



Supereulerian graphs with width s and s -collapsible graphs



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ABSTRACT

For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$ -trail-system of G is a subgraph H consisting of s edge-disjoint (u, v) -trails. A graph is **supereulerian with width s** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$ -trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for every integer k with $0 \leq k \leq s$. Thus a graph G with $\mu'(G) \geq 2$ has a spanning Eulerian subgraph. Catlin (1988) introduced collapsible graphs to study graphs with spanning Eulerian subgraphs, and showed that every collapsible graph G satisfies $\mu'(G) \geq 2$ (Catlin, 1988; Lai et al., 2009). Graphs G with $\mu'(G) \geq 2$ have also been investigated by Luo et al. (2006) as Eulerian-connected graphs. In this paper, we extend collapsible graphs to s -collapsible graphs and develop a new related reduction method to study $\mu'(G)$ for a graph G . In particular, we prove that $K_{3,3}$ is the smallest 3-edge-connected graph with $\mu' < 3$. These results and the reduction method will be applied to determine a best possible degree condition for graphs with supereulerian width at least 3, which extends former results in Catlin (1988) and Lai (1988).

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1. Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terminology and notation not defined here can be found in [3]. In particular, for a graph G , $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ represent the minimum degree, the maximum degree, the connectivity and the edge connectivity of a graph G , respectively. For subgraphs H_1, H_2 of G , $H_1 \cup H_2$ and $H_1 \cap H_2$ denote the union and intersection of H_1 and H_2 , respectively, as defined in [3]. For vertices $u, v \in V(G)$, a trail with end vertices being u and v will be called a (u, v) -trail. We use $O(G)$ to denote the set of all odd degree vertices in G . A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected, and is **supereulerian** if G has a spanning Eulerian subgraph.

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an $(s; u, v)$ -**trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v) -trails. A graph is **supereulerian with width s** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$ -trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \leq k \leq s$. Luo et al. in [19] defined graphs with $\mu'(G) \geq 2$ as **Eulerian-connected**

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graphs and investigated, for a given integer $r > 0$, the minimum value $\psi(r)$ such that if G is a $\psi(r)$ -edge-connected graph, then for any $X \subseteq E(G)$ with $|X| \leq r$, $\mu'(G - X) \geq 2$. Note that if for some vertices u and v , G does not have a spanning (u, v) -trail, then $\mu'(G) = 0$. The vertex counter-part of $\mu'(G)$, called the spanning connectivity of a graph, has been intensively studied, as can be seen in Chapters 14 and 15 of [11].

Following [3], if $V' \subseteq V(G)$ is a vertex subset, then $G[V']$ is the subgraph of G induced by V' ; if $X \subseteq E(G)$ is an edge subset, then $G[X]$ is the subgraph of G induced by X . If $v \in V(G)$, then $N_G(v)$ denotes the vertices of G adjacent to v in G . If H is a graph and Z is a set of edges such that the end vertices of each edge in Z are in $V(H)$, then $H + Z$ denotes the graph with vertex set $V(H)$ and edge set $E(H) \cup Z$.

In [2], Boesch et al. first raised the problem of characterizing supereulerian graphs. They remarked that such a problem would be difficult. In [20], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete. Jaeger [12] first proved that every 4-edge-connected graph is supereulerian. In [4], Catlin introduced collapsible graphs as a tool to study supereulerian graphs. Catlin [4] and Lai et al. [16] showed that if G is collapsible, then $\mu'(G) \geq 2$. (See also Chapter 3 of [21] and [26].) Most of the studies on supereulerian graphs with width at most 2 can be found in Catlin's survey [5] and its updates [9,15]. By definition, we have the obvious inequality

$$\mu'(G) \leq \kappa'(G), \quad \text{for any connected graph } G. \quad (1)$$

Determining when equality holds in (1) is one of the most natural questions. One purpose of this paper is to investigate graphs G such that for a given integer k , $\mu'(G) \geq k$ if and only if $\kappa'(G) \geq k$. Motivated by Catlin's work in [4], in Section 2 we extend the concept of collapsible graphs to s -collapsible graphs, and use it to develop a new reduction method using s -collapsible graphs. In Section 3, we study the s -collapsibility of complete graphs and some other dense graphs, and prove that $K_{3,3}$ is the smallest among all 3-edge-connected graphs G such that $\mu'(G) < \kappa'(G)$. In the last section, we apply the reduction method associated with s -collapsible graphs to study the structure of reduced graphs under a degree condition. These allow us to obtain a best possible degree condition for supereulerian graphs with width at least 3, extending former results in [4] and [13].

2. Reductions with s -collapsible graphs

Throughout this paper, we adopt the convention that any graph is 0-edge-connected, and so $\kappa'(G) \geq 0$ holds for any graph G , and let $s \geq 1$ denote an integer. For sets X and Y , the **symmetric difference** of X and Y is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Definition 2.1. A graph G is **s -collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, G has a spanning subgraph Γ_R such that

- (i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \geq s - 1$, and
- (ii) $G - E(\Gamma_R)$ is connected.

A spanning subgraph Γ_R of G with both properties in Definition 2.1 is an (s, R) -**subgraph** of G . Let \mathcal{C}_s denote the collection of s -collapsible graphs. Then \mathcal{C}_1 is the collection of all collapsible graphs, defined in [4]. By definition, any $(s+1, R)$ -subgraph of G is also an (s, R) -subgraph of G . This implies that

$$\mathcal{C}_{s+1} \subseteq \mathcal{C}_s, \quad \text{for any positive integer } s. \quad (2)$$

Proposition 2.2. Let G be a graph, and let $s \geq 1$ be an integer. Then the following are equivalent.

- (i) $G \in \mathcal{C}_s$.
- (ii) For any $X \subseteq V(G)$ with $|X| \equiv 0 \pmod{2}$, G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G - E(L_X)) \geq s - 1$.

Proof. (i) \implies (ii). Given $X \subseteq V(G)$ with $|X| \equiv 0 \pmod{2}$, let $R = O(G) \Delta X$. By the definition of R , it follows that $|R| \equiv 0 \pmod{2}$. Since $G \in \mathcal{C}_s$, G has a spanning subgraph Γ_R such that $O(\Gamma_R) = R$, $\kappa'(\Gamma_R) \geq s - 1$, and $G - E(\Gamma_R)$ is connected. Let $L_X = G - E(\Gamma_R)$. Then L_X is a spanning connected subgraph such that $O(L_X) = R \Delta O(G) = X \Delta O(G) \Delta O(G) = X$. Moreover $\kappa'(G - E(L_X)) = \kappa'(\Gamma_R) \geq s - 1$.

(ii) \implies (i). Given $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, let $X = R \Delta O(G)$. By the definition of X , it follows that $|X| \equiv 0 \pmod{2}$. By (ii), G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G - E(L_X)) \geq s - 1$. Let $\Gamma_R = G - E(L_X)$. Then both $\kappa'(\Gamma_R) \geq s - 1$ and $O(\Gamma_R) = O(G) \Delta X = R$. As $G - E(\Gamma_R) = L_X$ is connected, $G \in \mathcal{C}_s$. \square

For a graph G , and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G , then we write G/H for $G/E(H)$. When H is connected, we use v_H to denote the vertex in G/H onto which H is contracted.

Lemma 2.3. Suppose that H is a connected subgraph of G , and $R \subseteq V(G)$ is a subset with $|R| \equiv 0 \pmod{2}$. Define

$$R' = \begin{cases} R - V(H) & \text{if } |R \cap V(H)| \equiv 0 \pmod{2} \\ (R - V(H)) \cup \{v_H\} & \text{if } |R \cap V(H)| \equiv 1 \pmod{2}. \end{cases}$$

If G/H has an (s, R') -subgraph $\Gamma_{R'}$, and if $H \in \mathcal{C}_s$, then G has an (s, R) -subgraph Γ_R .

Proof. Let $\Gamma_{R'}$ be an (s, R') -subgraph of G/H . Define $R^* = V(H) \cap O(G[E(\Gamma_{R'})])$. Thus R^* consists of the vertices in H that are incident with an odd number of edges in $E(\Gamma_{R'})$. By the definition of R' , $|R^*| \equiv d_{\Gamma_{R'}}(v_H) \equiv |R \cap V(H)| \pmod{2}$. Define $R'' = R^* \Delta (R \cap V(H))$. By definition, $|R''| \equiv |R^*| + |R \cap V(H)| \equiv 0 \pmod{2}$ and $R'' \subseteq V(H)$. Since $H \in \mathcal{C}_s$, H has an (s, R'') -subgraph $\Gamma_{R''}$. Define

$$\Gamma_R = G \left[E(\Gamma_{R'}) \cup E(\Gamma_{R''}) \right].$$

Since $\kappa'(\Gamma_{R'}) \geq s - 1$ and $\kappa'(\Gamma_{R''}) \geq s - 1$, and as $\Gamma_R/\Gamma_{R''} = \Gamma_{R'}$ when $s \geq 2$, we conclude that $\kappa'(\Gamma_R) \geq s - 1$. By the definition of R' and R'' , we observe that $O(\Gamma_R) - V(H) = R - V(H)$; since $R \cap V(H) \subseteq V(H)$ and $R^* \subseteq V(H)$, we have $(R^* \Delta (R \cap V(H))) \cap V(H) = R^* \Delta (R \cap V(H))$, and so $O(\Gamma_R) \cap V(H) = (O(G[E(\Gamma_{R'})]) \cap V(H)) \Delta ((R^* \Delta (R \cap V(H))) \cap V(H)) = R^* \Delta ((R^* \Delta (R \cap V(H))) \cap V(H)) = R^* \Delta R^* \Delta (R \cap V(H)) = (R \cap V(H))$. Thus

$$O(\Gamma_R) = O(G[E(\Gamma_{R'})]) \Delta O(\Gamma_{R''}) = (R - V(H)) \cup (R \cap V(H)) = R.$$

Moreover, $G - E(\Gamma_R) = G[E(G/H - E(\Gamma_{R'})) \cup E(H - E(\Gamma_{R''}))]$. Since $\Gamma_{R'}$ is an (s, R') -subgraph of G/H , and since $\Gamma_{R''}$ is an (s, R'') -subgraph of H , $G/H - E(\Gamma_{R'})$ contains a spanning tree of G/H and $H - E(\Gamma_{R''})$ contains a spanning tree of H . It follows that $G - E(\Gamma_R)$ contains a spanning tree of G , and so by definition, Γ_R is an (s, R) -subgraph of G . \square

Corollary 2.4. Let $s \geq 1$ be an integer. Then \mathcal{C}_s satisfies the following.

- (C1) $K_1 \in \mathcal{C}_s$.
- (C2) If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.
- (C3) If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.

Proof. (C1) and (C2) follow immediately from definitions, and (C3) follows from Lemma 2.3. \square

Corollary 2.5. Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s + 1$.

Proof. Let u and v be two distinct vertices of G . Let $X = \emptyset$. Since $G \in \mathcal{C}_s$, by Proposition 2.2, G has a spanning connected subgraph L_X with $O(L_X) = \emptyset$ and $\kappa'(G - E(L_X)) \geq s - 1$. Since L_X is Eulerian, L_X can be partitioned into two edge-disjoint (u, v) -trails T_1, T_2 . By the edge version of Menger's Theorem, $G - E(L_X)$ has $s - 1$ edge-disjoint (u, v) -paths, T_3, T_4, \dots, T_{s+1} . Since $T_1 \cup T_2 = L_X$ is spanning, $\{T_1, T_2, \dots, T_{s+1}\}$ is spanning $(s + 1; u, v)$ -trail-system. \square

A subgraph H of G is \mathcal{C}_s -maximal if $H \in \mathcal{C}_s$ and if G has no subgraph in \mathcal{C}_s that properly contains H .

