On Compiling Structured CNFs to OBDDs

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Abstract. We present new results on the size of OBDD representations of structurally characterized classes of CNF formulas. First, we prove that *variable convex* formulas (that is, formulas with incidence graphs that are convex with respect to the set of variables) have polynomial OBDD size. Second, we prove an exponential lower bound on the OBDD size of a family of CNF formulas with incidence graphs of bounded degree.

We obtain the first result by identifying a simple sufficient condition—which we call the *few subterms* property—for a class of CNF formulas to have polynomial OBDD size, and show that variable convex formulas satisfy this condition. To prove the second result, we exploit the combinatorial properties of expander graphs; this approach allows us to establish an exponential lower bound on the OBDD size of formulas satisfying strong syntactic restrictions.

1 Introduction

The goal of *knowledge compilation* is to succinctly represent propositional knowledge bases in a format that supports a number of queries in polynomial time [8]. Choosing a representation language generally involves a trade-off between succinctness and the range of queries that can be efficiently answered. In this paper, we study ordered binary decision diagram (OBDD) representations of propositional theories given as formulas in conjunctive normal form (CNF). Binary decision diagrams (also known as branching programs) and their variants are widely used and well-studied representation languages for Boolean functions [24]. OBDDs in particular enjoy properties, such as polynomial-time equivalence testing, that make them the data structure of choice for a range of applications.

Perhaps somewhat surprisingly, the question of which classes of CNFs can be represented as (or *compiled* into, in the jargon of knowledge representation) OBDDs of polynomial size is largely unexplored [24, Chapter 4]. We approach this classification problem by considering *structurally* characterized CNF classes, more specifically, classes of CNF formulas defined in terms of properties of their *incidence graphs* (the incidence graph of a formula is the bipartite graph on clauses and variables where a variable is adjacent to the clauses it occurs in). Figure 1 depicts a hierarchy of well-studied bipartite graph classes as considered by Lozin and Rautenbach [19, Fig. 2]. This hierarchy is particularly well-suited

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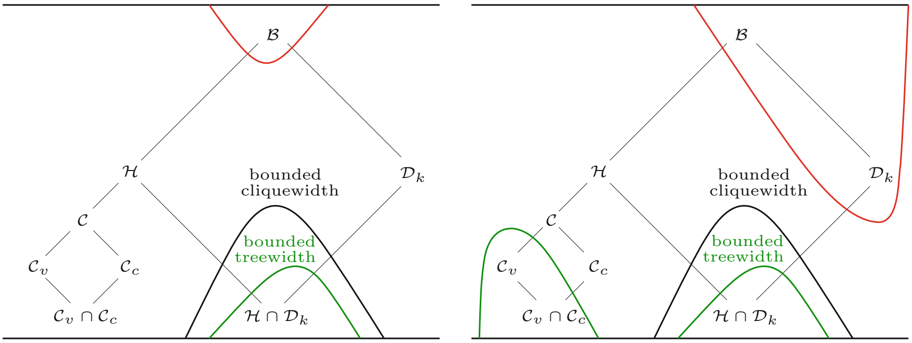

Fig. 1. The diagram depicts a hierarchy of classes of bipartite graphs under the inclusion relation (thin edges). $\mathcal{B}$, $\mathcal{H}$, $\mathcal{D}_k$, $\mathcal{C}$, $\mathcal{C}_v$, and $\mathcal{C}_c$ denote, respectively, bipartite graphs, chordal bipartite graphs (corresponding to beta acyclic CNFs), bipartite graphs of degree at most k ($k \geq 3$), convex graphs, left (variable) convex graphs, and right (clause) convex graphs. The class $\mathcal{C}_v \cap \mathcal{C}_c$ of biconvex graphs and the class $\mathcal{D}_k$ of bipartite graphs of degree at most k have unbounded clique-width. The class $\mathcal{H} \cap \mathcal{D}_k$ of chordal bipartite graph of degree at most k has bounded treewidth. The green and red curved lines enclose, respectively, classes of incidence graphs whose CNFs have polynomial time OBDD compilation, and classes of incidence graphs whose CNFs have exponential size OBDD representations; the right hand picture shows the compilability frontier, updated in light of Results 1 and 2.

for our classification project as it includes prominent cases such as beta acyclic CNFs [5] and bounded clique-width CNFs. When located within this hierarchy, the known bounds on the OBDD size of structural CNF classes leave a large gap (depicted *on the left* of Fig. 1):

- On the one hand, we have a polynomial upper bound on the OBDD size of bounded treewidth CNF classes proved recently by Razgon [22]. The corresponding graph classes are located at the bottom of the hierarchy.
- On the other hand, there is an exponential lower bound for the OBDD size of general CNFs, proved two decades ago by Devadas [9]. The corresponding graph class is not chordal bipartite, has unbounded degree and unbounded clique-width, and hence is located at the top of the hierarchy.

Contribution. In this paper, we tighten this gap as illustrated *on the right* in Fig. 1. More specifically, we prove new bounds for two structural classes of CNFs.

Result 1. CNF formulas with *variable convex* incidence graphs have polynomial OBDD size (Theorem 7).

Convexity is a property of bipartite graphs that has been extensively studied in the area of combinatorial optimization [13,14,23], and that can be detected in linear time [4,18].

