condition numbers are computed with (4.13) and 100 sample points for $z \in S^1$

$\widetilde{s_1} := s_1/s_0 = re^{i\phi} \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ with $r > 0$ and $\phi \in (-\pi, \pi)$. The numerical results suggest a behavior like

$$\mathrm{cond}(\mathscr{T}(a_{s_0,s_1})) \sim \max\left\{\left|\frac{s_0}{s_1}\right|, \left|\frac{s_1}{s_0}\right|\right\}.$$

In particular, it seems that it is independent of the angle between the parameters s_0 and s_1 as long as $s_1/s_0 \notin \mathbb{R}_{\leq 0}$. Since we used parameters s_0 and s_1 with negative real part, this is no restriction.

These results indicate that it seems reasonable to choose parameters with negative real part and almost the same absolute value under the constraint

$$\sqrt{-\frac{|s_0|^2\Im(s_1) + |s_1|^2\Im(s_0)}{\Im(s_0 + s_1)}} = \zeta,$$

where ζ is a wave-number with vanishing group velocity. If there is some knowledge about the exact solution, one might try to solve a minimization problem as above. Otherwise, due to the scaling of ζ with a scaling of the parameters, one could take the parameters given in the caption of Fig. 5 for $\zeta = 1$ and scale them to the correct value of ζ.

7 Conclusion

We have presented a method for solving scattering problems with solutions which consist of modes with phase velocities of different signs. To construct this method we have used the modes, but the method itself is independent of the modes. Hence, it can easily be applied to problems for which the modes are complicated to compute. Moreover, when solving resonance problems for which the frequency is the unknown resonance, the method leads to a linear matrix eigenvalue problem.

Section 4 contains all results on the method which are independent of our specific model problem. In particular, we have derived convenient matrix representations of the involved operators. When applying the method to other problems, only these matrices have to be used and implemented. In Sect. 6.2 and Remark 4.4 we have seen, how to choose the main parameters s_0 and s_1 of the method. With all these data it is not difficult to apply the method to other problems.

Acknowledgments We would like to thank Timo Weidl for suggesting the model problem to us. Moreover, the authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

Appendix

A.1 Hardy spaces

In this section we collect the definitions and the required properties of standard and non-standard Hardy spaces. Proofs which are not carried out here can be found in [6] or [11]. We denote complex contour integrals by $\int ds$ and line integrals by $\int |ds|$.

Definition A.1 *(Hardy space on the unit disk and its complement)* Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle. The Hardy spaces $H^{\pm}(S^1)$ are the sets of all functions $U_{bd} \in L^2(S^1)$ for which there exists a holomorphic function U_{vol} on $\{z \in \mathbb{C} : |z|^{\pm 1} < 1\}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |U_{vol}(r^{\pm 1}e^{i\theta})|^2 d\theta < \infty \quad \text{and} \quad \lim_{r \nearrow 1} \int_0^{2\pi} |U_{bd}(i\theta) U_{vol}(r^{\pm 1}e^{i\theta})|^2 d\theta = 0. \tag{A.1}$$

We say that U_{bd} are L^2-boundary values of U_{vol}.

It can be shown that U_{vol} is uniquely determined by U_{bd} and can be computed by Cauchy's integral formula. This one-to-one correspondence of U_{vol} and U_{bd} explains the terminology. Note that for the Hardy space $H^+(S^1)$ on the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ the first condition in (A.1) is not required.

Lemma A.2 (Properties of $H^+(S^1)$) *Equipped with the L^2-inner product, $H^+(S^1)$ is a Hilbert space. A complete orthogonal basis of $H^+(S^1)$ is given by the monomials z^n, $n = 0, 1, \ldots$. Moreover, $L^2(S^1) = H^+(S^1) \oplus^{\perp} H^-(S^1)$.*

For $\kappa_0 \in \mathbb{C} \setminus \{0\}$ the Moebius transformation $m_{\kappa_0} : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is defined by $m_{\kappa_0}(z) := i\kappa_0 \frac{z+1}{z-1}$. Note that $m_{\kappa_0}(S^1 \setminus \{1\}) = \kappa_0 \mathbb{R}$ with $\kappa_0 \mathbb{R} := \{\kappa_0 x : x \in \mathbb{R}\}$. Therefore, we can define the pull-back operators

$$\mathcal{M}_{\kappa_0} : L^2(\kappa_0 \mathbb{R}) \to L^2(S^1), \qquad (\mathcal{M}_{\kappa_0} f)(z) := \frac{(f \circ m_{\kappa_0})(z)}{z-1},$$

where the weight factor $\frac{1}{z-1}$ is chosen such that $\mathscr{M}_{\kappa_0}$ is well-defined and unitary up to the factor $\sqrt{-2i\kappa_0}$.