Lemma 2.6. Let G be a graph and let $s > 0$ be an integer. Each of the following holds.

- (i) Let L_1, L_2 be vertex induced subgraphs of G . If $V(L_1) \cap V(L_2) \neq \emptyset$ and if $L_1, L_2 \in \mathcal{C}_s$, then $L_1 \cup L_2 \in \mathcal{C}_s$.
- (ii) The graph G has a unique set of vertex disjoint \mathcal{C}_s -maximal subgraphs H_1, H_2, \dots, H_c such that $V(G) = \bigcup_{i=1}^c V(H_i)$, and if $G' = G / (\bigcup_{i=1}^c E(H_i))$, then G' contains no nontrivial subgraph in \mathcal{C}_s .

Proof. (i) Let J_1, J_2, \dots, J_t be the connected components of $L_1 \cap L_2$. Since $L_1 \in \mathcal{C}_s$, by Corollary 2.4(C2), $L_1 / (L_1 \cap L_2) \in \mathcal{C}_s$. Let v_{j_i} be the vertex in $L_1 / (L_1 \cap L_2)$ onto which J_i is contracted, $(1 \leq j \leq t)$, and let X be a set of $t - 1$ additional edges, (i.e. $X \cap E(G) = \emptyset$), such that the graph with vertices $\{v_{j_1}, \dots, v_{j_t}\}$ and edge set X is a tree. Since $L_1 / (L_1 \cap L_2) \in \mathcal{C}_s$, it follows by definition of s -collapsible graphs that $L_1 / (L_1 \cap L_2) + X \in \mathcal{C}_s$, and so by Corollary 2.4(C2), $(L_1 / (L_1 \cap L_2) + X) / X \in \mathcal{C}_s$. By definition of contraction and since L_1, L_2 are vertex induced connected subgraphs of G , we have

$$(L_1 \cup L_2) / L_2 = (L_1 / (L_1 \cap L_2) + X) / X \in \mathcal{C}_s.$$

It follows from $L_2 \in \mathcal{C}_s$ and by Corollary 2.4(C3) that $L_1 \cup L_2 \in \mathcal{C}_s$.

(ii) The existence and the uniqueness of this set of \mathcal{C}_s -maximal subgraphs H_1, H_2, \dots, H_c follow from Corollary 2.4(C1) and from (i). Let $V(G') = \{u_1, u_2, \dots, u_c\}$, where u_i is the vertex onto which the subgraph H_i is contracted, $(1 \leq i \leq c)$. Suppose that G' has a nontrivial subgraph $H' \in \mathcal{C}_s$. We may assume that $V(H') = \{u_1, u_2, \dots, u_t\}$ with $t \geq 2$. Then by repeated applications of Corollary 2.4(C3),

$$H = G \left[E(H') \cup \left(\bigcup_{i=1}^t E(H_i) \right) \right] \in \mathcal{C}_s,$$

contrary to the assumption that these H_i 's are \mathcal{C}_s -maximal. \square

A graph is \mathcal{C}_s -**reduced** if it contains no nontrivial subgraph in \mathcal{C}_s . By Lemma 2.6, the graph $G' = G/(\bigcup_{i=1}^c E(H_i))$ is \mathcal{C}_s -reduced; call it the \mathcal{C}_s -**reduction** of G .

Corollary 2.7. *Let $s \geq 1$ be an integer. Let T be a spanning tree of a graph G . If for any $e \in E(T)$, e lies in a subgraph $H_e \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Proof. The hypothesis implies that G has a nontrivial subgraph in \mathcal{C}_s . Let H be a subgraph of G such that $H \in \mathcal{C}_s$ with $|V(H)|$ being maximized. If $G = H$, then we are done. Assume that $|V(H)| < |V(G)|$. Since T is a spanning tree, there must be an edge $e \in E(T) - E(H)$ such that e is incident with a vertex in H . By assumption, G has a subgraph $H_e \in \mathcal{C}_s$ such that $e \in E(H_e)$. Since $V(H) \cap V(H_e) \neq \emptyset$, by Lemma 2.6(i), $H \cup H_e \in \mathcal{C}_s$, contrary to the maximality of H . Hence we must have $G = H$ in \mathcal{C}_s . \square

Lemma 2.8. *Let $s \geq 1$ be an integer. Suppose that H is a connected subgraph of a given graph G , and let v_H denote the vertex in G/H onto which H is contracted. For any $x \in V(G)$, define $x' = x$ if $x \in V(G) - V(H)$ and $x' = v_H$ if $x \in V(H)$. If $H \in \mathcal{C}_s$, then for any $u, v \in V(G)$ with $u \neq v$, the following are equivalent.*

- (i) G has a spanning $(s + 1; u, v)$ -trail-system.
- (ii) If $u' \neq v'$, then G/H has a spanning $(s + 1; u', v')$ -trail-system; and if $u' = v' = v_H$, then G/H is supereulerian.

Proof. (i) \implies (ii). Let T_1, T_2, \dots, T_{s+1} be edge-disjoint (u, v) -trails in G such that $\bigcup_{i=1}^{s+1} T_i$ is spanning in G . For $i \in \{1, 2, \dots, s + 1\}$, define T'_i to be the graph obtained from $(T_i \cup H)/H$ by deleting the possible isolated vertex v_H . Then in G/H , if $u' \neq v'$, $T'_1, T'_2, \dots, T'_{s+1}$ are edge-disjoint (u', v') -trails. Since $\bigcup_{i=1}^{s+1} T_i$ is spanning in G , $\{T'_1, T'_2, \dots, T'_{s+1}\}$ is a spanning $(s + 1; u', v')$ -trail-system of G/H . If $u' = v'$, then since $u \neq v$ in G , we must have $u' = v' = v_H$, and so $T'_1, T'_2, \dots, T'_{s+1}$ are edge-disjoint closed trails in G/H . Since $\bigcup_{i=1}^{s+1} T_i$ is spanning in G , $\bigcup_{i=1}^{s+1} T'_i$ is a spanning closed trail in G/H , and so G/H is supereulerian.

(ii) \implies (i). Suppose first that $u' = v' = v_H$, and G/H is supereulerian. Let T' denote a spanning closed trail in G/H and let $X' = O(G[E(T')])$. Since T' is an Eulerian subgraph of G/H , we conclude that $X' \subseteq V(H)$ and $|X'| \equiv 0 \pmod{2}$. Since $H \in \mathcal{C}_s$, by Proposition 2.2, H has a spanning connected subgraph $L_{X'}$ with $O(L_{X'}) = X'$ such that $\kappa'(H - E(L_{X'})) \geq s - 1$. Thus $H - E(L_{X'})$ has $s - 1$ edge-disjoint (u, v) -paths T_1, T_2, \dots, T_{s-1} . Let $\Gamma = G[E(T') \cup E(L_{X'})]$. Since T' is spanning and connected in G/H , and since $L_{X'}$ is spanning and connected in H , Γ is a spanning connected subgraph of G with $O(\Gamma) = O(G[E(T')]) \Delta O(L_{X'}) = X' \Delta X' = \emptyset$. Thus Γ is a spanning Eulerian subgraph of G , and so Γ can be partitioned into two edge-disjoint (u, v) -trails T_s and T_{s+1} , such that $T_s \cup T_{s+1} = \Gamma$ is spanning in G . Note that Γ is edge-disjoint from $H - E(L_{X'})$ and from T_1, T_2, \dots, T_{s-1} . It follows that $\{T_1, T_2, \dots, T_{s+1}\}$ is a spanning $(s + 1; u, v)$ -trail-system.

Therefore we may assume that $u' \neq v'$ and $u' \neq v_H$. Choose a spanning $(s + 1; u', v')$ -trail-system $\{T'_1, T'_2, \dots, T'_{s+1}\}$ of G/H such that $d_{T'_1}(v_H) \geq d_{T'_2}(v_H) \geq \dots \geq d_{T'_{s+1}}(v_H)$ and such that $d_{T'_1}(v_H)$ is maximized. Since the T'_i 's are trails, the maximality of $d_{T'_1}(v_H)$ implies that we must have $d_{T'_i}(v_H) \leq 2$ for each i with $2 \leq i \leq s + 1$. Since for each i , T'_i is a (u', v') -trail in G/H ,

$$O(G[E(T'_i)]) \subseteq V(H) \cup \{u, v\}, \quad 1 \leq i \leq s + 1. \tag{3}$$

Define $Y_i = O(G[E(T'_i)]) \cap V(H)$, $(1 \leq i \leq s + 1)$. Without loss of generality, we assume that t is an integer such that $Y_i \neq \emptyset$ when $1 \leq i \leq t$, and $Y_i = \emptyset$, for all $i > t$. (If $v_H \in \{u', v'\}$, then $\{u, v\} \cap V(H) \neq \emptyset$ and so $t = s + 1$.) For each i with $1 \leq i \leq t$, T'_i is an (u', v') -trail containing v_H , and so there must be $u_i, v_i \in Y_i$ such that $G[E(T'_i)]$ contains an (u, u_i) -trail J_i and a (v_i, v) -trail J'_i such that J_i and J'_i are edge-disjoint. (If $v' = v_H$, we choose $v_i = v$ and in this case, J'_i consists of only one vertex.)

Since T'_1 and T'_2 are edge disjoint, the maximality of $d_{T'_1}(v_H)$ implies that $J' = T'_1 \cup T'_2$ is an Eulerian subgraph of G/H containing $\{u', v', v_H\}$. Let $X = O(G[E(J')])$. As J' is an Eulerian subgraph of G/H , we have $X \subseteq V(H)$ and $|X| \equiv 0 \pmod{2}$. Since $H \in \mathcal{C}_s$, and since $X \subseteq V(H)$ with $|X| \equiv 0 \pmod{2}$, by Proposition 2.2, H has a spanning connected subgraph L_X with $O(L_X) = X$, such that $\kappa'(H - E(L_X)) \geq s - 1$.

Let $J = G[E(J') \cup E(L_X)]$. Then J is an Eulerian subgraph of G containing $V(H) \cup \{u, v\}$. Hence J can be partitioned into two edge disjoint (u, v) -trails T_1, T_2 .

Since $\kappa'(H - E(L_X)) \geq s - 1$, for some permutation π on $\{3, 4, \dots, t\}$, $H - E(L_X)$ has edge-disjoint $(u_i, v_{\pi(i)})$ -trails J''_i , $(3 \leq i \leq t)$. Define edge induced subgraphs as follows:

$$T_i = \begin{cases} G \left[E(J_i) \cup E(J_{\pi(i)'}) \cup E(J''_i) \right] & \text{if } 3 \leq i \leq t \\ G[E(T'_i)] & \text{if } t + 1 \leq i \leq s + 1. \end{cases}$$

Recall that $\{T'_1, T'_2, \dots, T'_{s+1}\}$ is a spanning $(s + 1; u', v')$ -trail-system of G/H , that J_i and J'_i are subgraphs of T'_i , and that the (u_i, v_i) -trails J''_i ($3 \leq i \leq t$) in $H - E(L_X)$ are edge-disjoint subgraphs. By the definition of the T_i 's, for all $1 \leq i \leq s + 1$, these T_i 's are edge-disjoint (u, v) -trails. Since $V(G/H) = \bigcup_{i=1}^{s+1} V(T'_i)$ and since $V(H) \subseteq V(T_1) \cup V(T_2)$, it follows that $\bigcup_{i=1}^{s+1} V(T_i) = V(G)$ and so $\{T_1, T_2, \dots, T_{s+1}\}$ is a spanning $(s + 1; u, v)$ -trail-system of G . \square

Corollary 2.9. Let G be a graph and H be a subgraph of G with $H \in \mathcal{C}_s$. Each of the following holds.