To prove Result 1, we define a property of CNF classes—called the *few subterms property*—that is sufficient for polynomial-size compilability (Theorem 4), and then prove that CNFs with variable convex incidence graphs have this property (Lemma 6). The few subterms property naturally arises as a sufficient condition for polynomial size compilability when considering OBDD representations of CNF formulas (cf. Oztok and Darwiche's recent work on *CV-width* [21], which explores a similar idea). Aside from its role in proving polynomial-size compilation for variable convex CNFs, the few subterms property can also be used to explain the (known) fact that classes of CNFs with incidence graphs of *bounded treewidth* have OBDD representations of polynomial size (Lemma 9), and as such offers a unifying perspective on these results. Both the result on variable convex CNFs and the result on bounded treewidth CNFs can be improved to polynomial *time* compilation by appealing to a stronger version of the few subterms property (Theorems 7 and 10).

In an attempt to push the few subterms property further, we adopt the language of *parameterized complexity* to formally capture the idea that CNFs "close" to a class with few subterms have "small" OBDD representations. More precisely, defining the *deletion distance* of a CNF from a CNF class as the number of its variables or clauses that have to be deleted in order for the resulting formula to be in the class, we prove that CNFs have fixed-parameter tractable OBDD size parameterized by the deletion distance from a CNF class with few subterms (Theorem 12). This result can again be improved to fixed-parameter *time* compilation under additional assumptions (Theorem 13), yielding for instance fixed-parameter tractable time compilation of CNFs into OBDDs parameterized by the *feedback vertex set* size (Corollary 14).

Result 2. There is a class of CNF formulas with incidence graphs of bounded degree such that every formula F in this class has OBDD size at least $2^{\Omega(\mathsf{size}(F))}$, where $\mathsf{size}(F)$ denotes the number of variable occurrences in F (Theorem 18).

This substantially improves on a $2^{\Omega(\sqrt{\mathsf{size}(F)})}$ lower bound for the OBDD size of a class of CNFs by Devadas [9]. Moreover, we establish this bound for a class that satisfies strong syntactic restrictions: every clause contains exactly two positive literals and each variable occurs at most 3 times.

The heavy lifting in our proof of this result is done by a family of *expander graphs*. Expander graphs have found applications in many areas of mathematics and computer science [15,20], including circuit and proof complexity [16]. In this paper, we show how they can be used to derive lower bounds for OBDDs.

Organization. The paper is organized as follows. In Sect. 2, we introduce basic notation and terminology. In Sect. 3, we prove that formulas with few subterms have polynomial OBDD size and show that variable-convex CNFs (as well as bounded treewidth CNFs) enjoy the few subterms property. Fixed-parameter tractable size and time compilability results based on the few subterms property are presented in Sect. 3.4. In Sect. 4, we prove a strongly exponential lower bound on the OBDD size of CNF formulas based on expander graphs. We conclude in Sect. 5.

Due to space constraints, several proofs have been omitted.

2 Preliminaries

Formulas. Let X be a countable set of *variables.* A *literal* is a variable x or a negated variable $\neg x$. If x is a variable we let $\mathsf{var}(x) = \mathsf{var}(\neg x) = x$. A *clause* is a finite set of literals. For a clause c we define $\mathsf{var}(c) = \{\mathsf{var}(l) \mid l \in c\}$. If a clause contains a literal negated as well as unnegated it is *tautological.* A *conjunctive normal form (CNF)* is a finite set of non-tautological clauses. If F is a CNF formula we let $\mathsf{var}(F) = \bigcup_{c \in F} \mathsf{var}(c)$. The *size* of a clause c is $|c|$, and the *size* of a CNF F is $\mathsf{size}(F) = \sum_{c \in F} |c|$. An *assignment* is a mapping $f \colon X' \to \{0,1\}$, where $X' \subseteq X$; we identify f with the set $\{\neg x \mid x \in X', f(x) = 0\} \cup \{x \mid x \in X', f(x) = 1\}$. An assignment f *satisfies* a clause c if $f \cap c \neq \emptyset$; for a CNF F, we let $F[f]$ denote the CNF containing the clauses in F not satisfied by f, restricted to variables in $X \setminus \mathsf{var}(f)$, that is, $F[f] = \{c \setminus \{x, \neg x \mid x \in \mathsf{var}(f)\} \mid c \in F, f \cap c = \emptyset\}$; then, f *satisfies* F if $F[f] = \emptyset$, that is, if it satisfies all clauses in F. If F is a CNF with $\mathsf{var}(F) = \{x_1, \ldots, x_n\}$ we define the Boolean function $F(x_1, \ldots, x_n)$ *computed by* F as $F(b_1, \ldots, b_n) = 1$ if and only if the assignment $f_{(b_1,\ldots,b_n)} : \mathsf{var}(F) \to \{0,1\}$ given by $f_{(b_1,\ldots,b_n)}(x_i) = b_i$ satisfies the CNF F.