Definition A.3 *(Hardy space on half planes)* For $\kappa \in \mathbb{C}\backslash\{0\}$ the Hardy spaces $H^{\pm}(\kappa_0\mathbb{R})$ on the half-planes $\{\kappa_0(x \pm iy) : x \in \mathbb{R}, y > 0\}$ are defined by $H^{\pm}(\kappa_0\mathbb{R}) := \mathscr{M}_{\mp\kappa_0}^{-1} H^+(S^1)$. ($\kappa_0\mathbb{R}$ has to be considered as an oriented curve to distinguish between these spaces!) They are equipped with the inner product $(f,g)_{H^{\pm}(\kappa_0\mathbb{R})} := \int_{\kappa_0\mathbb{R}} f(s)\overline{g(s)}|ds|$. For $\kappa_0 = 1$ we will omit the parameter and shortly write $H^{\pm}(\mathbb{R})$.

The definition differs from the one given in most text books where $H^-(\mathbb{R})$ is characterized as the set of all functions $U_{bd} \in L^2(\mathbb{R})$ that are L^2 boundary values of a function U_{vol} which is analytic in $\mathbb{C}^- := \{z \in \mathbb{C} : \Im z < 0\}$ and for which the integrals $\int_{\mathbb{R}} |U_{vol}(x-iy)|^2 dx$ are uniformly bounded for $y \in (0,\infty)$. Due to [11] it is clear that these definitions are equivalent. We stick to Definition A.3 because it is easier to generalize to other boundaries.

Lemma A.4 (Properties of $H^-(\mathbb{R})$)

(1) *$U_{bd} \in L^2(\mathbb{R})$ belongs to $H^-(\mathbb{R})$ if and only if there exists an analytic function U_{vol} in $\mathbb{C}^-$ and a sequence of rectifiable Jordan curves $C_1, C_2, \ldots$ tending to the boundary in $\mathbb{C}^-$ such that the integrals $\int_{C_n} |U_{vol}(z)|^2|dz|$ are uniformly bounded and $U_{vol}(s) \to U_{bd}(t)$ for $s \in \mathbb{C}^-$, $s \to t$ non-tangentially for almost all $t \in \mathbb{R}$. With C_n tending to the boundary we mean that C_n eventually surrounds each compact subset of $\mathbb{C}^-$.*

(2) *$\mathscr{M}_1$ is unitary from $H^-(\mathbb{R})$ to $H^+(S^1)$.*

(3) *Equipped with the $L^2(\mathbb{R})$-inner product $H^-(\mathbb{R})$ is a Hilbert space with orthonormal basis $\frac{2i}{s-i}\left(\frac{s+i}{s-i}\right)^n$, $n = 0, 1, \ldots$.*

(4) $L^2(\mathbb{R}) = H^+(\mathbb{R}) \oplus^{\perp} H^-(\mathbb{R})$.

(5) *Let $\Lambda \subset \mathbb{C}^+ := \{s \in \mathbb{C} : \Im s > 0\}$ be an infinite set with a cluster point in $\mathbb{C}^+$. Then $H^-(\mathbb{R}) = \overline{\operatorname{span}\{\frac{1}{\bullet-\lambda} : \lambda \in \Lambda\}}^{H^-(\mathbb{R})}$.*

Proof The first four points follow from [6, Sect. 11, Example 1] and the definition of $H^-(\mathbb{R})$. The last one can be found in [12, Lemma A.2]. □

Assumption A.5 Let $\Gamma = \gamma(\mathbb{R})$ be an oriented curve parameterized by a twice continuously differentiable function $\gamma : \mathbb{R} \to \mathbb{C}$ of the form

$$\gamma(\rho) = \rho\sigma(|\rho|), \qquad \rho \in \mathbb{R} \tag{A.2}$$

with $\sigma : [0,\infty) \to S^1$, and suppose that $\sup_{\rho\in\mathbb{R}} |\gamma'(\rho)| < \infty$ and there exists $\sigma_\infty \in S^1$ such that $\lim_{|\rho|\to\infty} |\gamma(\rho) - \sigma_\infty\rho| = 0$.

