

# Algorithmic Applications of Tree-Cut Width

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**Abstract.** The recently introduced graph parameter *tree-cut width* plays a similar role with respect to immersions as the graph parameter *treewidth* plays with respect to minors. In this paper we provide the first algorithmic applications of tree-cut width to hard combinatorial problems. Tree-cut width is known to be lower-bounded by a function of treewidth, but it can be much larger and hence has the potential to facilitate the efficient solution of problems which are not known to be fixed-parameter tractable (FPT) when parameterized by treewidth. We introduce the notion of nice tree-cut decompositions and provide FPT algorithms for the showcase problems CAPACITATED VERTEX COVER, CAPACITATED DOMINATING SET and IMBALANCE parameterized by the tree-cut width of an input graph  $G$ . On the other hand, we show that LIST COLORING, PRECOLORING EXTENSION and BOOLEAN CSP (the latter parameterized by the tree-cut width of the incidence graph) are  $W[1]$ -hard and hence unlikely to be fixed-parameter tractable when parameterized by tree-cut width.

## 1 Introduction

In their seminal work on graph minors, Robertson and Seymour have shown that all finite graphs are not only well-quasi ordered by the *minor* relation, but also by the *immersion* relation<sup>1</sup>, the Graph Immersion Theorem [19]. This verified another conjecture by Nash-Williams [17]. As a consequence of this theorem, each graph class that is closed under taking immersions can be characterized by a finite set of forbidden immersions, in analogy to a graph class closed under taking minors being characterized by a finite set of forbidden minors.

In a recent paper [21], Wollan introduced the graph parameter *tree-cut width*, which plays a similar role with respect to immersions as the graph parameter *treewidth* plays with respect to minors. Wollan obtained an analogue to the Excluded Grid Theorem for these notions: if a graph has bounded tree-cut width, then it does not admit an immersion of the  $r$ -wall for arbitrarily large  $r$

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<sup>1</sup> A graph  $H$  is an immersion of a graph  $G$  if  $H$  can be obtained from  $G$  by applications of vertex deletion, edge deletion, and edge lifting, i.e., replacing two incident edges by a single edge which joins the two vertices not shared by the two edges.

[21, Theorem 15]. Marx and Wollan [16] proved that for all  $n$ -vertex graphs  $H$  with maximum degree  $k$  and all  $k$ -edge-connected graphs  $G$ , either  $H$  is an immersion of  $G$ , or  $G$  has tree-cut width bounded by a function of  $k$  and  $n$ .

In this paper we provide the first algorithmic applications of tree-cut width to hard combinatorial problems. Tree-cut width is known to be lower-bounded by a function of treewidth, but it can be much larger than treewidth if the maximum degree is unbounded (see Subsect. 2.5 for an comparison of tree-cut width to other parameters). Hence tree-cut width has the potential to facilitate the efficient solution of problems which are not known or not believed to be fixed-parameter tractable (FPT) when parameterized by treewidth. For other problems it might allow the strengthening of parameterized hardness results.

We provide results for both possible outcomes: in Sect. 4 we provide FPT algorithms for the showcase problems CAPACITATED VERTEX COVER, CAPACITATED DOMINATING SET and IMBALANCE parameterized by the tree-cut width of an input graph  $G$ , while in Sect. 5 we show that LIST COLORING, PRECOLORING EXTENSION and BOOLEAN CSP parameterized by tree-cut width (or, for the third problem, by the tree-cut width of the incidence graph) are not likely to be FPT. Table 1 provides an overview of our results. The table shows how tree-cut width provides an intermediate measurement that allows us to push the frontier for fixed-parameter tractability in some cases, and to strengthen  $W[1]$ -hardness results in some other cases.

**Table 1.** Overview of results (*tw* stands for treewidth).

Problem	Parameter		
	tw	tree-cut width	max-degree and tw
CAPACITATED VERTEX COVER	$W[1]$ -hard [4]	FPT <sup>(Thm 3)</sup>	FPT
CAPACITATED DOMINATING SET	$W[1]$ -hard [4]	FPT <sup>(Thm 5)</sup>	FPT
IMBALANCE	Open [15]	FPT <sup>(Thm 4)</sup>	FPT[15]
LIST COLORING	$W[1]$ -hard [7]	$W[1]$ -hard <sup>(Thm 6)</sup>	FPT <sup>(Obs 4)</sup>
PRECOLORING EXTENSION	$W[1]$ -hard [7]	$W[1]$ -hard <sup>(Thm 6)</sup>	FPT <sup>(Obs 4)</sup>
BOOLEAN CSP	$W[1]$ -hard [20]	$W[1]$ -hard <sup>(Thm 7)</sup>	FPT [20]

Our FPT algorithms assume that a suitable decomposition, specifically a so-called *tree-cut decomposition*, is given as part of the input. Since the class of graphs of tree-cut width at most  $k$  is closed under taking immersions [21, Lemma 10], the Graph Immersion Theorem together with the fact that immersions testing is fixed-parameter tractable [10] gives rise to a non-uniform FPT algorithm for testing whether a graph has tree-cut width at most  $k$ . In a recent unpublished manuscript, Kim et al. [12] provide a uniform FPT algorithm which constructs a tree-cut decomposition whose width is at most twice the optimal one. Effectively, this result allows us to remove the condition that a tree-cut decomposition is supplied as part of the input from our uniform FPT algorithms.