- (i) $G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.
- (ii) If $\mu'(G) \geq s + 1$, then for any $e \in E(G)$, $\mu'(G/e) \geq s + 1$.
- (iii) $\mu'(G) \geq s + 1$ if and only if $\mu'(G/H) \geq s + 1$.

Proof. (i) follows from Corollary 2.4. To prove (ii), we assume that $e = xy$ and use v_e to denote the vertex in G/e onto which e is contracted. Let $u, v' \in V(G/e)$ such that $u \neq v'$. We may assume that $u \neq v_e$ and so $u \in V(G)$. Define $v = v'$ if $v' \neq v_e$ and $v = x$ if $v' = v_e$. Since $\mu'(G) \geq s + 1$, for any integer k with $1 \leq k \leq s + 1$, G has a spanning $(k; u, v)$ -trail system consisting of k edge-disjoint (u, v) -trails L_1, L_2, \dots, L_k . For each $1 \leq i \leq k$, define $L'_i = (L_i + e)/\{e\}$ if $x, y \in V(L_i)$ or $L'_i = L_i$ if $|\{x, y\} \cap V(L_i)| \leq 1$. By definition of the L'_i 's, $L'_1, L'_2, \dots, L'_{s+1}$ form a spanning $(k; u, v')$ -trail system in G/e . Thus (ii) must hold.

By (ii), if $\mu'(G) \geq s + 1$, then $\mu'(G/H) \geq s + 1$. Thus to prove (iii), we only need to assume that $\mu'(G/H) \geq s + 1$ to prove $\mu'(G) \geq s + 1$. Let k be an integer with $1 \leq k \leq s + 1$, and let v_H denote the vertex in G/H onto which H is contracted. For any $x \in V(G)$, define $x' = x$ if $x \notin V(H)$ and $x' = v_H$ if $x \in V(H)$. For any $u, v \in V(G)$, if $u' \neq v'$, then since $\mu'(G/H) \geq s + 1$, G/H has a spanning $(k; u'v')$ -trail system. If $u' = v'$, then as $\mu'(G/H) \geq s + 1 \geq 2$, by the definition of μ' , G/H is supereulerian. It follows by Lemma 2.8 that G has a spanning $(k; u, v)$ -trail system, and so as u, v are arbitrary vertices of G , $\mu'(G) \geq s + 1$. \square

For a graph G , let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G . By the well known spanning tree packing theorem of Nash-Williams [22] and Tutte [24], every $2k$ -edge-connected graph must have k edge-disjoint spanning trees. (For a direct proof of this fact, see [10], or Theorems 1.1 and 1.3 of [7]). Following Catlin's notation, let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph G' (possibly having multiple edges) with $\tau(G') \geq s$. The value of $F(G, s)$ has been studied and determined in [18], whose matroidal versions are proved in [14] and [17]. Catlin proved the following when $s = 2$.

Theorem 2.10 (Catlin, Theorem 7 of [4]). If $F(G, 2) \leq 1$, then $G \in \mathcal{C}_1$ if and only if $\kappa'(G) \geq 2$.

Further studies on $F(G, 2)$ can be found in [6]. We extend this theorem to all other values of s .

Theorem 2.11. Let $s \geq 1$ be an integer. If $F(G, s + 1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s + 1$.

Proof. Suppose first that $G \in \mathcal{C}_s$. By Corollary 2.5 and by (1), we have $\kappa'(G) \geq \mu'(G) \geq s + 1$.

Conversely, we assume that $\kappa'(G) \geq s + 1$ to prove that $G \in \mathcal{C}_s$. By Theorem 2.10, we may assume that $s > 1$. Let $n = |V(G)|$.

Since $F(G, s + 1) \leq 1$, G has spanning trees T_1, T_2, \dots, T_s such that $J = G - \bigcup_{i=1}^s E(T_i)$ is a spanning subgraph of G with at most two components. Let $X \subseteq V(G)$ be a subset with $|X| \equiv 0 \pmod{2}$. By Proposition 2.2, it suffices to show that G has a spanning connected subgraph L_X with $O(L_X) = X$ and with $\kappa'(G - E(L_X)) \geq s - 1$.

Claim 1. If for some i with $1 \leq i \leq s$, $T_i \cup J \in \mathcal{C}_1$, then $G \in \mathcal{C}_s$.

Suppose that $H = T_i \cup J \in \mathcal{C}_1$. Then $V(H) = V(T_i) = V(G)$. By Proposition 2.2, as $H \in \mathcal{C}_1$, H has a spanning connected subgraph L_X with $O(L_X) = X$. Note that $V(L_X) = V(H) = V(G)$. Since $G - E(L_X)$ contains spanning trees T_2, \dots, T_s , we have $\kappa'(G - E(L_X)) \geq s - 1$. By Proposition 2.2 again, $G \in \mathcal{C}_s$. This proves Claim 1.

By Theorem 2.10 and by Claim 1, if J is connected, then $G \in \mathcal{C}_s$ and we are done. Hence J has two components J' and J'' . For each i with $1 \leq i \leq s$, let $H_i = T_i \cup J$. By Claim 1, we may assume that for each i , $H_i \notin \mathcal{C}_1$. By definition, $F(H_i, 2) = 1$, for $1 \leq i \leq s$, and so by Theorem 2.10, we may assume that for all i , $\kappa'(H_i) = 1$. Thus for each i with $1 \leq i \leq s$, there must be an edge $e_i \in E(T_i)$ which is a cut edge of H_i , such that if T'_i, T''_i are the components of $T_i - e_i$, then $V(J') = V(T'_i)$ and $V(J'') = V(T''_i)$. It follows that $\{e_1, e_2, \dots, e_s\}$ is an edge cut of G separating $V(J')$ and $V(J'')$, contrary to the assumption that $\kappa'(G) \geq s + 1$. Hence we may assume that $\kappa'(H_1) \geq 2$. By Theorem 2.10, $H_1 \in \mathcal{C}_1$. By Claim 1, we conclude that $G \in \mathcal{C}_s$. \square

We need a theorem of Nash-Williams to derive a corollary of Theorem 2.11. For an explicit proof of this theorem, see Theorem 2.4 of [25].

Theorem 2.12 (Nash-Williams [23]). Let G be a graph. If $\frac{|E(G)|}{|V(G)|-1} \geq s + 1$, then G has a nontrivial subgraph L with $\tau(L) \geq s + 1$.

Corollary 2.13. Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.

- (i) If $\tau(G) \geq s + 1$, then $G \in \mathcal{C}_s$.
- (ii) If G is \mathcal{C}_s -reduced, then for any nontrivial subgraph H of G , $\frac{|E(H)|}{|V(H)|-1} < s + 1$.
- (iii) If $\kappa'(G) \geq s + 1$ and G is \mathcal{C}_s -reduced, then

$$F(G, s + 1) = (s + 1)(|V(G)| - 1) - |E(G)| \geq 2.$$

Proof. (i) If $\tau(G) \geq s + 1$, then $F(G, s + 1) = 0$ and $\kappa'(G) \geq \tau(G) \geq s + 1$. By [Theorem 2.11](#), $G \in \mathcal{C}_s$.

(ii) Assume that G is \mathcal{C}_s -reduced and for some connected subgraph H of G , $\frac{|E(H)|}{|V(H)|-1} \geq s + 1$. Then by [Theorem 2.12](#), H (and so G) has a nontrivial subgraph L with $\tau(L) \geq s + 1$. It follows from [Corollary 2.13](#)(i) that $L \in \mathcal{C}_s$, contrary to the assumption that G is \mathcal{C}_s -reduced.

(iii) The formula $F(G, s + 1) = (s + 1)(|V(G)| - 1) - |E(G)|$ follows from Lemma 3.1 of [[14](#)] (or indirectly, from Theorem 3.4 of [[18](#)]). Since G is nontrivial and \mathcal{C}_s -reduced, $G \notin \mathcal{C}_s$. Now the inequality follows from [Theorem 2.11](#). \square

The following theorem of Chen is useful when dealing with graphs with small order.

Theorem 2.14 ([Chen \[8\]](#)). *If G satisfies $\kappa'(G) \geq 3$ and $|V(G)| \leq 11$, then $G \in \mathcal{C}_1$ if and only if G cannot be contracted to the Petersen graph.*

3. Complete graphs and other examples

In this section, we shall study the \mathcal{C}_s membership and the μ' values of certain graphs, which will be useful in our arguments in later sections. For a graph G , if $X, Y \subseteq V(G)$ are disjoint vertex subsets, then $[X, Y]_G$ denotes the set of edges in G with one end in X and the other end in Y . We start with a simple example. For an integer $\ell > 1$, and a graph H , ℓH denotes the graph obtained from H by replacing each edge of H by a set of ℓ parallel edges joining the same pair of vertices. For example, ℓK_2 is the loopless connected graph with two vertices and ℓ edges. By [Corollaries 2.5](#) and [2.13](#) and as $\mu'(G) \leq \kappa'(G)$ for any graph G , we have

Corollary 3.1. *Let $\ell \geq 2, s \geq 1$ be integers. Then $\ell K_2 \in \mathcal{C}_s$ if and only if $\ell \geq s + 1$.*

Next we consider the problem to determine the values of n such that $K_n \in \mathcal{C}_s$, for a given integer $s \geq 1$.

Lemma 3.2. *Let $n \geq 2, s \geq 2$ be integers.*

- (i) *If both n and s are odd and if $sn > n^2 - 3n + 3$, then $K_n \notin \mathcal{C}_s$.*
- (ii) *If at least one of n and s is even, and if $sn > n^2 - 3n + 2$, then $K_n \notin \mathcal{C}_s$.*

Proof. In the proofs below, for each n satisfying the inequalities, we will choose a particular $R \subseteq V(K_n)$, and assume that if K_n has an (s, R) -subgraph Γ , then a contradiction will be obtained.

(i) Take $R \subset V(G)$ with $|R| = n - 1 \equiv 0 \pmod{2}$. Since Γ is an (s, R) -subgraph, by [Definition 2.1](#), we have $\kappa'(\Gamma) \geq s - 1$, $s - 1 \equiv 0 \pmod{2}$ and $O(\Gamma) = R$. Thus for any $v \in R$, we must have $d_\Gamma(v) \geq s$. It follows that $2|E(\Gamma)| = \sum_{v \in V(\Gamma)} d_\Gamma(v) \geq s(n - 1) + (s - 1) = sn - 1$. As $sn > n^2 - 3n + 3$, we have

$$|E(K_n) - E(\Gamma)| \leq \frac{n(n - 1)}{2} - \frac{sn - 1}{2} < \frac{(n^2 - n) - (n^2 - 3n + 3 - 1)}{2} = n - 1.$$

Hence $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R) -subgraph of K_n .

(ii) Set $R = V(K_n)$ if $s \equiv 1 \pmod{2}$, and $R = \emptyset$ if $s \equiv 0 \pmod{2}$. Then since $\kappa'(\Gamma) \geq s - 1$, we have $\delta(\Gamma) \geq s$, and so $2|E(\Gamma)| \geq sn$. Since $sn > n^2 - 3n + 2$, we have

$$|E(K_n) - E(\Gamma)| \leq \frac{n(n - 1)}{2} - \frac{sn}{2} < \frac{(n^2 - n) - (n^2 - 3n + 2)}{2} = n - 1.$$

Hence $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R) -subgraph of G . \square

Theorem 3.3. *Let $s \geq 2$ and $n \geq 2$ be integers. Then $K_n \in \mathcal{C}_s$ if and only if $n \geq s + 3$.*

Proof. By [Corollary 2.5](#) and (1), if $K_n \in \mathcal{C}_s$, then $\kappa'(K_n) \geq s + 1$. Thus if $n < s + 1$, then $K_n \notin \mathcal{C}_s$. Since $s \geq 2$, if $s + 1 \leq n \leq s + 2$, then by simple elementary computation in the respective two cases, we obtain $sn > n^2 - 3n + 3$, and so by [Lemma 3.2](#), $K_{s+1}, K_{s+2} \notin \mathcal{C}_s$. This completes the proof of necessity.