Binary Decision Diagrams. A *binary decision diagram (BDD)* D on variables $\{x_1, \ldots, x_n\}$ is a labelled directed acyclic graph satisfying the following conditions: D has at at most two vertices without outgoing edges, called *sinks* of D. Sinks of D are labelled with 0 or 1; if there are exactly two sinks, one is labelled with 0 and the other is labelled with 1. Moreover, D has exactly one vertex without incoming edges, called the *source* of D. Each non-sink node of D is labelled by a variable x_i, and has exactly two outgoing edges, one labelled 0 and the other labelled 1. Each node v of D represents a Boolean function $F_v = F_v(x_1, \ldots, x_n)$ in the following way. Let $(b_1, \ldots, b_n) \in \{0,1\}^n$ and let w be a node labelled with x_i. We say that $(b_1, \ldots, b_n)$ *activates* an outgoing edge of w labelled with $b \in \{0,1\}$ if $b_i = b$. Since $(b_1, \ldots, b_n)$ activates exactly one outgoing edge of each non-sink node, there is a unique sink that can be reached from v along edges activated by $(b_1, \ldots, b_n)$. We let $F_v(b_1, \ldots, b_n) = b$, where $b \in \{0,1\}$ is the label of this sink. The function *computed by* D is F_s, where s denotes the (unique) source node of D. The *size* of a BDD is the number of its nodes.

An *ordering* σ of a set $\{x_1, \ldots, x_n\}$ is a total order on $\{x_1, \ldots, x_n\}$. If σ is an ordering of $\{x_1, \ldots, x_n\}$ we let $\mathsf{var}(\sigma) = \{x_1, \ldots, x_n\}$. Let σ be the ordering of $\{1, \ldots, n\}$ given by $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$. For every integer $0 < j \leq n$, the *length* j *prefix* of σ is the ordering of $\{x_{i_1}, \ldots, x_{i_j}\}$ given by $x_{i_1} < \cdots < x_{i_j}$. A *prefix* of σ is a length j prefix of σ for some integer $0 < j \leq n$. For orderings $\sigma = x_{i_1} < \cdots < x_{i_n}$ of $\{x_1, \ldots, x_n\}$ and $\rho = y_{i_1} < \cdots < y_{i_m}$ of $\{y_1, \ldots, y_m\}$, we let $\sigma\rho$ denote the ordering of $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ given by $x_{i_1} < \cdots < x_{i_n} < y_{i_1} < \cdots < y_{i_m}$. Let D be a BDD on variables $\{x_1, \ldots, x_n\}$ and let $\sigma = x_{i_1} < \cdots < x_{i_n}$ be an ordering of $\{x_1, \ldots, x_n\}$. The BDD D is a *σ-ordered binary decision diagram (σ-OBDD)* if $x_i < x_j$ (with respect to σ) whenever D contains an edge from a node labelled with x_i to a node labelled with x_j. A BDD

D on variables $\{x_1, \ldots, x_n\}$ is an *ordered binary decision diagram (OBDD)* if there is an ordering σ of $\{x_1, \ldots, x_n\}$ such that D is a σ-OBDD. For a Boolean function $F = F(x_1, \ldots, x_n)$, the *OBDD size* of F is the size of the smallest OBDD on $\{x_1, \ldots, x_n\}$ computing F.

We say that a class $\mathcal{F}$ of CNFs has *polynomial-time compilation into OBDDs* if there is a polynomial-time algorithm that, given a CNF $F \in \mathcal{F}$, returns an OBDD computing the same Boolean function as F. We say that a class $\mathcal{F}$ of CNFs *has polynomial size compilation into OBDDs* if there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ such that, for all CNFs $F \in \mathcal{F}$, there exists an OBDD of size at most $p(\mathsf{size}(F))$ that computes the same function as F.

Graphs. For standard graph theoretic terminology, see [10]. Let $G = (V, E)$ be a graph. The *(open) neighborhood* of W in G, in symbols $\mathsf{neigh}(W, G)$, is defined by

$$\mathsf{neigh}(W, G) = \{v \in V \setminus W \mid \text{there exists } w \in W \text{ such that } vw \in E\}.$$

We freely use $\mathsf{neigh}(v, G)$ as a shorthand for $\mathsf{neigh}(\{v\}, G)$, and we write $\mathsf{neigh}(W)$ instead of $\mathsf{neigh}(W, G)$ if the graph G is clear from the context. A graph $G = (V, E)$ is *bipartite* if it its vertex set V can be partitioned into two blocks V' and V'' such that, for every edge $vw \in E$, we either have $v \in V'$ and $w \in V''$, or $v \in V''$ and $w \in V'$. In this case we may write $G = (V', V'', E)$. The *incidence* graph of a CNF F, in symbols $\mathsf{inc}(F)$, is the bipartite graph $(\mathsf{var}(F), F, E)$ such that $vc \in E$ if and only if $v \in \mathsf{var}(F)$, $c \in F$, and $v \in \mathsf{var}(c)$; that is, the blocks are the variables and clauses of F, and a variable is adjacent to a clause if and only if the variable occurs in the clause.

A bipartite graph $G = (V, W, E)$ is *left convex* if there exists an ordering σ of V such that the following holds: if wv and wv' are edges of G and $v < v'' < v'$ (with respect to the ordering σ) then wv'' is an edge of G. The ordering σ is said to *witness* left convexity of G. A CNF F is *variable convex* if $\mathsf{inc}(F) = (\mathsf{var}(F), F, E)$ is left convex.