We briefly outline the methodology used to obtain our algorithmic results. As a first step, in Sect. 3 we develop the notion of *nice tree-cut decompositions*<sup>2</sup> and show that every tree-cut decomposition can be transformed into a nice one in polynomial time. These nice tree-cut decompositions are of independent interest, since they provide a means of simplifying the complex structure of tree-cut decompositions. In Sect. 4 we introduce a general three-stage framework for the design of FPT algorithms on nice tree-cut decompositions and apply it to our problems. The crucial part of this framework is the computation of the “joins.” We show that the children of any node in a nice tree-cut decomposition can be partitioned into (i) a bounded number of children with complex connections to the remainder of the graph, and (ii) a potentially large set of children with only simple connections to the remainder of the graph. We then process these by a combination of branching techniques applied to (i) and integer linear programming applied to (ii). The specifics of these procedures differ from problem to problem.

## 2 Preliminaries

### 2.1 Basic Notation

We use standard terminology for graph theory, see for instance [3]. All graphs except for those used to compute the torso-size in Subsect. 2.4 are simple; the multigraphs used in Subsect. 2.4 have loops, and each loop increases the degree of the vertex by 2.

Given a graph  $G$ , we let  $V(G)$  denote its vertex set and  $E(G)$  its edge set. The (open) neighborhood of a vertex  $x \in V(G)$  is the set  $\{y \in V(G) : xy \in E(G)\}$  and is denoted by  $N_G(x)$ . The closed neighborhood  $N_G[v]$  of  $x$  is defined as  $N_G(v) \cup \{v\}$ . For a vertex subset  $X$ , the (open) neighborhood of  $X$  is defined as  $\bigcup_{x \in X} N(x) \setminus X$  and denoted by  $N_G(X)$ . The set  $N_G[X]$  refers to the closed neighborhood of  $X$  defined as  $N_G(X) \cup X$ . We refer to the set  $N_G(V(G) \setminus X)$  as  $\partial_G(X)$ ; this is the set of vertices in  $X$  which have a neighbor in  $V(G) \setminus X$ . The degree of a vertex  $v$  in  $G$  is denoted  $deg_G(v)$ . When the graph we refer to is clear, we drop the lower index  $G$  from the notation. We use  $[i]$  to denote the set  $\{0, 1, \dots, i\}$ .

### 2.2 Parameterized Complexity

We refer the reader to [5] for standard notions in parameterized complexity, such as the complexity classes FPT and W[1], FPT Turing reductions and the MULTI-COLORED CLIQUE (MCC) problem.

### 2.3 Integer Linear Programming

Our algorithms use an Integer Linear Programming (ILP) subroutine. ILP is a well-known framework for formulating problems and a powerful tool for the development of fpt-algorithms for optimization problems.

<sup>2</sup> We call them “nice” as they serve a similar purpose as the nice tree decompositions [13], although the definitions are completely unrelated.

**Definition 1 (*p*-Variable Integer Linear Programming Optimization).**

Let  $A \in \mathbb{Z}^{q \times p}$ ,  $b \in \mathbb{Z}^{q \times 1}$  and  $c \in \mathbb{Z}^{1 \times p}$ . The task is to find a vector  $x \in \mathbb{Z}^{p \times 1}$  which minimizes the objective function  $c \cdot \bar{x}$  and satisfies all  $q$  inequalities given by  $A$  and  $b$ , specifically satisfies  $A \cdot \bar{x} \geq b$ . The number of variables  $p$  is the parameter.

**Theorem 1** ([8,9,11,14]). *p*-OPT-ILP can be solved using  $O(p^{2.5p+o(p)} \cdot L)$  arithmetic operations in space polynomial in  $L$ ,  $L$  being the number of bits in the input.

**2.4 Tree-Cut Width**

The notion of tree-cut decompositions was first proposed by Wollan [21], see also [16]. A family of subsets  $X_1, \dots, X_k$  of  $X$  is a *near-partition* of  $X$  if they are pairwise disjoint and  $\bigcup_{i=1}^k X_i = X$ , allowing the possibility of  $X_i = \emptyset$ .