To prove sufficiency, we first consider $n > s + 3$. Note that K_n/K_{s+3} contains a spanning tree isomorphic to $K_{1, n-(s+3)}$ with the contraction image of K_{s+3} being a vertex of degree $n - (s + 3)$, such that every edge e of this spanning tree lies in a subgraph $H_e \cong (s + 3)K_2$. By [Corollaries 3.1](#) and [2.7](#), $K_n/K_{s+3} \in \mathcal{C}_s$. Thus if we can show $K_{s+3} \in \mathcal{C}_s$, then it follows from [Corollary 2.4](#)(C3) that $K_n \in \mathcal{C}_s$.

Let $n = s + 3$ and denote $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Then as $s \geq 2, n = s + 3 \geq 5$. Let $R \subseteq V(K_n)$ be a subset with $|R| \equiv 0 \pmod{2}$. We shall show that for any possible values of $|R|$, K_n always has an (s, R) -subgraph Γ_R .

In the arguments below, we will utilize the fact that if $n - 3 > \frac{n}{2}$, then the quadratic function $x(n - x) - 3x$ has minimum value $n - 4$ when $1 \leq x \leq \frac{n}{2}$. As for integer value n , we have $n - 3 > \frac{n}{2}$ if and only if $n \geq 7$, we first consider the cases when $n \geq 7$.

Case 1. $n = 2k + 1$, for some integer $k \geq 3$.

For each even subset $R \subset V(G)$ with $|R| = 2\ell \geq 0$ with $0 \leq \ell \leq k$, we will find an (s, R) -subgraph Γ_R below. By symmetry and since $n \geq 7$ is odd, we may assume that $v_1 \notin R$, and when $\ell > 0$, $R = \{v_i, v_{2k-i+3} : i = 2, 3, 4, \dots, \ell + 1\}$. Let $C_n = v_1 v_2 \dots v_n v_1$ be a hamiltonian cycle of K_n . Since $s = n - 3$, $K_n - E(C_n)$ is an s -edge-connected, s -regular graph. If $|R| = 0$, then define $\Gamma_R = K_n - E(C_n)$; if $\ell > 0$, then define $M_{(\ell)} = \{v_i v_{2k-i+2} : \text{with } i = 2, 3, \dots, \ell\} \cup \{v_{\ell+1} v_{2k}\}$. Note that $M_{(\ell)}$ is a perfect matching of $K_n - E(C_n) - v_1$, and observe that $M_{(\ell)} \cap E(C_n) = \emptyset$. Let $\Gamma_R = K_n - E(C_n) - M_{(\ell)}$. We claim that

$$\kappa'(\Gamma_R) \geq n - 4 = s - 1. \tag{4}$$

Let X, Y be a vertex partition of $V(K_n) = V(\Gamma_R)$ with $|X| = x$ and $|Y| = n - x$ such that $1 \leq x \leq n - x$. Then in $[X, Y]_{K_n}$, there are at most $2x$ edges in C_n and at most x edges in $M_{(\ell)}$. It follows that $|[X, Y]_{\Gamma_R}| \geq x(n - x) - 3x \geq n - 4$, where $1 \leq x \leq n/2$, and so (4) must hold.

By the definition of R , we have $O(\Gamma_R) = R$; as $G - E(\Gamma_R)$ contains the hamiltonian cycle C_n , it is connected. These, together with (4), imply that $K_n \in \mathcal{C}_s$.

Case 2. $n = 2k$, for some integer $k \geq 4$.

By symmetry and since n is even, we may assume that if $|R| = 2\ell > 0$, then $R = \{v_1, v_{k+1}, \dots, v_\ell, v_{k+1}\}$. Let $M_1 = \{v_i v_{k+i} : i = 1, 2, \dots, k\}$, $M_2 = \{v_i v_{k+i+1} : i = 1, 2, \dots, k - 1\} \cup \{v_k v_{k+1}\}$, and $M_3 = \{v_i v_{k+i+2} : i = 1, 2, \dots, k - 2\} \cup \{v_{k-1} v_{k+1}, v_k v_{k+2}\}$. Then M_1, M_2, M_3 are mutually edge disjoint perfect matchings of K_n . Let $L = G[M_1 \cup M_2 \cup M_3]$, and define

$$\Gamma_R = \begin{cases} K_n - E(L) & \text{if } |R| = 0, \\ K_n - E(L - \{v_i v_{k+i} : 1 \leq i \leq \ell\}) & \text{if } |R| = 2\ell \text{ for some } 0 < \ell \leq k. \end{cases}$$

We claim that

$$\kappa'(\Gamma_R) \geq \kappa'(\Gamma_R) \geq n - 4 = s - 1. \tag{5}$$

Let X, Y be a vertex partition of $V(K_n) = V(\Gamma_R)$ with $|X| = x$ and $|Y| = n - x$ such that $1 \leq x \leq n - x$. Then in $[X, Y]_{K_n}$, there are at most x edges in each M_i . It follows that $|[X, Y]_{\Gamma_R}| \geq x(n - x) - 3x \geq n - 4$, and so (5) must hold.

By the definition of Γ_R , we have $O(\Gamma_R) = R$; as $G - E(\Gamma_R)$ contains a hamiltonian cycle $v_1 v_{k+2} v_k v_{k+1} v_{k-1} v_{2k} v_{k-2} v_{2k-1} \dots v_2 v_{k+3} v_1$, whose edge set is $M_2 \cup M_3$, it is connected. These, together with (5), imply that $K_n \in \mathcal{C}_s$.

Case 3. $n \in \{5, 6\}$.

For $n = 5$, we have $s = 2$; let $C_5 = v_1 v_3 v_5 v_2 v_4 v_1$. Define

$$\Gamma_R = \begin{cases} C_5 & \text{if } R = \emptyset, \\ C_5 \cup \{v_3 v_4\} & \text{if } R = \{v_3, v_4\}, \\ (C_5 \cup \{v_3 v_4\}) - v_2 v_5 & \text{if } R = \{v_2, v_3, v_4, v_5\}. \end{cases}$$

In any case, $O(\Gamma_R) = R$ and both Γ_R and $G - E(\Gamma_R)$ are connected. By symmetry and by definition, $K_5 \in \mathcal{C}_2$.

Suppose that $n = 6$ and so $s = 3$. Let $C_6 = v_1 v_2 v_3 v_4 v_5 v_6 v_1$, and $H = C_6 + v_2 v_5$. Define

$$\Gamma_R = \begin{cases} C_6 & \text{if } R = \emptyset, \\ H & \text{if } R = \{v_2, v_5\}, \\ H \cup \{v_4 v_6\} & \text{if } R = \{v_2, v_4, v_5, v_6\}, \\ H \cup \{v_1 v_3, v_4 v_6\} & \text{if } R = V(K_6). \end{cases}$$

In any case, we have $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq 2$ such that $G - E(\Gamma_R)$ is connected. By symmetry and by definition, $K_6 \in \mathcal{C}_3$. \square

Example 3.1. We present some examples G with $\kappa'(G) = \mu'(G) = 3$. Let $C_n = v_1 v_2 \dots v_n v_1$ denote a cycle on n vertices and let $v_0 \notin \{v_1, v_2, \dots, v_n\}$ be a vertex. The **wheel** on $n + 1$ vertices, denoted by W_n , is obtained from C_n and v_0 by adding n new edges $v_0 v_i$, ($1 \leq i \leq n$). These new edges $v_0 v_i$, ($1 \leq i \leq n$), are referred to as spokes of W_n . The graph W'_n is obtained from W_n by contracting a spoke. Isomorphically, we can write $W'_n = W_n / \{v_0 v_n\}$. The following can be routinely verified (hint: apply Corollary 2.9(ii) for Example 3.1(ii)).

- (i) $\mu'(K_n) = \kappa'(K_n) = n - 1$.
- (ii) if $G \in \{W_n, W'_n\}$ for $n \geq 3$, then $\mu'(G) = \kappa'(G) = 3$.

4. $K_{3,3}$ is the smallest graph G with $\mu'(G) < \kappa'(G) = 3$

The main result of this section will determine the smallest graph G with $\mu'(G) < \kappa'(G) = 3$. For a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

We start by quoting a conditional reduction lemma; its proof is straightforward.

Table 1
 $\mu'(K_{2,3}^+) \geq 3$.

u	v	Spanning $(3; u, v)$ -trail system	Similar cases by symmetry
v_1	v_2	$\{e_1, \{e'_1, e_2, e'_2\}, \{e_3, v_4v_5, v_5v_2\}$	$v \in \{v_3, v_4\}, u = v_1$
v_1	v_5	$\{e_1, v_2v_5\}, \{e_2, v_3v_5\}, \{e_3, v_4v_5\}$	
v_2	v_3	$\{e_1, e_2\}, \{e'_1, e_3, e'_3, e'_2\}, \{v_2v_5, v_5v_3\}$	$(u, v) \in \{(v_2, v_4), (v_3, v_4)\}$
v_2	v_5	$\{v_2v_5\}, \{e_1, e_2, v_3v_5\}, \{e'_1, e_3, v_4v_5\}$	$(u, v) \in \{(v_3, v_5), (v_4, v_5)\}$

Lemma 4.1 (Lemma 5.4.1 of [17]). Let G be a graph and let $H = 2K_2$ be a subgraph of G . Denote $V(H) = \{z_1, z_2\}$ and $E(H) = \{e_1, e_2\}$. Suppose that

$$|E_G(z_i) - E(H)| \leq 2, \quad \text{for each } i = 1, 2. \tag{6}$$

Let v_H denote the vertex in G/H onto which H is contracted. For each vertex $v \in V(G)$, define $v' = v$ if $v \in V(G) - V(H)$ and $v' = v_H$ if $v \in V(H)$. Each of the following holds for any $u, v \in V(G)$.

- (i) If $\{u', v'\} - \{v_H\} \neq \emptyset$, and if G/H has a spanning $(3; u', v')$ -trail-system, then G has a spanning $(3; u, v)$ -trail-system.
- (ii) If $\{u, v\} = \{z_1, z_2\}$ and if $G - E(H)$ has a spanning (u, v) -trail, then G has a spanning $(3; u, v)$ -trail-system.

A subgraph $2K_2$ of G is a **contractible** $2K_2$ of G if it satisfies (6) and Lemma 4.1(ii).

Example 4.1. Let C_n be a cycle on $n \geq 3$ vertices. Then $\forall e \in E(2C_n)$, repeat the application of Lemma 4.1 to digons not containing e to result in a $4K_2$. This shows that $\mu'(2C_n - e) = 3$.

Lemma 4.2. Let $K_{3,3}, K_{2,3}^+, K'_{2,4}, K''_{2,4}, K'''_{2,4}$, and $S(2, 1)$ be the graphs depicted in Fig. 1A. Each of the following holds.

- (i) $\mu'(K_{3,3}) = 2$.
- (ii) For each $G \in \{K_{2,3}^+, K'_{2,4}, K''_{2,4}, K'''_{2,4}\}$, $\mu'(G) = 3$.
- (iii) If G is a non-hamiltonian graph spanned by a $S(2, 1)$, and if $\kappa'(G) \geq 3$, then $\mu'(G) = 3$.

Proof. We shall use the notations in Fig. 1A in the proofs.

(i) By Theorem 2.10, $K_{3,3} \in \mathcal{C}_1$, and so by Corollary 2.5, $\mu'(K_{3,3}) \geq 2$. It suffices to show that for some $u, v \in V(K_{3,3})$, $K_{3,3}$ does not have a spanning $(3; u, v)$ -trail-system.