Note that any curve Γ satisfying Assumption A.5 is point symmetric, i.e. $-\Gamma = \Gamma$ and separates $\mathbb{C}$ into the unbounded, simply connected sets

$$\Gamma^{\pm} := \{\gamma(\rho)\exp(\pm it) : \rho > 0, t \in (0,\pi)\}, \tag{A.3}$$

i.e. $\mathbb{C} = \Gamma^- \dot\cup \Gamma \dot\cup \Gamma^+$.

Lemma A.6 *If Γ satisfies Assumption A.5, there exists a bijective, continuously differentiable mapping $\eta\colon \mathbb{C}^-\cup\mathbb{R} \to \Gamma^-\cup\Gamma$ such that η is conformal on $\mathbb{C}^-$, $\eta(\mathbb{R}) = \Gamma$ and* log *can be defined analytically on $\eta'(\mathbb{C}^-)$.*

Proof Due to the Riemann mapping theorem, see e.g. [3] there exists a conformal bijective mapping $\eta\colon \mathbb{C}^- \to \Gamma^-$. Since $\mathbb{C}^-$ is simply connected, so is $\eta'(\mathbb{C}^-)$. As η is conformal, we have $0 \notin \eta'(\mathbb{C}^-)$, and the logarithm can be defined analytically. We need to show that η can be extended with the stated properties. If Γ^- would be bounded, [16, Sect. 3.3] would already give us the claimed extension. As the stated properties are of local nature, we can generalize the results in [16, Sect. 3.3]. □

Definition A.7 Let Γ fulfill Assumption A.5 and let η be a mapping as described in Lemma A.6. Let $\mathcal{N}_\eta : L^2(\Gamma) \to L^2(\mathbb{R})$ with $(\mathcal{N}_\eta f)(s) := (f\circ\eta)(s)\sqrt{\eta'(s)}$ for $s\in\mathbb{R}$ and $f\in L^2(\Gamma)$, and $\mathcal{N}_\eta^{-1} : L^2(\mathbb{R}) \to L^2(\Gamma)$ with $\mathcal{N}_\eta^{-1} g = \frac{g}{\sqrt{\eta'}}\circ\eta^{-1}$ for $g\in L^2(\mathbb{R})$.
Then we define the Hardy space $H(\Gamma)^\pm$ by $H(\Gamma)^\pm := \mathcal{N}_{\mp\eta}^{-1} H^-(\mathbb{R})$ with the inner product $(f,g)_{H^-(\Gamma)} = \int_\Gamma f(s)\bar{g}(s)|ds|$.

Due to Lemma A.6 $\mathcal{N}_\eta$ and $\mathcal{N}_\eta^{-1}$ are well defined. $H^-(\Gamma)$ is independent of the choice of η since for two such mappings η_1 and η_2 the composition $\eta_2^{-1}\circ\eta_1 : \mathbb{C}^- \to \mathbb{C}^l$ is an automorphism of $\mathbb{C}^-$ with ∞ as invariant point, and $H^-(\mathbb{R})$ is invariant under such transformations.