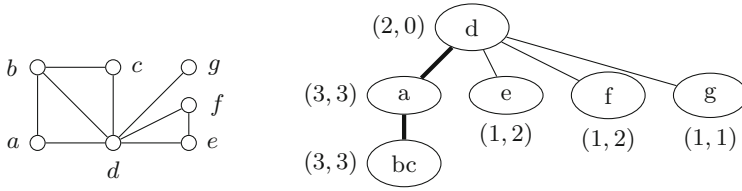
**Definition 2.** A tree-cut decomposition of  $G$  is a pair  $(T, \mathcal{X})$  which consists of a tree  $T$  and a near-partition  $\mathcal{X} = \{X_t \subseteq V(G) : t \in V(T)\}$  of  $V(G)$ . A set in the family  $\mathcal{X}$  is called a *bag* of the tree-cut decomposition.

For any edge  $e = (u, v)$  of  $T$ , let  $T_u$  and  $T_v$  be the two connected components in  $T - e$  which contain  $u$  and  $v$  respectively. Note that  $(\bigcup_{t \in T_u} X_t, \bigcup_{t \in T_v} X_t)$  is a partition of  $V(G)$ , and we use  $cut(e)$  to denote the set of edges with one endpoint in each partition. A tree-cut decomposition is *rooted* if one of its nodes is called the root, denoted by  $r$ . For any node  $t \in V(T) \setminus \{r\}$ , let  $e(t)$  be the unique edge incident to  $t$  on the path between  $r$  and  $t$ . We define the *adhesion* of  $t$  ( $adh_T(t)$  or  $adh(t)$  in brief) as  $|cut(e(t))|$ ; if  $t$  is the root, we set  $adh_T(t) = 0$ .

The *torso* of a tree-cut decomposition  $(T, \mathcal{X})$  at a node  $t$ , written as  $H_t$ , is the graph obtained from  $G$  as follows. If  $T$  consists of a single node  $t$ , then the torso of  $(T, \mathcal{X})$  at  $t$  is  $G$ . Otherwise let  $T_1, \dots, T_\ell$  be the connected components of  $T - t$ . For each  $i = 1, \dots, \ell$ , the vertex set  $Z_i$  of  $V(G)$  is defined as the set  $\bigcup_{b \in V(T_i)} X_b$ . The torso  $H_t$  at  $t$  is obtained from  $G$  by *consolidating* each vertex set  $Z_i$  into a single vertex  $z_i$ . Here, the operation of consolidating a vertex set  $Z$  into  $z$  is to substitute  $Z$  by  $z$  in  $G$ , and for each edge  $e$  between  $Z$  and  $v \in V(G) \setminus Z$ , adding an edge  $zv$  in the new graph. We note that this may create parallel edges.

The operation of *suppressing* a vertex  $v$  of degree at most 2 consists of deleting  $v$ , and when the degree is two, adding an edge between the neighbors of  $v$ . Given a graph  $G$  and  $X \subseteq V(G)$ , let the *3-center* of  $(G, X)$  be the unique graph obtained from  $G$  by exhaustively suppressing vertices in  $V(G) \setminus X$  of degree at most two. Finally, for a node  $t$  of  $T$ , we denote by  $\tilde{H}_t$  the 3-center of  $(H_t, X_t)$ , where  $H_t$  is the torso of  $(T, \mathcal{X})$  at  $t$ . Let the *torso-size*  $tor(t)$  denote  $|\tilde{H}_t|$ .

**Definition 3.** The *width* of a tree-cut decomposition  $(T, \mathcal{X})$  of  $G$  is  $\max_{t \in V(T)} \{adh(t), tor(t)\}$ . The *tree-cut width* of  $G$ , or  $tcw(G)$  in short, is the minimum width of  $(T, \mathcal{X})$  over all tree-cut decompositions  $(T, \mathcal{X})$  of  $G$ .



**Fig. 1.** A graph  $G$  and a width-3 tree-cut decomposition of  $G$ , including the torso-size (left value) and adhesion (right value) of each node.

We conclude this subsection with some notation related to tree-cut decompositions. For  $t \in V(T) \setminus \{r\}$ , we let  $p_T(t)$  (or  $p(t)$  in brief) denote the parent of  $t$  in  $T$ . For two distinct nodes  $t, t' \in V(T)$ , we say that  $t$  and  $t'$  are *siblings* if  $p(t) = p(t')$ . Given a tree node  $t$ , let  $T_t$  be the subtree of  $T$  rooted at  $t$ . Let  $Y_t = \bigcup_{b \in V(T_t)} X_b$ , and let  $G_t$  denote the induced subgraph  $G[Y_t]$ . The *depth* of a node  $t$  in  $T$  is the distance of  $t$  from the root  $r$ . The vertices of  $\partial(Y_t)$  are called the *borders* at node  $t$ . A node  $t \neq r$  in a rooted tree-cut decomposition is *thin* if  $adh(t) \leq 2$  and *bold* otherwise (see Fig. 1).