Suppose that $K_{3,3}$ has a spanning $(3; v_1, v_3)$ -trail-system $\{P_1, P_2, P_3\}$. Let $e_1 = v_1v_2, e_2 = v_1v_4$, and $e_3 = v_1v_6$; and $f_1 = v_3v_2, f_2 = v_3v_4$ and $f_3 = v_3v_6$. Since P_1, P_2, P_3 are edge-disjoint, we must have

$$|\{e_1, e_2, e_3\} \cap E(P_i)| = 1 = |\{f_1, f_2, f_3\} \cap E(P_i)|, \quad \forall i \in \{1, 2, 3\}. \tag{7}$$

By (7), we may assume that $e_i \in E(P_i)$, ($1 \leq i \leq 3$). If $f_1 \notin E(P_1)$, then since $K_{3,3}$ is 3-regular, P_1 must use v_2v_5 , which will force f_1 lying in no P_i 's, contrary to (7). Therefore, we must have $f_1 \in E(P_1)$. Similarly, we must have $f_2 \in E(P_2)$ and $f_3 \in E(P_3)$. Since $v_5 \notin V(P_i)$, ($1 \leq i \leq 3$), it follows that $K_{3,3}$ does not have a spanning $(3; v_1, v_3)$ -trail-system, and so $\mu'(K_{3,3}) = 2$. This proves (i).

(ii) To show that $\mu'(K_{2,3}^+) = 3$, by (1), it suffices to show that for any distinct $u, v \in V(K_{2,3}^+)$ and any integer $1 \leq s \leq 3$, there will always be a spanning $(s; u, v)$ -trail system. Since $\tau(K_{2,3}^+) = 2$, it follows by Corollaries 2.13 and 2.5 that $\mu'(K_{2,3}^+) \geq 2$. Table 1 shows that we can always find spanning $(3; u, v)$ -trail systems for any $u, v \in V(K_{2,3}^+)$. This proves that $\mu'(K_{2,3}^+) = 3$. The proofs for the cases when $G \in \{K'_{2,4}, K''_{2,4}, K'''_{2,4}\}$ are similar but somewhat more elaborate, and will thus be omitted. This proves (ii).

(iii) Let G be a minimally 3-edge-connected non-hamiltonian graph spanned by an $S(2, 1)$, and let \tilde{G} be the underlying simple graph of G . We adopt the labels of $S(2, 1)$ in Fig. 1A, and denote $e_1 = v_1v_2, e_2 = v_1v_3, e_3 = v_3v_5, e_4 = v_1v_4, e_5 = v_5v_6$. If e_i has a duplicated edges, then we assume that e_i, e'_i are parallel edges in the discussions below. Since G is not hamiltonian,

$$v_2v_3 \notin E(G), \quad \text{and for any } i \in \{2, 3\} \text{ and for any } j \in \{4, 6\}, v_iv_j \notin E(G). \tag{8}$$

Since G is minimally 3-edge-connected, and by (8),

$$\text{for every } i \in \{2, 3\}, \text{ there exists exactly one } j \in \{1, 5\} \text{ such that } v_iv_j \text{ is a parallel edge in } G. \tag{9}$$

By (9) and by symmetry, we assume that v_1, v_2 are joined by parallel edges e_1 and e'_1 .

Case 1. $\tilde{G} = S(2, 1)$ and v_1, v_3 are joined by parallel edges e_2, e'_2 .

If v_1, v_4 are also joined by parallel edges, then by $\kappa'(G) \geq 3$, either $G[\{v_4, v_6\}]$ or $G[\{v_5, v_6\}]$ is a contractible $2K_2$; and contracting this $2K_2$ results in a graph isomorphic to $K_{2,3}^+$. By Lemma 4.2(ii), and by Lemma 4.1, $\mu'(G) = 3$. Hence we assume that $G[\{v_1, v_4\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, we have $G[\{v_4, v_6\}] \cong G[\{v_5, v_6\}] \cong 2K_2$, and both are contractible $2K_2$. Contracting these $2K_2$ results in a graph $J(4)$, depicted in Fig. 1B, with

$$V(J(4)) = \{v_1, v_2, v_3, v_4\} \quad \text{and} \quad E(J(4)) = \{e_1, e'_1, e_2, e'_2, v_1v_4, v_2v_4, v_3v_4\}. \tag{10}$$

It is routine to verify that $\mu'(J(4)) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$. This proves Case 1.

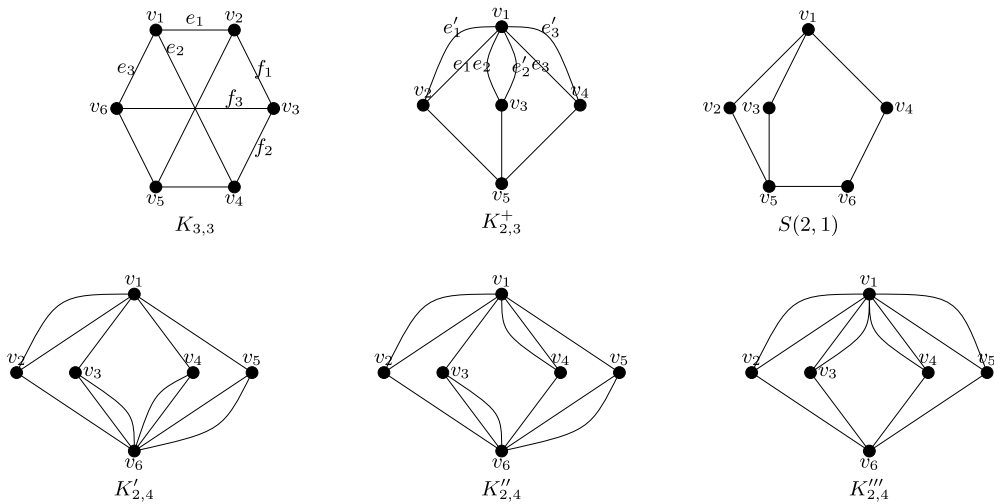


Fig. 1A. Graphs $K_{3,3}$, $K_{2,3}^+$, $S(2, 1)$, $K'_{2,4}$, $K''_{2,4}$ and $K'''_{2,4}$.

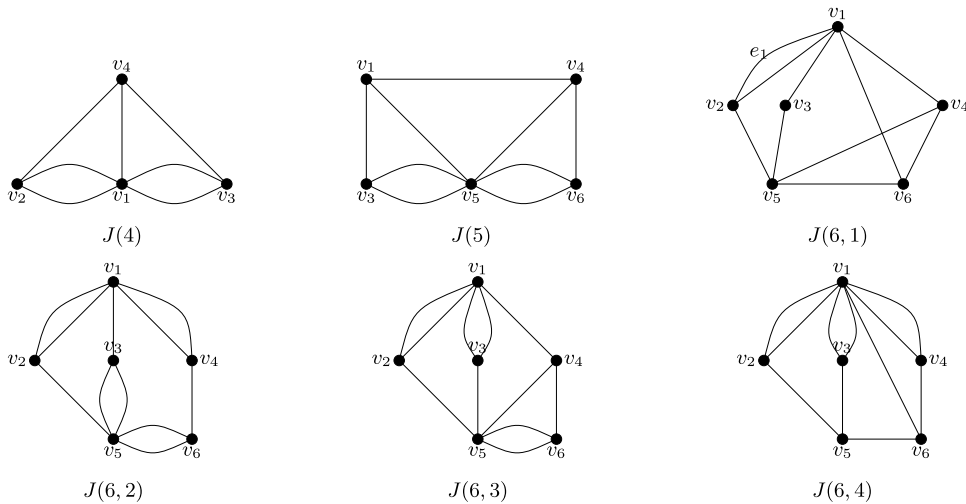


Fig. 1B. Graphs $J(4)$, $J(5)$ and $J(6, i)$, $1 \leq i \leq 4$.

Case 2. $\tilde{G} = S(2, 1)$ and v_1, v_3 are not joined by parallel edges.

By (9), v_3, v_5 are joined by parallel edges e_3, e'_3 . If $G[\{v_4, v_6\}] \cong 2K_2$, then as G is minimally 3-edge-connected, either $G[\{v_1, v_4\}] \cong 2K_2$ or $G[\{v_5, v_6\}] \cong 2K_2$. In the first case, $G[\{v_3, v_5\}]$ and $G[\{v_4, v_6\}]$ are contractible $2K_2$'s; in the second case, $G[\{v_1, v_2\}]$ and $G[\{v_4, v_6\}]$ are contractible $2K_2$'s. As contracting the corresponding $2K_2$'s results in a graph isomorphic to $J(4)$ defined in (10), and as $\mu'(J(4)) = 3$, it follows by Lemma 4.1 that $\mu'(G) = 3$. Hence we may assume that $G[\{v_4, v_6\}] \cong K_2$, and so by $\kappa'(G) \geq 3$, we have both $G[\{v_1, v_4\}] \cong 2K_2$ and $G[\{v_5, v_6\}] \cong 2K_2$. In order for $G[\{v_1, v_4\}]$ not to be a contractible $2K_2$, we must have $G[\{v_1, v_2\}] \cong 2K_2$. Thus $G \cong J(6, 2)$ depicted in Fig. 1B. Now it is routine to verify that $\mu'(G) = 3$. This proves Case 2.

In the cases of Cases 3, 4, and 5, \tilde{G} differs from $S(2, 1)$ but contains $S(2, 1)$ as a spanning subgraph.

Case 3. $\tilde{G} \neq S(2, 1)$ and $v_1v_6, v_4v_5 \in E(\tilde{G})$.

Then either e_2, e'_2 are parallel edges joining v_1, v_3 or e_3, e'_3 are parallel edges joining v_3, v_5 in G . Define $J(6, 1)$, depicted in Fig. 1B, as follows:

$$V(J(6, 1)) = V(S(2, 1)), \quad \text{and} \quad E(J(6, 1)) = E(S(2, 1)) \cup \{e'_1, v_1v_6, v_4v_5\}, \tag{11}$$

and define $G'_2 = J(6, 1) + e'_2$ and $G''_2 = J(6, 1) + e'_3$. By the assumption of Case 3, and since G is minimally 3-edge-connected, we have $G \in \{G'_2, G''_2\}$. It is routine to verify that $\mu'(G) = 3$. This proves Case 3.

Case 4. $\tilde{G} \neq S(2, 1)$ and $v_4v_5 \in E(\tilde{G})$ and $v_1v_6 \notin E(\tilde{G})$.

If $G[\{v_4, v_6\}] \cong 2K_2$, then $G[\{v_4, v_6\}]$ is always a contractible $2K_2$. It follows that either $G[\{v_1, v_3\}] \cong 2K_2$, whence $\{v_4v_5, v_5v_6\}$ induces another contractible $2K_2$ in $G/G[\{v_4, v_6\}]$; or $G[\{v_1, v_3\}] \cong K_2$, whence $G[\{v_3, v_5\}] \cong 2K_2$ and

$G[\{v_1, v_2\}]$ is a contractible $2K_2$ in G . After contracting these contractible $2K_2$'s, we obtain a graph isomorphic to $J(4)$ defined in (10). As we already know that $\mu'(J(4)) = 3$, by Lemma 4.1, $\mu'(G) = 3$.