For an integer d, a CNF F has *degree* d if $\mathsf{inc}(F)$ has degree at most d. A class $\mathcal{F}$ of CNFs has *bounded degree* if there exists an integer d such that every CNF in $\mathcal{F}$ has degree d.

3 Polynomial Time Compilability

In this section, we introduce the *few subterms* property, a sufficient condition for a class of CNFs to admit polynomial size compilation into OBDDs (Sect. 3.1). We prove that the classes of variable convex CNFs and bounded treewidth CNFs have the few subterms property (Sects. 3.2 and 3.3). Finally, we establish fixed-parameter tractable size and time OBDD compilation results for CNFs, where the parameter is the deletion distance to a few subterms CNF class (Sect. 3.4).

3.1 The Few Subterms Property

Definition 1 (Subterm width). *Let F be a CNF formula and let $V \subseteq \mathsf{var}(F)$. The set of V-*subterms *of F is defined* $\mathsf{st}(F, V) = \{F[f] \mid f : V \to \{0, 1\}\}$*. Given an ordering σ of* $\mathsf{var}(F)$*, the* subterm width of F with respect to σ *is*

$$\mathsf{stw}(F, \sigma) = \max\{|\mathsf{st}(F, \mathsf{var}(\pi))| \mid \pi \textit{ is a prefix of } \sigma\}.$$

The subterm width *of F is the minimum subterm width of F with respect to σ, where σ ranges over all orderings of* $\mathsf{var}(F)$*.*

Definition 2 (Subterm Bound). *Let $\mathcal{F}$ be a class of CNF formulas. A function $b : \mathbb{N} \to \mathbb{N}$ is a* subterm bound *of $\mathcal{F}$ if, for all $F \in \mathcal{F}$, the subterm width of F is bounded from above by $b(\mathsf{size}(F))$. Let $b : \mathbb{N} \to \mathbb{N}$ be a subterm bound of $\mathcal{F}$, let $F \in \mathcal{F}$, and let σ be an ordering of $\mathsf{var}(F)$. We call σ a* witness *of the subterm bound b with respect to F if $\mathsf{stw}(F, \sigma) \leq b(\mathsf{size}(F))$.*

Definition 3 (Few Subterms). *A class $\mathcal{F}$ of CNF formulas has* few subterms *if it has a polynomial subterm bound $p : \mathbb{N} \to \mathbb{N}$; if, in addition, for all $F \in \mathcal{F}$, an ordering σ of $\mathsf{var}(F)$ witnessing p with respect to F can be computed in polynomial time, $\mathcal{F}$ is said to have* constructive few subterms*.*

The few subterms property naturally presents itself as a sufficient condition for a polynomial size construction of OBDDs from CNFs.

Theorem 4. *There exists an algorithm that, given a CNF F and an ordering σ of $\mathsf{var}(F)$, returns a σ-OBDD for F of size at most $|\mathsf{var}(F)|\,\mathsf{stw}(F, \sigma)$ in time polynomial in $|\mathsf{var}(F)|$ and $\mathsf{stw}(F, \sigma)$.*

Proof. Let F be a CNF and $\sigma = x_1 < \cdots < x_n$ be an ordering of $\mathsf{var}(F)$. The algorithm computes a σ-OBDD D for F as follows.

At step $i = 1$, create the source of D, labelled by F, at level 0 of the diagram; if $\emptyset \in F$ (respectively, $F = \emptyset$), then identify the source with the 0-sink (respectively, 1-sink) of the diagram, otherwise make the source an x_1-node.

At step $i + 1$ for $i = 1, \ldots, n - 1$, let $v_1, \ldots, v_l$ be the x_i-nodes at level $i - 1$ of the diagram, respectively labelled $F_1, \ldots, F_l$. For $j = 1, \ldots, l$ and $b = 0, 1$, compute $F_j[x_i = b]$, where $x_i = b$ denotes the assignment $f : \{x_i\} \to \{0, 1\}$ mapping x_i to b. If $F_j[x_i = b]$ is equal to some label of an x_{i+1}-node v already created at level i, then direct the b-edge leaving the x_i-node labelled F_j to v; otherwise, create a new x_{i+1}-node v at level i, labelled $F_j[x_i = b]$, and direct the b-edge leaving the x_i-node labelled F_j to v. If $\emptyset \in F_j[x_i = b]$, then identify v with the 0-sink of D, and if $\emptyset = F_j[x_i = b]$, then identify v with the 1-sink of D.

At termination, the diagram obtained computes F and respects σ. We analyze the runtime. At step $i + 1$ $(0 \leq i < n)$, the nodes created at level i are labelled by CNFs of the form $F[f]$, where f ranges over all assignments of $\{x_1, \ldots, x_i\}$ not falsifying F; that is, these nodes correspond exactly to the $\{x_1, \ldots, x_i\}$-subterms $\mathsf{st}(F, \{x_1, \ldots, x_i\})$ of F not containing the empty clause, whose number is bounded above by $\mathsf{stw}(F, \sigma)$. As level i is processed in time bounded above by its size times the size of level $i-1$, and $|\mathsf{var}(F)|$ levels are processed, the diagram D has size at most $|\mathsf{var}(F)| \cdot \mathsf{stw}(F, \sigma)$ and is constructed in time bounded above by a polynomial in $|\mathsf{var}(F)|$ and $\mathsf{stw}(F, \sigma)$. □

Corollary 5. *Let $\mathcal{F}$ be a class of CNFs with constructive few subterms. Then $\mathcal{F}$ admits polynomial time compilation into OBDDs.*

3.2 Variable Convex CNF Formulas

In this section, we prove that the class of variable convex CNFs has the constructive few subterms property (Lemma 6), and hence admits polynomial time compilation into OBDDs (Theorem 7); as a special case, CNFs whose incidence graphs are cographs admit polynomial time compilation into OBDDs (Example 8).