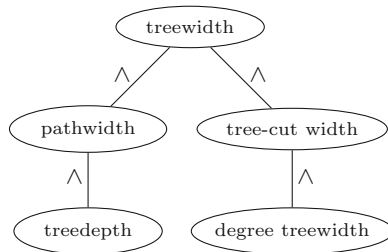
### 2.5 Relations to Other Width Parameters

Here we review the relations between the tree-cut width and other width parameters, specifically *treewidth* (**tw**), *pathwidth* (**pw**), and *treedepth* (**td**) [18]. We also compare to the maximum over treewidth and maximum degree, which we refer to as *degree treewidth* (**degtw**).

**Proposition 1** ([16, 21]). *There exists a function  $h$  such that  $\mathbf{tw}(G) \leq h(\mathbf{tcw}(G))$  and  $\mathbf{tcw}(G) \leq 4\mathbf{degtw}(G)^2$  for any graph  $G$ .*

Below, we provide an explicit bound on the relationship between treewidth and tree-cut width, and show that it is incomparable with pathwidth and treedepth (see Fig. 2).

**Proposition 2.** *For any graph  $G$  it holds that  $\mathbf{tw}(G) \leq 2\mathbf{tcw}(G)^2 + 3\mathbf{tcw}(G)$ .*



**Fig. 2.** Relationships between selected graph parameters ( $A > B$  means that every graph class of bounded  $A$  is also of bounded  $B$ , but there are graph classes of bounded  $B$  which are not of bounded  $A$ ).

**Proposition 3.** *There exists a graph class  $\mathcal{H}_1$  of bounded pathwidth and treedepth but unbounded tree-cut width, and there exists a graph class  $\mathcal{H}_2$  of bounded tree-cut width but unbounded pathwidth and treedepth.*

### 3 Nice Tree-Cut Decompositions

In this section we introduce the notion of a *nice tree-cut decomposition* and present an algorithm to transform any tree-cut decomposition into a nice one without increasing the width. A tree-cut decomposition  $(T, \mathcal{X})$  rooted at  $r$  is *nice* if it satisfies the following condition for every thin node  $t \in V(T)$ :  $N(Y_t) \cap \bigcup_{b \text{ is a sibling of } t} Y_b = \emptyset$ .

The intuition behind nice tree-cut decompositions is that we restrict the neighborhood of thin nodes in a way which facilitates dynamic programming.

**Lemma 1.** *There exists a cubic-time algorithm which transforms any rooted tree-cut decomposition  $(T, \mathcal{X})$  of  $G$  into a nice tree-cut decomposition of the same graph, without increasing its width or number of nodes.*

The proof of Lemma 1 is based on the following considerations. Let  $(T, \mathcal{X})$  be a rooted tree-cut decomposition of  $G$  whose width is at most  $w$ . We say that a node  $t$ ,  $t \neq r$ , is *bad* if it violates the above condition, i.e.,  $\text{adh}(t) \leq 2$  and there is a sibling  $b$  of  $t$  such that  $N(Y_t) \cap Y_b \neq \emptyset$ . For a bad node  $t$ , we say that  $b$  is a *bad neighbor* of  $t$  if  $N(Y_t) \cap X_b \neq \emptyset$  and  $b$  is either a sibling of  $t$  or a descendant of a sibling of  $t$ .

**REROUTING( $t$ ):** let  $t$  be a bad node and let  $b$  be a bad neighbor of  $t$  of maximum depth (resolve ties arbitrarily). Then remove the tree edge  $e(t)$  from  $T$  and add a new tree edge  $\{b, t\}$ .

**TOP-DOWN REROUTING:** as long as  $(T, \mathcal{X})$  is not a nice tree-cut decomposition, pick any bad node  $t$  of minimum depth. Perform REROUTING( $t$ ).

The proof of Lemma 1 then follows by showing that REROUTING does not increase the width of the decomposition and that TOP-DOWN REROUTING terminates after performing at most a quadratic number of REROUTING calls.

The following Theorem 2 builds upon Lemma 1 by additionally giving a bound on the size of the decomposition.