Hence we assume that $G[\{v_4, v_6\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, $G[\{v_5, v_6\}] \cong 2K_2$. Thus either $G[\{v_3, v_5\}] \cong 2K_2$, or $G[\{v_1, v_3\}] \cong 2K_2$. If $G[\{v_3, v_5\}] \cong 2K_2$, then $G[\{v_1, v_2\}]$ is a contractible $2K_2$, and $G/G[\{v_1, v_2\}] \cong J(5)$, depicted in Fig. 1B, with

$$V(J(5)) = \{v_1, v_3, v_4, v_5, v_6\} \quad \text{and} \quad E(J(5)) = \{v_1v_3, e_3, e'_3, e_5, e'_5, v_1v_4, v_1v_5, v_4v_5\}. \tag{12}$$

If $G[\{v_1, v_3\}] \cong 2K_2$, then $G = S(2, 1) + \{e'_1, e'_2, e'_5, v_4v_5\}$, which is the graph $J(6, 3)$ depicted in Fig. 2B. It is routine to verify that $\mu'(G) = 3$.

Case 5. $\tilde{G} \neq S(2, 1)$ and $v_4v_5 \notin E(\tilde{G})$ and $v_1v_6 \in E(\tilde{G})$.

If $G[\{v_4, v_6\}] \cong 2K_2$, then $G[\{v_4, v_6\}]$ is a contractible $2K_2$. By $\kappa'(G) \geq 3$, either $G[\{v_3, v_5\}] = 2K_2$ or $G[\{v_1, v_3\}] = 2K_2$. If $G[\{v_3, v_5\}] = 2K_2$, then all the $2K_2$'s in G are contractible, and contracting all these contractible $2K_2$'s results in a $J(4)$. Thus by $\mu'(J(4)) \geq 3$ and Lemma 4.1, $\mu'(G) = 3$ in this case. If $G[\{v_1, v_3\}] = 2K_2$, then $G/G[\{v_4, v_6\}] \cong K_{2,3}^+$. By Lemma 4.2(ii), $\mu'(K_{2,3}^+) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$.

Therefore, we assume that $G[\{v_4, v_6\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, $G[\{v_1, v_4\}] \cong 2K_2$. By $\kappa'(G) \geq 3$, either $G[\{v_3, v_5\}] = 2K_2$ or $G[\{v_1, v_3\}] = 2K_2$. If $G[\{v_3, v_5\}] = 2K_2$, then $G[\{v_3, v_5\}]$ is contractible, and $G/G[\{v_3, v_5\}] \cong J(5)$ defined in (12). As we already know that $\mu'(J(5)) = 3$, by Lemma 4.1, $\mu'(G) = 3$. If $G[\{v_1, v_3\}] = 2K_2$, then $G \cong (J(6, 1) + \{e'_2, e'_4\}) - v_4v_5$, where $J(6, 1)$ is defined in (11). We denote $J(6, 4) = (J(6, 1) + \{e'_2, e'_4\}) - v_4v_5$, as depicted in Fig. 1B. It is routine to verify that $\mu'(J(6, 4)) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$.

By (8) and (9), these cases cover all the possibilities and so the proof of (iii) is complete. \square

Lemma 4.3. *If $e \notin E(K_{3,3})$ is an edge whose ends are in $V(K_{3,3})$, and if $G = K_{3,3} + e$, then $\mu'(G) \geq 3$.*

Proof. We use the notation of Fig. 1A for $K_{3,3}$ and let $G = K_{3,3} + e$. By symmetry, we may assume that $e = v_1v_i$. If $G[\{v_1, v_i\}]$ is a contractible $2K_2$ of G , then $i \in \{2, 4, 6\}$ and $G/G[\{v_1, v_i\}]$ is isomorphic to W_4 , the wheel on 5 vertices. By Example 3.1, $\mu'(W_4) = 3$ and so by Lemma 4.1, $\mu'(G) \geq 3$. Now assume that $i \in \{3, 5\}$. It is routine to show that $\mu'(G) \geq 3$. (Detailed verification can be found in Chapter 5 of [17].) \square

Before proving the next theorem, we observe that, for every integer $k \geq 1$,

$$\mu'(G) \geq k \quad \text{if and only if} \quad \text{every block } H \text{ of } G \text{ satisfies } \mu'(H) \geq k. \tag{13}$$

Theorem 4.4. *Let G be a graph on n vertices.*

- (i) (Lemma 5 of [4]) *If $n \leq 4$, and if $\kappa'(G) \geq 2$, then $\mu'(G) \geq 2$ if and only if $G \neq K_{2,2}$.*
- (ii) *If $n \leq 6$, and if $\kappa'(G) \geq 3$, then $\mu'(G) \geq 3$ if and only if $G \neq K_{3,3}$.*

Proof of (ii). By Lemma 4.2(i), $\mu'(K_{3,3}) < 3$. It suffices to show that if $G \neq K_{3,3}$, then $\mu'(G) \geq 3$. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |E(G)| + |V(G)| \text{ minimized.} \tag{14}$$

If $n \leq 3$, then $\kappa'(G) \geq 3$ implies that $F(G, 3) \leq 1$, and so in (ii), it follows from Theorem 2.11 for $s = 2$ and from Corollary 2.5 that $n \geq 4$. We claim that

$$4 \leq n \leq 6, \kappa(G) \geq 2, \quad G \text{ is } \mathcal{C}_2\text{-reduced and minimally 3-edge-connected.} \tag{15}$$

As $n \geq 4$, by assumption, $n \leq 6$, hence $4 \leq n \leq 6$. By (13) and by (14), we conclude that $\kappa(G) \geq 2$. If G has a nontrivial subgraph H with $H \in \mathcal{C}_2$, then G/H satisfies both $|V(G/H)| < 6$ and $\kappa'(G/H) \geq 3$. It follows from $|V(G/H)| \leq 5$ that $G/H \neq K_{3,3}$ and so by (14), we have $\mu'(G/H) \geq 3$. By Corollary 2.9(iii) with $s = 2$, and by $H \in \mathcal{C}_2$, we conclude that $\mu'(G) \geq 3$, contrary to (14). Thus G must be \mathcal{C}_2 -reduced. If there exists an edge $e \in E(G)$ such that $\kappa'(G-e) \geq 3$, then by (14), we have $\mu'(G-e) \geq 3$. But $\mu'(G) \geq \mu'(G-e) \geq 3$, contrary to (14). Therefore, G must be minimally 3-edge-connected. This justifies (15).

If G has a subgraph H which is a contractible $2K_2$, then as $\kappa'(G/H) \geq \kappa'(G) \geq 3$, by (14), $\mu'(G/H) \geq 3$. By Lemma 4.1, $\mu'(G) \geq 3$, contrary to (14). Thus

$$G \text{ has no contractible } 2K_2. \tag{16}$$

By (15) and (16), we make the following observations.

Observation 1. *Let \tilde{G} denote the underlying simple graph of G , and suppose that \tilde{G} has a hamiltonian cycle C .*

- (i) *If \tilde{G} has at most one vertex of degree at least 4, then the vertices of degree 2 in \tilde{G} must be an independent set of \tilde{G} .*
- (ii) *Every edge of \tilde{G} not lying in a 2-edge-cut of \tilde{G} is not a parallel edge in G . For every edge cut X of size 2 in \tilde{G} , exactly one edge in X is a parallel edge in G .*
- (iii) *Every chord of C in \tilde{G} cannot have parallel edges in G .*
- (iv) *Every edge of G must be in a 3-edge-cut of G .*

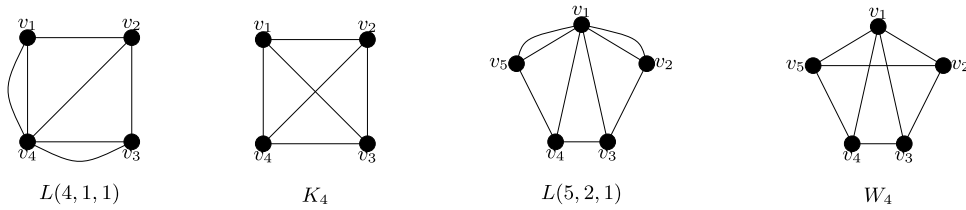


Fig. 2. Graphs in Claim 2.

In fact, if \tilde{G} has two adjacent vertices (say v_1, v_2) of degree 2 in \tilde{G} , then since \tilde{G} has at most one vertex of degree at least 4, we may assume that v_1 is not incident with a vertex of degree at least 4 in \tilde{G} . Since $\kappa'(G) \geq 3$, at least one edge incident with v_1 must be a parallel edge, and so by definition, G has a contractible $2K_2$, violating (16). This justifies Observation 1(i). Observation 1(ii) and (iv) follow from the assumption that G is minimally 3-edge-connected, stated in (15). Since any chord of C is not lying in a 2-edge-cut of \tilde{G} , Observation 1(iii) follows from Observation 1(ii).

Note that by Theorem 2.14, every such graph has a spanning Eulerian subgraph. By (15) and by $n \leq 6$, we further claim that

$$\text{every such graph } G \text{ has a Hamilton cycle } C = v_1 v_2 \cdots v_n v_1. \tag{17}$$

To justify (17), we observe that every 2-connected graph on 4 vertices must be hamiltonian, and so we assume that $n \in \{5, 6\}$. Now we proceed by contradiction. Let c be the length of a longest cycle of G . Since $\kappa(G) \geq 2$ and $n \geq 5$, we have $n > c \geq 4$.

Assume first that $c = 4$. Hence G has a $K_{2,2}$. Let $K \cong K_{2,t}$ be a subgraph of G with t maximized. For any $v \in V(G) - V(K)$, by $\kappa(G) \geq 2$, v must have two internally disjoint paths from v to K . As $c = 4$, v must be adjacent to the two vertices of degree t in $K \cong K_{2,t}$, violating the maximality of K . Hence G is spanned by a $K_{2,3}$ or a $K_{2,4}$. Since $c = 4$, G must be obtained from a $K_{2,3}$ or a $K_{2,4}$ by duplicating some edges in the $K_{2,3}$ or $K_{2,4}$, as otherwise G has a cycle longer than 4.

If G is spanned by a $K_{2,3}$, then by (16) and (15), we conclude that $G \cong K_{2,3}^+$, and so by Lemma 4.2, $\mu'(G) = 3$, contrary to (14). Now assume that G is spanned by a $K_{2,4}$. By $\kappa'(G) \geq 3$ and $c = 4$, one of the two edges incident with a vertex of degree 2 in this $K_{2,4}$ must be a parallel edge. It follows from (16) and (15) that $G \in \{K_{2,4}', K_{2,4}'', K_{2,4}'''\}$. By Lemma 4.2(ii), we have $\mu'(G) = 3$, contrary to (14). This finishes the case when $c = 4$.

Next, we assume that $c = 5$; $n = 6$ follows from necessity. By $\kappa(G) \geq 2$, and by $c = 5$, we conclude that G is a non-hamiltonian graph spanned by an $S(2, 1)$ with $\kappa'(G) \geq 3$, and so by Lemma 4.2(iii), $\mu'(G) = 3$, contrary to (14). This justifies (17).

Recall that \tilde{G} denotes the underlying simple graph of G . Let C be a hamiltonian cycle of \tilde{G} . Let $f(G, C) = |E(\tilde{G})| - n$ denote the number of chords of C in \tilde{G} . If $f(G, C) = 0$, then $G = 2C_n - e$ by (15), and so by Example 4.1, $\mu'(G) = 3$, contrary to (14). Hence $f(G, C) \geq 1$. If $n \geq 5$ and $f(G, C) = 1$, then by $\kappa'(G) \geq 3$ and by (15), it is straightforward to verify that G must have a contractible $2K_2$, violating (16). Therefore, we have

Claim 1. When $n \geq 5$, $f(G, C) \geq 2$.

Claim 2. Theorem 4.4(ii) holds if $4 \leq n \leq 5$.