Lemma A.8 (Properties of $H^-(\Gamma)$) *Let Assumption A.5 hold true. Then*

(1) *$U_{bd}\in L^2(\Gamma)$ belongs to $H^-(\Gamma)$ if and only if there exists an analytic function U_{vol} in Γ^- and a sequence of rectifiable Jordan curves $C_1, C_2, \dots$ tending to the boundary in Γ^- such that the integrals $\int_{C_n}|U_{vol}(s)|^2|ds|$ are uniformly bounded and $U_{vol}(s)\to U_{bd}(t)$ for $s\in\Gamma^-$, $s\to t$ non-tangentially for almost all $t\in\Gamma$.*
(2) *$\mathcal{N}_\eta$ is an isometry between $H^-(\Gamma)$ and $H^-(\mathbb{R})$.*
(3) *Equipped with the $L^2(\Gamma)$ inner product $H^-(\Gamma)$ is a Hilbert space.*
(4) *$\frac{1}{\bullet-\lambda}\in H^-(\Gamma)$ for all $\lambda\in\Gamma^+$.*
(5) *For $U_{bd}\in H^-(\Gamma)$ there holds $U_{vol}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{U_{bd}(s)}{s-z}ds$ for all $z\in\Gamma^-$ and $\frac{1}{2\pi i}\int_\Gamma\frac{U_{bd}(s)}{s-z}ds = 0$ for all $z\in\Gamma^+$.*

Proof The first three points are obvious. (4) We use A.8(1) as characterization of $H^-(\Gamma)$. Since $\frac{1}{s-\lambda}$ is analytic in Γ^-, we need to state a sequence of rectifiable Jordan curves C_n tending to the boundary in Γ^- such that the integrals $\int_{C_n}\frac{1}{|z-\lambda|^2}|dz|$ are uniformly bounded. Such curves can be constructed by choosing

$$\gamma_n(\rho) := \gamma(\rho)\exp\left(-\frac{i}{n}\frac{\rho}{1+\rho^2}\right), \qquad \rho\in\mathbb{R},\quad n\in\mathbb{N}.$$

The curves $C_n^{(1)} := \gamma_n([-n,n])$ are closed by circular arcs $C_n^{(2)} := \{\gamma(n)\exp(-it) : \frac{n}{1+n^2} \le t \le \pi - \frac{n}{1+n^2}\}$, i.e. $C_n := C_n^{(1)}\cup C_n^{(2)}$. It is easy to see that $C_n \subset \Gamma^-$

and that the curves C_n eventually surround any compact subset of Γ^-. The integrals $\int_{C_n^{(1)}} |s-\lambda|^{-2}\,|ds| = \int_{-n}^{n} |\gamma_n(\rho)-\lambda|^{-2}|\gamma_n'(\rho)|d\rho$ are uniformly bounded since $\sup_{n\in\mathbb{N}} |\gamma_n(\rho)-\lambda|^{-2} = O(\rho^{-2})$ for $|\rho|\to\infty$, $\sup_{n\in\mathbb{N},\rho\in\mathbb{R}} |\gamma_n(\rho)-\lambda|^{-2} < \infty$, and $\sup_{n\in\mathbb{N},\rho\in\mathbb{R}} |\gamma_n'(\rho)| \le \sup_{n\ge n_0,\rho\in\mathbb{R}} |\gamma'(\rho)| + |\gamma(\rho)|^2 n^{-1}(1-\rho^2)(1+\rho^2)^{-2} < \infty$. Similarly, it is easy to see that $\int_{C_n^{(2)}} |s-\lambda|^{-2}\,|ds|$ is uniformly bounded.

(5) Let $z\in\Gamma^-$ be given. Using the mapping $\eta:\mathbb{R}\to\Gamma$ defined in Lemma A.6 Cauchy's integral Theorem leads together with Lemma A.8 to

$$\begin{aligned}
U_{vol}(z) &= \lim_{y\nearrow 0}\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\eta([-R+iy,R+iy]\cup\{R\exp(i[0,\pi])+iy\})}\frac{U_{vol}(s)}{s-z}ds\\
&= \frac{1}{2\pi i}\lim_{y\nearrow 0}\int_{\mathbb{R}}(\mathcal{N}_\eta U_{vol})(x+iy)\left(\mathcal{N}_\eta\left\{(\bullet-z)^{-1}\right\}\right)(x+iy)\,dx\\
&= \frac{1}{2\pi i}\int_{\mathbb{R}}(\mathcal{N}_\eta U_{bd})(x)\left(\mathcal{N}_\eta\left\{(\bullet-z)^{-1}\right\}\right)(x)\,dx = \frac{1}{2\pi i}\int_{\Gamma}\frac{U_{bd}(s)}{s-z}ds.
\end{aligned}$$

The second statement follows analogously since for $z\in\Gamma^+$ the integrand is holomorphic in Γ^-. □

Lemma A.9 *Let Γ satisfy Assumption A.5.*

(1) *The generalized Hilbert transform $\mathcal{H}_\Gamma : L^2(\Gamma)\to L^2(\Gamma)$,*

$$(\mathcal{H}_\Gamma U)(z) := P.V.\frac{1}{\pi i}\int_\Gamma\frac{U(s)}{s-z}ds := \lim_{\epsilon\searrow 0,R\to\infty}\left(\frac{1}{\pi i}\int_{(\Gamma\cap B_R(z)\setminus B_\epsilon(z)}\frac{U(s)}{s-z}ds\right)$$

is a well-defined bounded linear operator.