**Theorem 2.** *If  $\text{tcw}(G) = k$ , then there exists a nice tree-cut decomposition  $(T, \mathcal{X})$  of  $G$  of width  $k$  with at most  $2|V(G)|$  nodes. Furthermore,  $(T, \mathcal{X})$  can be computed from any width- $k$  tree-cut decomposition of  $G$  in cubic time.*

### 4 FPT Algorithms

In this section we will introduce a general dynamic programming framework for the design of FPT algorithms on nice tree-cut decompositions. The framework is based on leaf-to-root processing of the decompositions and can be divided into three basic steps:

- **Data Table:** definition of a data table  $\mathcal{D}_T(t)$  ( $\mathcal{D}(t)$  in brief) for a problem  $\mathcal{P}$  associated with each node  $t$  of a nice tree-cut decomposition  $(T, \mathcal{X})$ .
- **Initialization and Termination:** computation of  $\mathcal{D}(t)$  in FPT time for any leaf  $t$ , and solution of  $\mathcal{P}$  in FPT time if  $\mathcal{D}(r)$  is known for the root  $r$ .
- **Inductive Step:** an FPT algorithm for computing  $\mathcal{D}(t)$  for any node  $t$  when  $\mathcal{D}(t')$  is known for every child  $t'$  of  $t$ .

Let  $t$  be a node in a nice tree-cut decomposition. We use  $B_t$  to denote the set of thin children  $t'$  of  $t$  such that  $N(Y_{t'}) \subseteq X_t$ , and we let  $A_t$  contain every child of  $t$  not in  $B_t$ . The following lemma is a crucial component of our algorithms, since it bounds the number of children with nontrivial edge connections to other parts of the decomposition.

**Lemma 2.** *Let  $t$  be a node in a nice tree-cut decomposition of width  $k$ . Then  $|A_t| \leq 2k + 1$ .*

In the remainder of this section we employ this high-level framework on the design of FPT algorithms for the following problems: CAPACITATED VERTEX COVER, IMBALANCE, and CAPACITATED DOMINATING SET.

### 4.1 Capacitated Vertex Cover

The CAPACITATED VERTEX COVER is a generalization of the classical VERTEX COVER problem. Unlike its uncapacitated variant, CAPACITATED VERTEX COVER is known to be  $W[1]$ -hard when parameterized by treewidth [4].

A capacitated graph is a graph  $G = (V, E)$  together with a capacity function  $c : V \rightarrow \mathbb{N}_0$ . Then we call  $C \subseteq V$  a *capacitated vertex cover* of  $G$  if there exists a mapping  $f : E \rightarrow C$  which maps every edge to one of its endpoints so that the total number of edges mapped by  $f$  to any  $v \in C$  does not exceed  $c(v)$ . We say that  $f$  *witnesses*  $C$ .

tcw-CAPACITATED VERTEX COVER (tcw-CVC)  
*Instance:* A capacitated graph  $G$  on  $n$  vertices together with a width- $k$  tree-cut decomposition  $(T, \mathcal{X})$  of  $G$ , and an integer  $d$ .  
*Parameter:*  $k$ .  
*Task:* Decide whether there exists a capacitated vertex cover  $C$  of  $G$  containing at most  $d$  vertices.

**Data Table, Initialization and Termination.** Informally, we store for each node  $t$  two pieces of information: the “cost” of covering all edges inside  $G[Y_t]$ , and how much more it would cost to additionally cover edges incident to  $Y_t$ . We formalize below.

For any graph  $G = (V, E)$  and  $U \subseteq V$ , we let  $cvc(G, U)$  denote the minimum cardinality of a capacitated vertex cover  $C \subseteq U$  of  $G$ ; if no such capacitated vertex cover exists, we instead let  $cvc(G, U) = \infty$ . For any node  $t$  in a nice tree-cut decomposition of a capacitated graph  $G = (V, E)$ , we then use  $a_t$  to denote  $cvc(G[Y_t], Y_t)$ .

Let  $E_t$  denote the set of all edges with both endpoints in  $Y_t$  and let  $K_t$  denote the set of edges with exactly one endpoint in  $Y_t$ . Then  $\mathcal{Q}_t = \{H = (Y_t \cup N(Y_t), E_t \cup E') \mid E' \subseteq K_t\}$ . Finally, we define  $\beta_t : \mathcal{Q}_t \rightarrow \mathbb{N}_0$  such that  $\beta_t(H) = \text{cvc}(H, Y_t) - a_t$  (whereas  $\infty$  acts as an absorbing element).

**Definition 4.**  $\mathcal{D}(t) = (a_t, \beta_t)$ .

Next, we show that the number of possible functions  $\beta_t$  is bounded.