Lemma 6. *The class $\mathcal{F}$ of variable convex CNFs has the constructive few subterms property.*

Proof. Let $F \in \mathcal{F}$, so that $\mathsf{inc}(F)$ is left convex, and let σ be an ordering of $\mathsf{var}(F)$ witnessing the left convexity of $\mathsf{inc}(F)$. Let π be any prefix of σ. Call a clause $c \in F$ *π-active* in F if $\mathsf{var}(c) \cap \mathsf{var}(\pi) \neq \emptyset$ and $\mathsf{var}(c) \cap (\mathsf{var}(F) \setminus \mathsf{var}(\pi)) \neq \emptyset$. Let A denote the set of π-active clauses of F. For all $c \in A$, let $\mathsf{var}_\pi(c) = \mathsf{var}(c) \cap \mathsf{var}(\pi)$.

Claim 1. Let $c, c' \in A$. Then, $\mathsf{var}_\pi(c) \subseteq \mathsf{var}_\pi(c')$ or $\mathsf{var}_\pi(c') \subseteq \mathsf{var}_\pi(c)$.

Proof (of Claim). Let $c, c' \in A$. Assume for a contradiction that the statement does not hold, that is, there exist variables $v, v' \in \mathsf{var}(\pi)$, $v \neq v'$, such that $v \in \mathsf{var}_\pi(c) \setminus \mathsf{var}_\pi(c')$ and $v' \in \mathsf{var}_\pi(c') \setminus \mathsf{var}_\pi(c)$. Assume that $\sigma(v) < \sigma(v')$; the other case is symmetric. Since c is π-active, by definition there exists a variable $w \in \mathsf{var}(F) \setminus \mathsf{var}(\pi)$ such that $w \in \mathsf{var}(c)$. It follows that $\sigma(v') < \sigma(w)$. Therefore, we have $\sigma(v) < \sigma(v') < \sigma(w)$, where $v, w \in \mathsf{var}(c)$ and $v' \notin \mathsf{var}(c)$, contradicting the fact that σ witnesses the left convexity of $\mathsf{inc}(F)$. □

We now argue that there is a function g with domain A such that the image of A under g contains the set $\{A[f] \mid f \text{ does not satisfy } A\}$ of terms induced by assignments not satisfying A. Let $L = \{x, \neg x \mid x \in \mathsf{var}(\pi)\}$ denote the set of literals associated with variables in $\mathsf{var}(\pi)$. The function g is defined as follows. For $c \in A$, we let

$$g(c) = \{c' \setminus L \mid c' \in A, c' \cap L \subseteq c \cap L\}.$$

Let $f : \mathsf{var}(\pi) \to \{0, 1\}$ be an assignment that does not satisfy A. Let $c \in A$ be a clause not satisfied by f such that $\mathsf{var}_\pi(c)$ is maximal with respect to inclusion. We claim that $g(c) = A[f]$. To see this, let $c' \in A$ be an arbitrary clause. It follows from the claim proved above that either $\mathsf{var}_\pi(c) \subsetneq \mathsf{var}_\pi(c')$ or $\mathsf{var}_\pi(c') \subseteq \mathsf{var}_\pi(c)$. In the first case, c' is satisfied by choice of c. In the second case, c' is not satisfied by f if and only if $c' \cap L \subseteq c \cap L$. The formula $A[f]$ is precisely the set of clauses in A not satisfied by f, restricted to variables not in $\mathsf{var}(\pi)$, so $g(c) = A[f]$ as claimed.

Taking into account that an assignment might satisfy A, this implies

$$|\mathsf{st}(A, \mathsf{var}(\pi))| \leq |A| + 1 \leq \mathsf{size}(F) + 1.$$

Let $A' = \{c \in F \mid \mathsf{var}(c) \subseteq \mathsf{var}(\pi)\}$ and $A'' = \{c \in F \mid \mathsf{var}(c) \cap \mathsf{var}(\pi) = \emptyset\}$, so that $F = A \cup A' \cup A''$. For every assignment $f : \mathsf{var}(\pi) \to \{0,1\}$ we have $A''[f] = A''$ and either $A'[f] = \emptyset$ or $A'[f] = \{\emptyset\}$. Since $F[f] = A[f'] \cup A'[f] \cup A''[f]$ for every assignment $f : \mathsf{var}(\pi) \to \{0,1\}$, the number of subterms of F under assignments to $\mathsf{var}(\pi)$ is bounded as

$$|\mathsf{st}(F, \mathsf{var}(\pi))| \leq 2 \cdot (\mathsf{size}(F) + 1).$$

This proves that the class of variable convex CNFs has few subterms. Moreover, an ordering witnessing the left convexity of $\mathsf{inc}(F)$ can be computed in polynomial (even linear) time [4, 18], so the class of variable convex CNFs even has the constructive few subterms property. □