(2) *For every $U\in L^2(\Gamma)$ the Sokhotski-Plemelj jump relations*

$$\lim_{\lambda\to z,\lambda\in\Gamma^\pm}\frac{1}{2\pi i}\int_\Gamma\frac{U(s)}{s-\lambda}ds = \frac{1}{2}\left(U(z)\mp\mathcal{H}_\Gamma U(z)\right),\quad z\in\Gamma$$

hold true in the L^2 sense (see Lemma A.8.)

(3) *The operators $\mathcal{P}^\pm := \frac{1}{2}(\mathrm{id}\mp\mathcal{H}_\Gamma)$ are bounded linear projections of $L^2(\Gamma)$ onto $H^\pm(\Gamma)$ along $H^\mp(\Gamma)$. In particular, we have the topological sum*

$$L^2(\Gamma) = H^+(\Gamma)\oplus H^-(\Gamma).$$

Proof (1) Since $1\le|\gamma'(\rho)|\le C<\infty$ for all $\rho\in\mathbb{R}$, the pull-back $L^2(\Gamma)\to L^2(\mathbb{R})$, $U\mapsto\widetilde{U} := U\circ\gamma$ is bounded and boundedly invertible. Therefore, it suffices to show that the linear operator $\tilde{\mathcal{H}}_\Gamma : L^2(\mathbb{R})\to L^2(\mathbb{R})$, $\tilde{U}\mapsto(\mathcal{H}_\Gamma U)\circ\gamma$ is well-defined and bounded. In the numerator of the kernel of $\mathcal{H}_\Gamma$ we add and subtract a smooth, symmetric cut-off function $\chi:\mathbb{R}\to[0,1]$ satisfying $\chi(\rho)=1$ for $|\rho|\le 1$ and $\chi(\rho)=0$ for $|t|\ge 2$. Then

$$\begin{aligned}\pi i(\widetilde{\mathscr{H}_\Gamma}\widetilde{U})(r) &= P.V.\int_{\mathbb{R}}\frac{\chi(\rho-r)}{\gamma(\rho)-\gamma(r)}\widetilde{U}(\rho)\gamma'(\rho)d\rho\\ &\quad+P.V.\int_{\mathbb{R}}\frac{1-\chi(\rho-r)}{\gamma(\rho)-\gamma(r)}\widetilde{U}(\rho)\gamma'(\rho)d\rho\\ &=: (T_1\widetilde{U})(r)+(T_2\widetilde{U})(r).\end{aligned}$$

To show L^2-boundedness of T_1 we will use smoothness of Γ and for L^2-boundedness of T_2 the behavior of Γ at infinity. We split off the singularity of T_1 as a convolution operator by writing $\gamma(\rho)-\gamma(r)=\gamma'(\rho)(\rho-r+R(\rho,r))$ with the scaled Taylor remainder $R(\rho,r):=\frac{\gamma(\rho)-\gamma(r)}{\gamma'(\rho)}-(\rho-r)$. Then

$$\begin{aligned}(T_1\widetilde{U})(r) &= P.V.\int_{|\rho-r|\le 2}\frac{\chi(\rho-r)}{\rho-r}\widetilde{U}(\rho)d\rho\\ &\quad-\int_{|\rho-r|\le 2}\frac{\chi(\rho-r)R(\rho,r)}{(\rho-r)(\gamma(\rho)-\gamma(r))}\gamma'(\rho)\widetilde{U}(\rho)d\rho.\end{aligned}\tag{A.4}$$

The L^2-boundedness of the convolution operator can e.g. be derived from the L^2-boundedness of the classical Hilbert transform since the Fourier transforms of the convolution kernel is a convolution of the (distributional) Fourier transform of Hilbert transform kernel $t\mapsto 1/t$ and the Fourier transform of χ. Hence, we have the Fourier transform of an L^∞ and an L^1-function, which belongs to L^∞. As $|\gamma(\rho)-\gamma(r)|\ge|\rho-r|$ and $|R(\rho,r)|\le C|\rho-r|^2$ with a constant $C>0$ independent of t,r with $|\rho-r|\le 2$, the kernel of the second integral operator is uniformly bounded. Together with boundedness of the integration domain this easily yields L^2-boundedness.