We shall use the notations in Fig. 2 in our arguments below. By (16), G cannot have a contractible $2K_2$. Therefore, if $n = 4$, G must be either K_4 or $L(4, 1, 1)$ as depicted in Fig. 2. In fact, as $n = 4$, $1 \leq f(G, C) \leq 2$, where $f(G, C) = 2$ if and only if $G = K_4$. By Example 3.1, $\mu'(K_4) = 3$. We assume that $f(G, C) = 1$, and without loss of generality, that $v_2 v_4 \in E(G)$ and $v_1 v_3 \notin E(G)$ (see Fig. 2). By $\kappa'(G) \geq 3$, one of the two edges incident with v_1 or v_3 must have parallel edges. By (16) and (15), these parallel edges must be all incident with v_2 or all incident with v_4 , and so $G \cong L(4, 1, 1)$. It is straightforward to verify that $\mu'(L(4, 1, 1)) = 3$, and so we assume $n = 5$.

By Claim 1 and (15), $2 \leq f(G, C) \leq 4$. If $f(G, C) = 4$, then one of the chords of C may be removed and the resulting graph is still 3-edge-connected, contrary to (15). Next we assume $f(G, C) = 3$. As G is spanned by a 5-cycle, \tilde{G} has a vertex of degree 4. We assume that v_1 has degree 4 in \tilde{G} , and so $v_1 v_3, v_1 v_4 \in E(\tilde{G})$. By symmetry, we assume that the third chord of C in \tilde{G} is $v_2 v_5$, resulting in a wheel W_4 . As W_4 is already 3-edge-connected, we conclude that if $f(G, C) = 3$, then $G = W_4$, (see Fig. 2). By Example 3.1, $\mu'(W_4) = 3$. Finally we assume that $f(G, C) = 2$. If these two chords of C are not incident with the same vertex in C , then $\Delta(\tilde{G}) = 3$. By $\kappa'(G) \geq 3$, any vertex of degree 2 in \tilde{G} must be incident with parallel edges in G . As $\Delta(\tilde{G}) = 3$, G must have a contractible $2K_2$, contrary to (16). Hence we may assume that v_1 has degree 4 in \tilde{G} and $v_1 v_3, v_1 v_4 \in E(\tilde{G})$. As v_1 is the only vertex of \tilde{G} with degree 4, any parallel edge not incident with v_1 must be a contractible $2K_2$. By (15) and (16), G must be isomorphic to a $L(5, 2, 1)$, (see Fig. 2). It is routine to verify that $\mu'(L(5, 2, 1)) = 3$. (Detailed verifications can be found in Chapter 5 of [17].) This completes the proof for Claim 2.

We are now ready to complete the proof of Theorem 4.4(ii). By Claim 2 and Lemma 4.3, we may assume that $n = 6$ and G is not spanned by a $K_{3,3}$. If $f(G, C) \leq 1$, then $\Delta(\tilde{G}) = 3$ with 4 vertices of degree 2, which cannot be independent, contrary

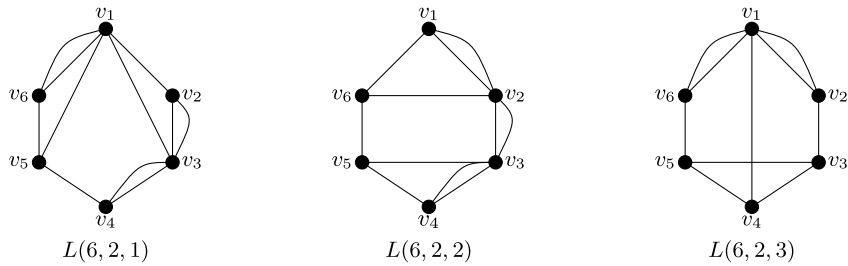


Fig. 3A. The graphs $L(6, 2, j)$ with $1 \leq j \leq 3$ in Case 1.

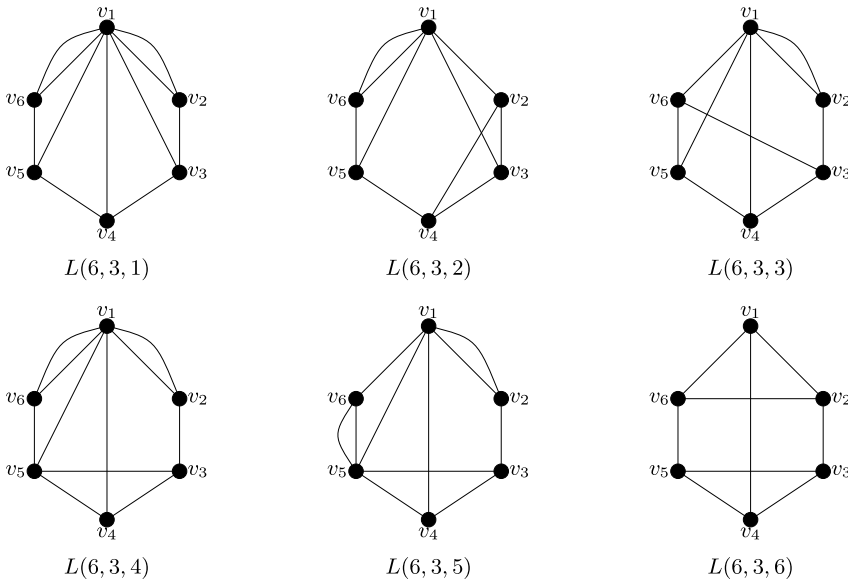


Fig. 3B. G has 6 vertices with 3 chords of C in Case 2.

to **Observation 1**(i). If $f(G, C) \geq 5$, then \tilde{G} is not minimally 3-edge-connected, violating (15). Hence $2 \leq f(G, C) \leq 4$. Let $d = \Delta(\tilde{G})$.

Case 1. $f(G, C) = 2$. Then $3 \leq d \leq 4$.

If $d = 4$, we may assume that v_1 has degree 4. By **Observation 1**(i), we must have $v_1v_3, v_1v_5 \in E(\tilde{G})$. By $\kappa'(G) \geq 3$, we may assume that $G[\{v_3, v_4\}] \cong 2K_2$. By (16), we must have $G[\{v_2, v_3\}] \cong 2K_2$. By $\kappa'(G) \geq 3$, either that $G[\{v_5, v_6\}] \cong 2K_2$, which is a contractible $2K_2$ of G ; or $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 2, 1)$, (see Fig. 3A).

If $d = 3$, then by symmetry and by **Observation 1**(i), we may assume either $v_1v_4, v_2v_5 \in E(\tilde{G})$, or $v_2v_6, v_3v_5 \in E(\tilde{G})$ or $v_1v_4, v_3v_5 \in E(\tilde{G})$. If $v_1v_4, v_2v_5 \in E(\tilde{G})$, then by **Observation 1**(ii), both v_1v_2 and v_4v_5 are not parallel edges in G . It follows that G will always have a contractible $2K_2$, contrary to (16). Next we assume that $v_2v_6, v_3v_5 \in E(\tilde{G})$. By $\kappa'(G) \geq 3$ and by symmetry, we may assume that $G[\{v_1, v_2\}] \cong 2K_2$. As $G[\{v_1, v_2\}]$ cannot be a contractible $2K_2$, we must have $G[\{v_2, v_3\}] \cong 2K_2$. By (15) and (16), either both $G[\{v_4, v_5\}] \cong 2K_2$ and $G[\{v_5, v_6\}] \cong 2K_2$, whence $\kappa'(G - v_3v_5) \geq 3$, contrary to (15); or $G[\{v_3, v_4\}] \cong 2K_2$, whence $G = L(6, 2, 2)$, (see Fig. 3A).

Finally we assume that $d = 3$ and $v_1v_4, v_3v_5 \in E(\tilde{G})$. It is straightforward to verify that if $G[\{v_2, v_3\}] \cong 2K_2$, then it will be a contractible $2K_2$. Thus we must have $G[\{v_1, v_2\}] \cong 2K_2$. By symmetry and (16), we also have $G[\{v_1, v_6\}] \cong 2K_2$. Hence $G = L(6, 2, 3)$, (see Fig. 3A).

Therefore, if $f(G, C) = 2$, then $G \in \{L(6, 2, 1), L(6, 2, 2), L(6, 2, 3)\}$. It is routine to verify that in any of these cases, $\mu'(G) \geq 3$. This proves Case 1.

Case 2. $f(G, C) = 3$. Then $3 \leq d \leq 5$.

If $d = 5$, then we may assume that $v_1v_3, v_1v_4, v_1v_5 \in E(\tilde{G})$. As before, it is routine to verify that if $G[\{v_2, v_3\}] \cong 2K_2$, then $G[\{v_2, v_3\}]$ is a contractible $2K_2$. Hence by **Observation 1**(ii), $G[\{v_1, v_2\}] \cong 2K_2$. By symmetry, $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 3, 1)$ (depicted in Fig. 3B).

If $d = 3$, then C has 3 independent chords in \tilde{G} , forcing $G \in \{K_{3,3}, L(6, 3, 6)\}$. However, $G \neq K_{3,3}$ by hypothesis, and so $G = L(6, 3, 6)$, (see Fig. 3B).

Next we suppose that $d = 4$ and v_1 has degree 4 in \tilde{G} . Assume first that v_1 is adjacent to v_2, v_3, v_5, v_6 . If $v_3v_5 \in E(\tilde{G})$, then v_3v_5 is not in any 3-edge-cut of G ; if $v_3v_6 \in E(\tilde{G})$, then v_1v_3 is not in any 3-edge-cut of G . By **Observation 1**(iv), neither

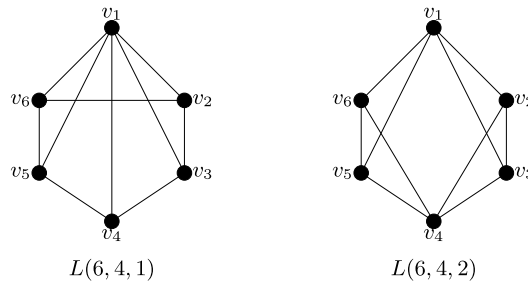


Fig. 4. G has at least 4 chords of C in Case 3.

possibility holds. By symmetry, we must have $v_2v_4 \in E(\tilde{G})$. By **Observation 1(ii)** and by (16), we must have $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 3, 2)$ (depicted in Fig. 3B).

Therefore, by symmetry, we may assume that v_1 is adjacent to v_2, v_4, v_5, v_6 . To avoid a contractible $2K_2$, v_3 must have degree 3 in \tilde{G} . Hence either $v_3v_6 \in E(\tilde{G})$ or $v_3v_5 \in E(\tilde{G})$. If $v_3v_6 \in E(\tilde{G})$, then by (15) and (16), $G[\{v_1, v_2\}] \cong 2K_2$, and so $G = L(6, 3, 3)$ (depicted in Fig. 3B).

Suppose that $v_3v_5 \in E(\tilde{G})$. By (15) and (16), we must have $G[\{v_1, v_2\}] \cong 2K_2$, and either $G[\{v_1, v_6\}] \cong 2K_2$ or $G[\{v_5, v_6\}] \cong 2K_2$. It follows that $G \in \{L(6, 3, 4), L(6, 3, 5)\}$ (depicted in Fig. 3B). However, v_1v_5 is not in any 3-edge-cut of G if $G \in \{L(6, 3, 4), L(6, 3, 5)\}$, contrary to **Observation 1(iv)**.

Therefore, if $f(G, C) = 3$, then $G \in \{L(6, 3, j) : j = 1, 2, 3, 6\}$. It is routine to verify that in any of these cases, $\mu'(G) \geq 3$. (Detailed verifications can be found in Chapter 5 of [17].)

Case 3. $f(G, C) = 4$. Then as $n = 6$ and C has at least 4 chords, $4 \leq d \leq 5$.