Theorem 7. *The class of variable convex CNF formulas has polynomial time compilation into OBDDs.*

Proof. Immediate from Corollary 5 and Lemma 6. □

Example 8 (Bipartite Cographs). Let F be a CNF such that $\mathsf{inc}(F)$ is a cograph. Note that $\mathsf{inc}(F)$ is a complete bipartite graph. Indeed, cographs are characterized as graphs of clique-width at most 2 [7], and it is readily verified that if a bipartite graph has clique-width at most 2, then it is a complete bipartite graph. A complete bipartite graph is trivially left convex. Then Theorem 7 implies that CNFs whose incidence graphs are cographs have polynomial time compilation into OBDDs.

3.3 Bounded Treewidth CNF Formulas

In this section, we prove that if a class of CNFs has *bounded treewidth*, then it has the constructive few subterms property (Lemma 9), and hence admits polynomial time compilation into OBDDs (Theorem 10).

Let G be a graph. A *tree decomposition* of G is a triple $\mathcal{T} = (T, \chi, r)$, where $T = (V(T), E(T))$ is a tree rooted at r and $\chi : V(T) \to 2^{V(G)}$ is a labeling of the vertices of T by subsets of $V(G)$ (called *bags*) such that

1. $\bigcup_{t \in V(T)} \chi(t) = V(G)$,
2. for each edge $uv \in E(G)$, there is a node $t \in V(T)$ with $\{u, v\} \subseteq \chi(t)$, and
3. for each vertex $v \in V(G)$, the set of nodes t with $v \in \chi(t)$ forms a connected subtree of T.

The *width* of a tree decomposition (T, χ, r) is the size of a largest bag $\chi(t)$ minus 1. The *treewidth* of G is the minimum width of a tree decomposition of G. The *pathwidth* of G is the minimum width of a tree decomposition (T, χ, r) such that T is a path.

Let F be a CNF. We say that $\mathsf{inc}(F) = (\mathsf{var}(F), F, E)$ has treewidth (respectively, pathwidth) k if the graph $(\mathsf{var}(F) \cup F, E)$ has treewidth (respectively, pathwidth) k. We identify the pathwidth (respectively, treewidth) of a CNF with the pathwidth (respectively, treewidth) of its incidence graph.

The next lemma essentially follows from a result by Razgon [22, Lemma 5].

Lemma 9. *Let $\mathcal{F}$ be a class of CNFs of bounded treewidth. Then $\mathcal{F}$ has the constructive few subterms property.*

Theorem 10. *Let $\mathcal{F}$ be a class of CNFs of bounded treewidth. Then, $\mathcal{F}$ has polynomial time compilation into OBDDs.*

Proof. Immediate from Lemma 9 and Corollary 5. □

3.4 Almost Few Subterms

In this section, we use the language of *parameterized complexity* to formalize the observation that CNF classes "close" to CNF classes with few subterms have "small" OBDD representations [11,12].

Let F be a CNF and D a set of variables and clauses of F. Let E be the formula obtained by deleting D from F, that is,

$$E = \{c \setminus \{l \in c \mid \mathsf{var}(l) \in D\} \mid c \in F \setminus D\};$$

we call D the *deletion set* of F with respect to E.

The following lemma shows that adding a few variables and clauses does not increase the subterm width of a formula too much.

Lemma 11. *Let F and E be CNFs such that D is the deletion set of F with respect to E. Let π be an ordering of $\mathsf{var}(E)$ and let σ be an ordering of $\mathsf{var}(F) \cap D$. Then $\mathsf{stw}(F, \sigma\pi) \leq 2^k \cdot \mathsf{stw}(E, \pi)$, where $k = |D|$.*

In this section, the standard of efficiency we appeal to comes from the framework of *parameterized complexity* [11,12]. The parameter we consider is defined as follows. Let $\mathcal{F}$ be a class of CNF formulas. We say that $\mathcal{F}$ is *closed under variable and clause deletion* if $E \in \mathcal{F}$ whenever E is obtained by deleting variables or clauses from $F \in \mathcal{F}$. Let $\mathcal{F}$ be a CNF class closed under variable and clause deletion. The *$\mathcal{F}$-deletion distance of F* is the minimum size of a deletion set of F from any $E \in \mathcal{F}$. An $\mathcal{F}$-deletion set of F is a deletion set of F with respect to some $E \in \mathcal{F}$.

Let $\mathcal{F}$ be a class of CNF formulas with few subterms closed under variable and clause deletion. We say that CNFs have fixed-parameter tractable OBDD size, parameterized by $\mathcal{F}$-deletion distance, if there is a computable function $f : \mathbb{N} \to \mathbb{N}$, a polynomial $p : \mathbb{N} \to \mathbb{N}$, and an algorithm that, given a CNF F having $\mathcal{F}$-deletion distance k, computes an OBDD equivalent to F in time $f(k)\, p(\mathsf{size}(F))$.