The operator T_2 can be decomposed into

$$\begin{aligned}(T_2\widetilde{U})(r) &= P.V.\int\limits_{|\rho-r|\ge 1}\frac{1-\chi(\rho-r)}{\sigma_\infty\rho-\sigma_\infty r}\gamma'(\rho)\widetilde{U}(\rho)d\rho\\ &\quad-\int\limits_{|\rho-r|\ge 1}(1-\chi(\rho-r))\frac{(\gamma(\rho)-\sigma_\infty\rho)-(\gamma(r)-\sigma_\infty r)}{(\gamma(\rho)-\gamma(r))(\rho-r)\sigma_\infty}\gamma'(\rho)\widetilde{U}(\rho)d\rho.\end{aligned}$$

As the first integral equals $\frac{1}{\sigma_\infty}P.V.\int\frac{1}{\rho-r}\gamma'(\rho)\widetilde{U}(\rho)d\rho-\frac{1}{\sigma_\infty}P.V.\int\frac{\chi(\rho-r)}{\rho-r}\gamma'(\rho)\widetilde{U}(\rho)d\rho$, it defines a linear operator on $L^2(\mathbb{R})$ due to the L^2 boundedness of the convolution operator in (A.4) and boundedness of the classical Hilbert transform. As $\sup_{\rho\in\mathbb{R}}|\gamma(\rho)-\sigma_\infty\rho|<\infty$ by Assumption A.5 and $|\gamma(\rho)-\gamma(r)|>|\rho-r|$, the kernel $k(\rho,r)$ of the second integral operator in the decomposition of T_2 satisfies $|k(\rho,r)|\le C|\rho-r|^{-2}$ for some $C>0$ and all $|\rho-r|\ge 1$. This bound implies L^2-boundedness since by the Cauchy-Schwarz inequality and the identity $\int_{|\rho|\ge 1}\frac{1}{\rho^2}d\rho=2$ we have

$$\int \left| \int_{|\rho-r|\geq 1} k(\rho,r)\widetilde{U}(\rho)d\rho \right|^2 dr \leq C^2 \int \int_{|\rho-r|\geq 1} \frac{1}{(\rho-r)^2} d\rho \int_{|\rho-r|\geq 1} \frac{|\widetilde{U}(\rho)|^2}{(\rho-r)^2} d\rho\, dr$$
$$= 2C^2 \int \int_{|\rho-r|\geq 1} \frac{1}{(\rho-r)^2} dr |\widetilde{U}(\rho)|^2 d\rho$$
$$= 4C^2 \|\widetilde{U}\|_{L^2}^2.$$

(2) This easily follows from the corresponding result for closed curves in [3] using a partition of unity.
(3) It follows from part 2 that $\mathscr{P}^{\pm}U \in H^{\pm}(\Gamma)$ for all $U \in L^2(\Gamma)$. Together with Lemma A.8 we obtain $\mathscr{P}^{\pm}U_{\pm} = U_{\pm}$ and $\mathscr{P}^{\mp}U_{\pm} = 0$ for $U_{\pm} \in H^{\pm}(\Gamma)$. □

Lemma A.10 *Let Assumption A.5 hold true and let $\Lambda^{\pm} \subset \Gamma^{\pm}$ be two infinite sets with cluster points in $\Gamma^{\pm}$. Then $H^{\mp}(\Gamma) = \overline{\mathrm{span}\{\frac{1}{\bullet-\lambda} : \lambda \in \Lambda^{\pm}\}}^{H^{\mp}(\Gamma)}$.*

Proof Let $\Lambda^{\pm} \subset \Gamma^{\pm}$ be two infinite sets with cluster points in $\Gamma^{\pm}$. We prove that $\mathrm{span}\{(\bullet-\lambda)^{-1} : \lambda \in \Lambda^+ \cup \Lambda^-\}$ is dense in $L^2(\Gamma)$. Then the assertion follows with A.9(3) and $(\bullet-\lambda)^{-1} \in H^{\mp}(\Gamma)$ for $\lambda \in \Lambda^{\pm}$ due to A.8(4).