**Lemma 3.** *Let  $k$  be the width of a nice tree-cut decomposition  $(T, \mathcal{X})$  of  $G$  and let  $t$  be any node of  $T$ . Then  $\beta_t(H) \in [k] \cup \{\infty\}$  for every  $H \in \mathcal{Q}_t$ .*

**Lemma 4.** *Let  $t$  be a leaf in a nice tree-cut decomposition  $(T, \mathcal{X})$  of a capacitated graph  $G$ , and let  $k$  be the width of  $(T, \mathcal{X})$ . Then  $\mathcal{D}(t)$  can be computed in time  $2^{O(k \cdot \log k)}$ .*

**Observation 1.** *Let  $(G, d)$  be an instance of tcw-CVC and let  $r$  be the root of a nice tree-cut decomposition of  $G$ . Then  $(G, d)$  is a yes-instance if and only if  $a_r \leq d$ .*

**Inductive Step.** Our next and final goal is to show how to compute  $\mathcal{D}(t)$  of a node  $t$  once we have  $\mathcal{D}(t')$  for each child  $t'$  of  $t$ . We call this problem CVC JOIN, and we use a two-step approach to solve it. First, we reduce the problem to a simplified version, which we call REDUCED CVC JOIN and which has the following properties:  $A_t$  is empty,  $\text{adh}(t) = 0$ , and  $G[X_t]$  is edgeless.

**Lemma 5.** *There is an FPT Turing reduction from CVC JOIN to  $2^{O(k^2)}$  instances of REDUCED CVC JOIN which runs in time  $2^{O(k^2)} \cdot (|B_t| + 1)$ .*

**Lemma 6.** *There exists an algorithm which solves REDUCED CVC JOIN in time  $k^{O(k^2)} \cdot (|B_t| + 1)$ .*

*Proof (Sketch).* We develop an ILP formulation by partitioning  $B_t$  into types, which contain nodes with the same  $\beta_t$  and the same neighborhood in  $X_t$ . We use variables to express how edges are assigned between types and  $X_t$ .  $\square$

**Corollary 1.** *There exists an algorithm which solves CVC JOIN in time  $k^{O(k^2)} \cdot (|B_t| + 1)$ .*

**Theorem 3.** *tcw-CVC can be solved in time  $k^{O(k^2)} \cdot n + |T|^3$ .*

*Proof.* We use Theorem 2 to transform  $(T, \mathcal{X})$  into a nice tree-cut decomposition with at most  $2n$  nodes. We then use Lemma 4 to compute  $\mathcal{D}(t)$  for each leaf  $t$  of  $T$ , and proceed by computing  $\mathcal{D}(t)$  for nodes in a bottom-to-top fashion by Corollary 1. The total running time is dominated by  $\sum_{t \in T} (k^{O(k^2)} \cdot (|B_t| + 1))$ , which is bounded by  $n \cdot k^{O(k^2)}$  since each node  $t'$  appears in at most a single set  $B_t$ . Once we obtain  $\mathcal{D}(r)$ , we can correctly output by Observation 1.  $\square$



### 4.2 Imbalance

The IMBALANCE problem was introduced by Biedl et al. [1]. The problem is FPT when parameterized by **deg<sub>tw</sub>** [15]. In this subsection we prove that IMBALANCE remains FPT even when parameterized by the more general tree-cut width.

Given a linear order  $R$  of vertices in a graph  $G$ , let  $\triangleleft_R(v)$  and  $\triangleright_R(v)$  denote the number of neighbors of  $v$  which occur, respectively, before (“to the left of”)  $v$  in  $R$  and after (“to the right of”)  $v$  in  $R$ . The *imbalance* of a vertex  $v$ , denoted  $imb_R(v)$ , is then defined as the absolute value of  $\triangleright_R(v) - \triangleleft_R(v)$ , and the imbalance of  $R$ , denoted  $imb_R$ , is equal to  $\sum_{v \in V(G)} imb_R(v)$ .

tcw-IMBALANCE (tcw-IMB)  
*Instance:* A graph  $G = (V, E)$  with  $|V| = n$ , and a width- $k$  tree-cut decomposition  $(T, \mathcal{X})$  of  $G$ , and an integer  $d$ .  
*Parameter:*  $k$ .  
*Task:* Decide whether there exists a linear order  $R$  of  $V$  such that  $imb_R \leq d$ .

**Data Table, Initialization and Termination.** Let  $A \subseteq B$  be sets and let  $f_A, f_B$  be linear orders of  $A, B$  respectively. We say that  $f_A$  is a *linear suborder* of  $f_B$  if the elements of  $A$  occur in the same order in  $f_A$  as in  $f_B$  (similarly,  $f_B$  is a *linear superorder* of  $f_A$ ). The information we remember in our data tables can be informally summarized as follows. First, we remember the minimum imbalance which can be achieved by any linear order in  $Y_t$ . Second, for each linear order  $f$  of vertices which have neighbors outside of  $Y_t$  and for a restriction on the imbalance on these vertices, we remember how much the imbalance grows when considering only linear superorders of  $f$  which satisfy these restrictions. The crucial ingredient is that the restrictions mentioned above are “weak” and we only care about linear superorders of  $f$  which do not increase over the optimum “too much”; this allows the second, crucial part of our data tables to remain bounded in  $k$ .