Theorem 12. *Let $\mathcal{F}$ be a class of CNF formulas with few subterms closed under variable and clause deletion. CNFs have fixed-parameter tractable OBDD size parameterized by $\mathcal{F}$-deletion distance.*

The assumption that $\mathcal{F}$ is closed under variable and clause deletion ensures that the deletion distance from $\mathcal{F}$ is defined for every CNF. It is a mild assumption though, as it is readily verified that if $\mathcal{F}$ has few subterms with polynomial subterm bound $p : \mathbb{N} \rightarrow \mathbb{N}$, then also the closure of $\mathcal{F}$ under variable and clause deletion has few subterms with the same polynomial subterm bound.

Analogously, we say that CNFs have fixed-parameter tractable time computable OBDDs (respectively, $\mathcal{F}$-deletion sets), parameterized by $\mathcal{F}$-deletion distance, if an OBDD (respectively, a $\mathcal{F}$-deletion set) for a given CNF F of $\mathcal{F}$-deletion distance k is computable in time bounded above by $f(k)\, p(\mathsf{size}(F))$.

Theorem 13. *Let $\mathcal{F}$ be a class of CNFs closed under variable and clause deletion satisfying the following:*

- *$\mathcal{F}$ has the constructive few subterms property.*
- *CNFs have fixed-parameter tractable time computable $\mathcal{F}$-deletion sets, parameterized by $\mathcal{F}$-deletion distance.*

CNFs have fixed-parameter tractable time computable OBDDs parameterized by $\mathcal{F}$-deletion distance.

Corollary 14 (Feedback Vertex Set). *Let $\mathcal{F}$ be the class of formulas whose incidence graphs are forests. CNFs have fixed-parameter tractable time computable OBDDs parameterized by $\mathcal{F}$-deletion distance.*

4 Polynomial Size Incompilability

In this section, we introduce the *subfunction width* of a graph CNF, to which the OBDD size of the graph CNF is exponentially related (Sect. 4.1), and prove that *expander graphs* yield classes of graph CNFs of *bounded degree* with linear subfunction width, thus obtaining an exponential lower bound on the OBDD size for graph CNFs in such classes (Sect. 4.2).

4.1 Many Subfunctions

In this section, we introduce the *subfunction width* of a graph CNF (Definition 15), and prove that the OBDD size of a graph CNF is bounded below by an exponential function of its subfunction width (Theorem 16).

A *graph CNF* is a CNF F such that $F = \{\{u, v\} \mid uv \in E\}$ for some graph $G = (V, E)$ without isolated vertices.

Definition 15 (Subfunction Width). *Let F be a graph CNF. Let σ be an ordering of $\mathsf{var}(F)$ and let π be a prefix of σ. We say that a subset $\{c_1, \ldots, c_e\}$ of clauses in F is* subfunction productive *relative to π if there exist $\{a_1, \ldots, a_e\} \subseteq \mathsf{var}(\pi)$ and $\{u_1, \ldots, u_e\} \subseteq \mathsf{var}(F) \setminus \mathsf{var}(\pi)$ such that for all $i, j \in \{1, \ldots, e\}$, $i \neq j$, and all $c \in F$,*

– $c_i = \{a_i, u_i\}$;
– $c \neq \{a_i, a_j\}$ *and* $c \neq \{a_i, u_j\}$.

The subfunction width *of* F*, in symbols* $\mathsf{sfw}(F)$*, is defined by*

$$\mathsf{sfw}(F) = \min_{\sigma} \max_{\pi} \{|M| \mid M \textit{ is subfunction productive relative to } \pi\},$$

where σ *ranges over all orderings of* $\mathsf{var}(F)$ *and* π *ranges over all prefixes of* σ.

Intuitively, in the graph G underlying the graph CNF F in Definition 15, there is a matching of the form $a_i u_i$ with $a_i \in \mathsf{var}(\pi)$ and $u_i \in \mathsf{var}(F) \setminus \mathsf{var}(\pi)$, $i \in \{1, \ldots, e\}$; such a matching is "almost" induced, in that G can contain edges of the form $u_i u_j$, but no edges of the form $a_i a_j$ or $a_i u_j$, $i, j \in \{1, \ldots, e\}$, $i \neq j$.

Theorem 16. *Let* F *be a graph CNF. The OBDD size of* F *is at least* $2^{\mathsf{sfw}(F)}$.

4.2 Bounded Degree

In this section, we use the existence of a family of *expander graphs* to obtain a class of graph CNFs with linear subfunction width (Lemma 17), thus obtaining an exponential lower bound on the OBDD size of a class of CNFs of *bounded degree* (Theorem 18).

Let n and d be positive integers, $d \geq 3$, and let $\epsilon < 1$ be a positive real. A graph $G = (V, E)$ is a (n, d, ϵ)-*expander* if G has n vertices, degree at most d, and for all subsets $W \subseteq V$ such that $|W| \leq n/2$, the inequality

$$|\mathsf{neigh}(W)| \geq \epsilon |W|. \tag{1}$$

It is known that for all integers $d \geq 3$, there exists a real $0 < \epsilon$, and a sequence

$$\{G_i \mid i \in \mathbb{N}\} \tag{2}$$

such that $G_i = (V_i, E_i)$ is an (n_i, d, ϵ)-expander $(i \in \mathbb{N})$, and n_i tends to infinity as i tends to infinity [1, Sect. 9.2].