Take $U \in L^2(\Gamma)$ such that $U \perp_{L^2(\Gamma)} (s-\lambda)^{-1}$ for all $\lambda \in \Lambda^+ \cup \Lambda^-$. Hence with the tangent function $\mathfrak{t}(s) \neq 0$ there holds $\int_{\Gamma} \overline{U(s)\mathfrak{t}(s)}(s-\lambda)^{-1}ds = 0$ for all $\lambda \in \Lambda^+ \cup \Lambda^-$. Hence, the function $g : \Lambda^+ \cup \Lambda^- \to \mathbb{C}$ with $g(\lambda) := \int_{\Gamma} \overline{U(s)\mathfrak{t}(s)}(s-\lambda)^{-1}ds$ is holomorphic in $\Gamma^+ \cup \Gamma^+$ and vanishes in $\Lambda^+ \cup \Lambda^-$. Therefore it vanishes in $\mathbb{C}\backslash\Gamma$ and with A.9(2) it holds $U = 0$. □

A.2 Toeplitz operators

When working with operator theory on Hardy spaces, Toeplitz operators are a most natural class of operators to deal with. From the rich theory of Toeplitz operator we only need a few rather simple results, which can all be found, e.g., in [5].

Definition A.11 (*Multiplication and Toeplitz operator*) Let H be a Hilbert space and $\mathscr{B}(H)$ the set of bounded linear operators on H. For a function $a \in L^{\infty}(S^1)$ we define the multiplication operator $\mathscr{Q}(a) \in \mathscr{B}(L^2(S^1))$ point-wise by $(\mathscr{Q}(a)u)(z) := a(z)u(z)$ for all $z \in S^1$ and $u \in L^2(S^1)$.

Moreover, if $\mathscr{P}$ denotes the orthogonal projection from $L^2(S^1)$ to $H^+(S^1)$, the Toeplitz operator $\mathscr{T}(a) \in \mathscr{B}(H^+(S^1))$ with symbol a is defined by $\mathscr{T}(a) := \mathscr{P}\mathscr{Q}(a)$.

Definition A.12 *(Toeplitz matrix)* An infinite matrix $T \in \mathscr{B}(l^2(\mathbb{N}_0))$, $(Tf)_n := \sum_{m=0}^{\infty} T_{n,m} f_m$ is called an (infinite) Toeplitz matrix if $T_{n,m} = T_{n+1,m+1}$ for all $n, m \in \mathbb{N}_0$. For a Toeplitz operator $\mathscr{T}(a) : H^+(S^1) \to H^+(S^1)$ with symbol $a \in L^{\infty}(S^1)$ the associated infinite Toeplitz matrix $T(a)$ is given as $T(a)_{n,m} := \int_{S^1} a(z)z^{n-m}|dz|$ for all $n, m \in \mathbb{N}_0$.

Note that $T(a)$ is the matrix representation of $\mathscr{T}(a)$ with respect to the orthonormal basis $\{z^k,\ k \in \mathbb{N}_0\}$ of $H^+(S^1)$.

Vice versa, let $T \in \mathscr{B}(l^2(\mathbb{N}_0))$ be an infinite Toeplitz matrix. Note that $T_{n,m}$ only depends on the difference of the indices, i.e. $T_{n,m} = a_{n-m}$, $n, m \in \mathbb{N}_0$ for some numbers $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$. Then it is easy to show that $T = T(a)$ is the Toeplitz matrix associated to the Toeplitz operator $\mathscr{T}(a) : H^+(S^1) \to H^+(S^1)$ with symbol $a \in L^\infty(S^1)$ given by $a(z) := \sum_{k\in\mathbb{Z}} a_k z^k$, $z \in S^1$.

Due to this one-to-one correspondence of Toeplitz operators and Toeplitz matrices the following results can be formulated both for Toeplitz operators and Toeplitz matrices.