For brevity, for  $v \in Y_t$  we let  $n_t(v)$  denote  $|N_{V \setminus Y_t}(v)|$ , i.e., the number of neighbors of  $v$  outside  $Y_t$ . Let  $f$  be a linear order of  $\partial(Y_t)$  and let  $\tau$  be a mapping such that  $\tau(v \in \partial(Y_t)) \in \{-\infty, -n_t(v), -n_t(v) + 1, \dots, n_t(v), \infty\}$ . We then call a tuple of the form  $(f, \tau)$  an *extract* (of  $Y_t$ ), and let  $\mathcal{L}$  denote the set of all extracts (for nodes with adhesion at most  $k$ ). The extract  $\alpha = (f, \tau)$  is realized in  $Y_t$  (by  $R$ ) if there exists a linear order  $R$  of  $Y_t$  such that

1.  $R$  is a linear superorder of  $f$ , and
2. for each  $v \in \partial(Y_t)$ :
  - if  $\tau(v) \in \mathbb{Z}$  then  $imb_R(v) = \tau(v)$ ,
  - if  $\tau(v) = -\infty$  then  $\triangleright_R(v) - \triangleleft_R(v) < -n_t(v) - 1$ ,
  - if  $\tau(v) = \infty$  then  $\triangleright_R(v) - \triangleleft_R(v) > n_t(v) + 1$ .

The cost of a realized extract  $\alpha$ , denoted  $c(\alpha)$ , is the minimum value of  $\sum_{v \in Y_t} imb_R(v)$  over all  $R$  which realize  $\alpha$  (notice that edges with only one endpoint in  $Y_t$  do not contribute to  $c(\alpha)$ ). If  $\alpha$  is not realized in  $Y_t$ , we let  $c(\alpha) = \infty$ .

We store the following information in our data table: the cost of a minimum extract realized in  $Y_t$ , and the cost of every extract whose cost is not much larger than the minimum cost. We formalize below; let  $e_t$  denote the number of edges with one endpoint in  $Y_t$ .

**Definition 5.**  $\mathcal{D}(t) = (a_t, \beta_t)$  where  $a_t = \min_{\alpha \in \mathcal{L}} c(\alpha)$  and  $\beta_t : \mathcal{L} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  such that  $\beta_t(\alpha) = c(\alpha) - a_t$  if  $c(\alpha) - a_t \leq 4e_t$  and  $\beta_t(\alpha) = \infty$  otherwise.

Notice that we are deliberately discarding information about the cost of extracts whose cost exceeds the optimum by over  $4e_t$ .

**Observation 2.** The cardinality of  $\mathcal{L}$  is bounded by  $k^{O(k)}$ , and hence the number of possible functions  $\beta_t$  is bounded by  $k^{O(k^2)}$ . Additionally, these may be enumerated in the same time.

**Lemma 7.** Let  $t$  be a leaf in a nice tree-cut decomposition  $(T, \mathcal{X})$  of a graph  $G$ , and let  $k$  be the width of  $(T, \mathcal{X})$ . Then  $\mathcal{D}(t)$  can be computed in time  $k^{O(k)}$ .

**Observation 3.** Let  $(G, d)$  be an instance of *tcw-IMB* and let  $r$  be the root of a nice tree-cut decomposition of  $G$ . Then  $(G, d)$  is a *yes-instance* if and only if  $a_r \leq d$ .

**Inductive Step.** What remains is to show how to compute  $\mathcal{D}(t)$  for a node  $t$  once  $\mathcal{D}(t')$  is known for each child  $t'$  of  $t$ . Once again, we use the two-step approach of first reducing to a “simpler” problem and then applying a suitable ILP encoding. We call the problem we reduce to **REDUCED IMB JOIN**.

**Lemma 8.** There is an *FPT Turing reduction* from **IMB JOIN** to  $k^{O(k^2)}$  instances of **REDUCED IMB JOIN** which runs in time  $k^{O(k^2)} \cdot (|B_t| + 1)$ .

*Proof (Sketch).* We branch over all linear orders of  $f(k)$ -many “important vertices” in  $A_t$ , over all possible extracts in  $\mathcal{D}(t)$ , and over all extracts in  $A_t$  which are compatible with the above. This gives sufficient information to compute the imbalance of vertices in  $X_t$  along with constraints on the placement of  $\beta_t$ , which yield an instance of **REDUCED IMB JOIN**.  $\square$

**Lemma 9.** There exists an algorithm which solves **REDUCED IMB JOIN** in time  $k^{O(k^4)} \cdot (|B_t| + 1)$ .

*Proof (Sketch).* The  $k$  absolute values can be translated into  $2^k$ -many ILP instances by branching on whether they end up being positive or negative. Vertices in  $X_t$  separate the linear order into  $k + 1$  “regions”, and we can again partition  $B_t$  into types. Variables express how many children of type  $i$  have a border vertex placed in region  $j$ .  $\square$

The proof of the theorem below is then analogous to the proof of Theorem 3.