Lemma 17. *Let* F *be a graph CNF whose underlying graph is an* (n, d, ϵ)*-expander, where* $n \geq 2$*,* $\epsilon > 0$*, and* $d \geq 3$*. Then*

$$\mathsf{sfw}(F) \geq \frac{\epsilon}{16d} n.$$

Proof. Let σ be any ordering of $\mathsf{var}(F)$ and let π be the length $\lfloor n/2 \rfloor$ prefix of σ.

Claim. There exists a subset $\{c_1, \ldots, c_l\}$ of clauses of F, subfunction productive relative to π, such that $l \geq \frac{\epsilon}{16d} n$.

Proof (of Claim). We will construct a sequence $(a_1, b_1), \ldots, (a_l, b_l)$ of pairs $(a_i, b_i) \in \mathsf{var}(\pi) \times (\mathsf{var}(F) \setminus \mathsf{var}(\pi))$ of vertices such that $a_i \notin \mathsf{neigh}(a_j)$, and such that $\{a_i, b_j\} \in F$ if and only if $i = j$, for $1 \leq i, j \leq l$. Letting $c_i = \{a_i, b_i\}$ for $1 \leq i \leq l$, this yields a set $\{c_1, \ldots, c_l\}$ of clauses that are subfunction productive relative to π. Assume we have chosen a (possibly empty) sequence $(a_1, b_1), \ldots, (a_j, b_j)$ of such pairs. For a vertex v in the underlying graph of F, let $N[v] = \{v\} \cup \mathsf{neigh}(v)$ denote its solid neighborhood. Let $V = \bigcup_{i=1}^{j}(N[a_i] \cup N[b_i])$ and $A = \mathsf{var}(\pi) \setminus V$. Then $|A| \leq n/2$ and we can use the expansion property (1) to conclude that $|\mathsf{neigh}(A)| \geq \epsilon|A|$. Let $B = \mathsf{neigh}(A) \setminus V$. If both A and B are nonempty we pick $(a_{j+1}, b_{j+1}) \in A \times B$ so that $a_{j+1}b_{j+1}$ is an edge. We have $A \subseteq \mathsf{var}(\pi)$ as well as $B \subseteq \mathsf{var}(F) \setminus (A \cup V) \subseteq \mathsf{var}(F) \setminus \mathsf{var}(\pi)$, so $(a_{j+1}, b_{j+1}) \in \mathsf{var}(\pi) \times (\mathsf{var}(F) \setminus \mathsf{var}(\pi))$. By construction, $\{a_{j+1}, b_{j+1}\}$ is a clause in F; moreover, $a_i \notin \mathsf{neigh}(b_{j+1})$ as well as $b_i \notin \mathsf{neigh}(a_{j+1})$, for $1 \leq i \leq j$. We conclude that the sequence $(a_1, b_1), \ldots, (a_{j+1}, b_{j+1})$ has the desired properties. Otherwise, if either of the sets A or B is empty, we stop.

We now give a lower bound on the length l of a sequence constructed in this manner. Let $(a_1, b_1), \ldots, (a_j, b_j)$ be such that one of the sets A and B as defined in the previous paragraph is empty, so that $j = l$. Since the degree of the underlying graph is bounded by d, we have $|V| \leq 2dj$ and $|A| \geq \lfloor n/2 \rfloor - 2dj$. If A is empty, we must have $2dj \geq \lfloor n/2 \rfloor$ and thus

$$j \geq \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{2d} \geq \frac{n-1}{4d} \geq \frac{n}{8d}, \tag{3}$$

where the last inequality follows from $n \geq 2$. Now suppose B is empty. We have $|B| \geq \epsilon|A| - |V|$, so

$$0 \geq \epsilon(\lfloor n/2 \rfloor - 2dj) - 2dj = \epsilon(\lfloor n/2 \rfloor) - 2dj(1+\epsilon).$$

From this, we get

$$j \geq \frac{\epsilon(n-1)}{4d(1+\epsilon)} \geq \frac{\epsilon(n-1)}{8d} \geq \frac{\epsilon n}{16d}. \tag{4}$$

Here, the last inequality follows again follows from $n \geq 2$. Recalling that $\epsilon < 1$ and taking the minimum of the bounds in (3) and (4), we obtain the lower bound stated in the claim. □

The lemma is an immediate consequence of the above claim. □

Theorem 18. *There exist a class $\mathcal{F}$ of CNF formulas and a constant $c > 0$ such that, for every $F \in \mathcal{F}$, the OBDD size of F is at least $2^{c \cdot \mathsf{size}(F)}$. In fact, $\mathcal{F}$ is a class of read 3 times, monotone, 2-CNF formulas.*

5 Conclusion

In closing, we briefly explain why completing the classification task laid out in this paper (and thus closing the gap depicted in Fig. 1) seems to require new ideas.

On the one hand, our upper bound for variable convex CNFs appears to push the few subterms property to its limits – natural variable orderings cannot be used to witness few subterms for (clause) convex CNFs and CNF classes of bounded clique-width. On the other hand, our lower bound technique based on expander graphs essentially requires bounded degree, but the candidate classes for improving lower bounds in our hierarchy, bounded clique-width CNFs and beta acyclic CNFs, have unbounded degree. In fact, in both cases, imposing a degree bound leads to classes of bounded treewidth [17] and thus polynomial bounds on the size of OBDD representations.

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