Proposition A.13 *If* $a \in C(S^1)$ *and* $\{a(z), z \in S^1\}$ *is a line segment, then* $\operatorname{spec}\mathscr{T}(a) = \{a(z), z \in S^1\}$.

Proof Follows from [5, Theorem 1.17]. □

In the following we consider matrix valued operators.

Definition A.14 (*Block operator*) For $a \in [L^\infty(S^1)]^{2\times 2}$ we define the block multiplication operator $\mathscr{Q}(a) \in \mathscr{B}([L^2(S^1)]^2)$ by $(\mathscr{Q}(a)u)\,(z) := a(z)\,u(z)$ for $u \in [L^2(S^1)]^2$ and $z \in S^1$.

Using the orthogonal projection $\mathscr{P}\colon [L^2(S^1)]^2 \to [H^+(S^1)]^2$, we define the block Toeplitz operator $\mathscr{T}(a) \in \mathscr{B}([H^+(S^1)]^2)$ by $\mathscr{T}(a)u := \mathscr{P}\mathscr{Q}(a)u$ for $u \in [H^+(S^1)]^2$.

Definition A.15 (*Block Toeplitz matrix*) An infinite matrix $T \in \mathscr{B}(l^2(\mathbb{N}_0))$ with $(Tf)_n := \sum_{m=0}^\infty T_{n,m} f_m$ is called an (infinite) 2×2 block Toeplitz matrix if $T_{n,m} = T_{n+2,m+2}$ for all $n, m \in \mathbb{N}_0$. For the block Toeplitz operator $\mathscr{T}(a) : [H^+(S^1)]^2 \to [H^+(S^1)]^2$ with symbol $a \in [L^\infty(S^1)]^{2\times 2}$ the associated infinite 2×2 block Toeplitz matrix $T(a)\colon l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ is given by

$$\begin{pmatrix} T(a)_{2n,2m} & T(a)_{2n,2m+1} \\ T(a)_{2n+1,2m} & T(a)_{2n+1,2m+1} \end{pmatrix} := \int_{S^1} z^{n-m} a(z)|dz|, \qquad n, m \in \mathbb{N}_0. \tag{A.5}$$

Note that $T(a)$ is the matrix representation of $\mathscr{T}(a)$ with respect to the orthonormal basis $\left\{ \binom{z^0}{0}, \binom{0}{z^0}, \binom{z^1}{0}, \binom{0}{z^1}, \dots \right\}$ of $[H^+(S^1)]^2$. If T is an infinite 2×2 block Toeplitz matrix, there exists 2×2 matrices $a_k \in \mathbb{C}^{2\times 2}$ for $k \in \mathbb{Z}$ such that (A.5) holds true with the right hand side replaced by a_{n-m}. Then $T = T(a)$ can be shown to be associated to the block Toeplitz operator $\mathscr{T}(a)$ with symbol $a(z) := \sum_{k\in\mathbb{Z}} z^k a_k$ for $z \in S^1$. Hence, there is again a one-to-one correspondence of block Toeplitz operators and block Toeplitz matrices.

Theorem A.16 *If* $a \in [C(S^1)]^{2\times 2}$ *and* $a(z)$ *is Hermitian positive definite for all* $z \in S^1$, *then* $\operatorname{spec}\mathscr{T}(a)$ *is contained in the convex hull of* $S := \bigcup_{z\in S^1}\{spec a(z)\}$.

Proof This follows from the Lemma of Lax-Milgram: If $\lambda > \sup(S)$, then

$$\begin{aligned} \langle u, (\lambda \mathrm{id} - \mathscr{T}(\mathrm{a}))\mathrm{u}\rangle_{[H^+(S^1)]^2} &= \int_{S^1} u(z)^H (\lambda \mathrm{id} - \mathrm{a(z)})\mathrm{u(z)}|\mathrm{dz}| \\ &\geq (\lambda - \sup(S))\|u\|^2_{[L^2(S^2)]^2} \end{aligned}$$

so $\lambda \notin \operatorname{spec}\mathscr{T}(a)$. Here v^H denotes the Hermitian transpose of a vector v. The cases $\lambda < \inf(S)$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ can be treated similarly. □

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