**Theorem 4.** *tcw-IMB* can be solved in time  $k^{O(k^4)} \cdot n + |T|^3$ .

### 4.3 Capacitated Dominating Set

CAPACITATED DOMINATING SET is a generalization of the classical DOMINATING SET problem by the addition of vertex capacities. It is known to be  $W[1]$ -hard when parameterized by treewidth [4].

Let  $G = (V, E)$  be a capacitated graph with a capacity function  $c : V(G) \rightarrow \mathbb{N}_0$ . We say that  $D \subseteq V(G)$  is a *capacitated dominating set* of  $G$  if there exists a mapping  $f : V \setminus D \rightarrow D$  which maps every vertex to one of its neighbors so that the total number of vertices mapped by  $f$  to any  $v \in D$  does not exceed  $c(v)$ .

tcw-CAPACITATED DOMINATING SET (tcw-CDS)  
*Instance:* A capacitated graph  $G$  on  $n$  vertices together with a width- $k$  tree-cut decomposition  $(T, \mathcal{X})$  of  $G$ , and an integer  $d$ .  
*Parameter:*  $k$ .  
*Task:* Decide whether there exists a capacitated dominating set  $D$  of  $G$  containing at most  $d$  vertices.

The methods used to solve tcw-CDS are similar to those used to prove Theorems 3 and 4, and hence we only provide a high-level description of our approach here. For tcw-CDS, the table  $\mathcal{D}(t)$  stores information about whether vertices in  $X_t$  occur in a dominating set, the residual capacities in  $\partial(Y_t)$ , and the size of a minimum capacitated dominating set which has these properties. The following steps are then analogous to those above.

**Theorem 5.** *tcw-CDS can be solved in time  $k^{O(k^2)} \cdot n + |T|^3$ .*

## 5 Lower Bounds

We show that LIST COLORING [6] and PRECOLORING EXTENSION [2] are  $W[1]$ -hard parameterized by tree-cut width, strengthening the known  $W[1]$ -hardness results with respect to treewidth [7].

tcw-LIST COLORING  
*Instance:* A graph  $G = (V, E)$ , a width- $k$  tree-cut decomposition  $(T, \mathcal{X})$  of  $G$ , and for each vertex  $v \in V$  a list  $L(v)$  of permitted colors.  
*Parameter:*  $k$ .  
*Task:* Decide whether there exists a proper vertex coloring  $c$  such that  $c(v) \in L(v)$  for each  $v \in V$ .

The tcw-PRECOLORING EXTENSION problem may be defined analogously as LIST COLORING; the only difference is that in PRECOLORING EXTENSION lists are restricted to either contain a single color or all possible colors.

**Observation 4.** LIST COLORING and PRECOLORING EXTENSION parameterized by **degtw** are FPT.

**Theorem 6.** *tcw-LIST COLORING and tcw-PRECOLORING EXTENSION are  $W[1]$ -hard.*

We also show that the CONSTRAINT SATISFACTION PROBLEM (CSP) is  $W[1]$ -hard when parameterized by the tree-cut width of the incidence graph, even when restricted to the Boolean domain; this is not the case for **deg $tw$**  [20].

tcw-CSP

*Instance:* A CSP instance  $I = (X, D, \mathcal{C})$  together with a width- $k$  tree-cut decomposition  $(T, \mathcal{X})$  of the incidence graph  $G_I$  of  $I$ .

*Parameter:*  $k$ .

*Task:* Decide whether  $I$  is satisfiable.

**Theorem 7.** *tcw-BOOLEAN CSP is  $W[1]$ -hard.*

The proofs of Theorems 6 and 7 are based on a reduction from MCC.

## 6 Concluding Notes

We have provided the first algorithmic applications of the new graph parameter tree-cut width, considering a variety of hard combinatorial problems. In some cases we could establish fixed-parameter tractability, in some cases we could establish  $W[1]$ -hardness, staking off the potentials and limits of this parameter (see Table 1). The FPT algorithms make use of our new notion of nice tree-cut decompositions, which we believe to be of independent interest. We hope that our results and methods stimulate further work on problems parameterized by tree-cut width, which will result in a more refined parameterized complexity landscape; natural candidate problems include further graph layout problems or the General Factor Problem.

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