If \tilde{G} has a vertex v of degree 2, then at least 4 edges in $E(\tilde{G}) - E(C)$ will be joining the vertices of $V(C) - \{v\}$, and so G must have at least one edge e , both of whose ends are of degree at least 4 in \tilde{G} , such that $\kappa'(G - e) \geq 3$. Thus G is not minimally 3-edge-connected, contrary to (15). This, together with **Lemma 4.3**, implies that

$$\delta(\tilde{G}) \geq 3, \text{ and } G \text{ is not spanned by a } K_{3,3} \text{ or any } L(6, 3, j) \text{ with } 1 \leq j \leq 6. \tag{18}$$

If $d = 5$, then we assume that v_1 is adjacent to all other 5 vertices of \tilde{G} . By (18), $\delta(\tilde{G}) \geq 3$, and so $v_2v_6 \in E(\tilde{G})$. Thus $G = L(6, 4, 1)$ (depicted in Fig. 4). Assume that $d = 4$ and that v_1 is a vertex of degree 4 in \tilde{G} .

Case 3.1. v_1 is adjacent to all but v_4 .

By (18), $\delta(\tilde{G}) \geq 3$, and so by symmetry, we may assume that $v_2v_4 \in E(\tilde{G})$, and either v_2v_6 or $v_4v_6 \in E(\tilde{G})$. If $v_2v_6 \in E(\tilde{G})$, then $\kappa'(G - v_1v_2) \geq 3$, violating (15). Hence we have $v_4v_6 \in E(\tilde{G})$ and so $G = L(6, 4, 2)$ (depicted in Fig. 4).

Case 3.2. v_1 is adjacent to v_2, v_i, v_4, v_6 , where $i \in \{3, 5\}$.

By symmetry, we may assume that $i = 3$. By (18), $\delta(\tilde{G}) \geq 3$. By **Observation 1(iv)**, $v_2v_4 \notin E(\tilde{G})$; but also $v_3v_5, v_3v_6, v_4v_6 \notin E(\tilde{G})$, whence $v_2v_5, v_2v_6 \in E(\tilde{G})$, contrary to **Observation 1(iv)**.

Thus in Case 3, when $f(G, C) = 4$, we must have $G \in \{L(6, 4, 1), L(6, 4, 2)\}$. It is routine to show that $\mu'(L(6, 4, 1)) = \mu'(L(6, 4, 2)) = 3$. Detailed verifications can be found in Chapter 5 of [17].

This completes the proof of the theorem. \square

5. Degree condition for supereulerian graphs with larger width

Settling three open problems of Bauer in [1], Catlin and Lai proved the following.

Theorem 5.1. Let G be a 2-edge-connected simple graph G on n vertices.

- (i) (Catlin, Theorem 9 of [4]) If $\delta(G) > \frac{n}{5} - 1$, then for sufficiently large n , G is supereulerian.
- (ii) (Lai, Theorem 5 of [13]) If G is bipartite, or G is triangle free, and if $\delta(G) > \frac{n}{10}$, then for sufficiently large n , G is supereulerian.

Both bounds in **Theorem 5.1** are best possible in the sense that there exist an infinite family of non-supereulerian 2-edge-connected graphs G on n vertices with $\delta(G) = \frac{n}{5} - 1$ (for **Theorem 5.1(i)**) and an infinite family of non-supereulerian bipartite graphs on n vertices with $\delta(G) = \frac{n}{10}$ (for **Theorem 5.1(ii)**). The main purpose of this section is to extend the theorem above, by using a more general argument than in the proofs in both [4] and [13]. We start with some additional notations and a preparatory lemma before presenting our main arguments. If G is a graph and G' is the \mathcal{C}_s -reduction of G , then for any vertex $u \in V(G')$, G has a maximal \mathcal{C}_s -subgraph H_u such that u is the vertex onto which H_u is contracted. The subgraph H_u is called the **preimage** of u in G . It is possible that H_u consists of a single vertex, in which case u is a **trivial vertex** of the contraction. If H is a subgraph of G , then define

$$A_G(H) = \{v \in V(H) : N_G(v) - V(H) \neq \emptyset\}.$$

Lemma 5.2. Let n, p, c be positive integers, and $f(n, p)$ be a function of n and p such that for every fixed $p > 0, \lim_{n \rightarrow \infty} f(n, p) = \infty$. Suppose that G is a simple graph on n vertices such that one of the following holds:

- (i) $\delta(G) \geq f(n, p) - 1$;
- (ii) G is triangle free and $\delta(G) \geq \frac{f(n,p)}{2}$.

Then for sufficiently large n (such that $f(n, p) \geq 2c + 2$, say), any vertex u in the \mathcal{C}_s -reduction of G whose degree is at most c has as its preimage the maximal \mathcal{C}_s -subgraph H_u with

$$|V(H_u)| \geq f(n, p). \tag{19}$$

Proof. Let G' be the \mathcal{C}_s -reduction of G . Define $W = \{u \in V(G') : d_{G'}(u) \leq c\}$ and for each $u \in W$, choose $v \in V(H_u)$. Then $V(H_u)$ contains all vertices in $N_G(v)$ except at most c vertices in $A_G(H) \cup (V(G) - V(H_u))$. Hence

$$\left| \left(V(H_u) \cap N_G(v) \right) - A_G(H) \right| \geq d_G(v) - c. \tag{20}$$

By assumption, there exists an N such that for any $n \geq N, f(n, p) \geq 2c + 2$. We assume that $n \geq N$ in the rest of the proof.

Suppose first that (i) holds. By (20), $|V(H_u) \cap N_G(v) - A_G(H)| \geq d_G(v) - c \geq f(n, p) - 1 - c \geq (2c + 2) - 1 - c = c + 1$. It follows that there exists a vertex $z \in V(H_u) \cap N_G(v) - A_G(H)$ such that $N_G(z) \subseteq V(H_u)$. By (i), we have $|V(H_u)| \geq |N_G(z) \cup \{z\}| \geq d_G(z) + 1 \geq f(n, p)$.

Now suppose that (ii) holds and so G is triangle free and $\delta(G) \geq \frac{f(n,p)}{2}$. Again by (20), $|V(H_u) \cap N_G(v) - A_G(H)| \geq d_G(v) - c \geq \frac{f(n,p)}{2} - c \geq \frac{2c+2}{2} - c > 0$. It follows that there exists a vertex $z' \in V(H_u) \cap N_G(v) - A_G(H)$ such that $N_G(z') \subseteq V(H_u)$. By (20) again with v replaced by z' , we have $|N_G(z') - A_G(H_u)| \geq d_G(z') - c > 0$. This implies that there exists a $z'' \in N_G(z') - A_G(H_u) \subseteq V(H_u) - A_G(H_u)$. By the choices of z' and z'' , we have $N_G(z') \cup N_G(z'') \subseteq V(H_u)$. Since G is triangle free and since $z'z'' \in E(G)$, we have $N_G(z') \cap N_G(z'') = \emptyset$. It follows that $|V(H_u)| \geq |N_G(z') \cup N_G(z'')| \geq d_G(z') + d_G(z'') \geq 2\delta(G) \geq f(n, p)$. This completes the proof of the lemma. \square

Theorem 5.3. Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices.

- (i) If n is sufficiently large (say $n \geq 2p((2s + 2)p - 2)$) and if

$$\delta(G) \geq \frac{n}{p} - 1, \tag{21}$$

then the \mathcal{C}_s -reduction of G has at most p vertices.

- (ii) If G is triangle free, n is sufficiently large (say $n \geq 2p((2s + 2)p - 2)$), and if

$$\delta(G) \geq \frac{n}{2p}, \tag{22}$$

then the \mathcal{C}_s -reduction of G has at most p vertices.

Proof. As the arguments to prove both conclusions are similar, we shall prove them simultaneously.

For given $p > 0$ and $s > 0$, choose an integer $c = (2s + 2)p - 3$. Let G' be the \mathcal{C}_s -reduction of G , and assume that $n' = |V(G')| > 1$. Define

$$W = \{u \in V(G') : d_{G'}(u) \leq c\}.$$

Choose $f(n, p) = \frac{n}{p}$. Then as $c = (2s + 2)p - 3$ and as $n \geq 2p((2s + 2)p - 2) = 2p(c + 1)$, we have $f(n, p) \geq 2c + 2$. Choose any $u \in W$ and any $z \in V(H_u)$. By Lemma 5.2, (19) must hold, and so,

$$n \geq \sum_{u \in W} |V(H_u)| \geq |W| \cdot f(n, p) = \frac{n|W|}{p}.$$

This implies that

$$|W| \leq p. \tag{23}$$

Since G' is \mathcal{C}_s -reduced, by Corollary 2.13(iii), we have

$$|E(G')| \leq (s + 1)n' - (s + 3). \tag{24}$$

By the definition of W , we have

$$2|E(G')| = \sum_{v \in V(G')} d_{G'}(v) = \sum_{v \in V(G') - W} d_{G'}(v) + \sum_{v \in W} d_{G'}(v) \geq \sum_{v \in V(G') - W} d_{G'}(v) \geq c|V(G') - W|.$$

This, together with (23) and (24), implies that $cn' - cp \leq c|V(G') - W| \leq 2|E(G')| \leq 2(s + 1)n' - 2(s + 3)$. Hence

$$n' \leq \frac{cp - 2(s + 3)}{c - 2(s + 1)}. \tag{25}$$

As $c > p(2s + 2) - 4 = 2p(s + 1) - 2(s + 3) + 2(s + 1)$, it follows that $c(p + 1) > cp - 2(s + 3) + 2(p + 1)(s + 1)$, and so algebraic manipulations lead to $(c - 2(s + 1))(p + 1) > cp - 2(s + 3)$. This, together with (25), implies

$$n' \leq \frac{cp - 2(s + 3)}{c - 2(s + 1)} < p + 1.$$

Hence $n' \leq p$, and so the theorem follows. \square

The theorem above can be applied to study the supereulerian width of some dense graphs, as shown in Corollary 5.4. By definition of $\mu'(G)$, $\mu'(G) \geq 2$ implies that G is supereulerian. It follows that when $s = 1$ and $p = 5$, Corollary 5.4 yields the results as stated in Theorem 5.1.

Corollary 5.4. *Let n, s be positive integers such that $1 \leq s \leq 2$. Suppose that G is a simple graph on n vertices with $\kappa'(G) \geq s + 1$. Let $p(s) = 2s + 3$. Each of the following holds for sufficiently large n .*

(i) *If*

$$\delta(G) \geq \frac{n}{p(s)} - 1, \tag{26}$$

then $\mu'(G) \geq s + 1$ if and only if the \mathcal{C}_s -reduction of G is not a $K_{s+1, s+1}$.

(ii) *If G is triangle free, and if*

$$\delta(G) \geq \frac{n}{2p(s)}, \tag{27}$$

then $\mu'(G) \geq s + 1$ if and only if the \mathcal{C}_s -reduction of G is not a $K_{s+1, s+1}$.

Proof. Let $p = p(s)$. Let G be a simple graph G satisfying (26) or a triangle free graph satisfying (27). Let G' denote the \mathcal{C}_s -reduction of G .

If $|V(G')| = 1$, then $G' = K_1 \in \mathcal{C}_s$. By Corollary 2.4, $G \in \mathcal{C}_s$. By Corollary 2.5, $\mu'(G) \geq s + 1$. Hence we may assume that $|V(G')| > 1$.

By Theorem 5.3, there exists an integer $N_1(s)$ such that if $n \geq N_1(s)$, $|V(G')| \leq p$. We shall further show that $|V(G')| \leq p - 1$, for all sufficiently large n . Assume by contradiction that we always have $|V(G')| = p$. By Lemma 5.2 with $c = p$ and $f(n, p) = \frac{n+1}{p}$, we conclude that there exists an integer $N = N_2(s) \geq N_1$ such that when $n \geq N$, every vertex v in G' has a nontrivial preimage H_v with at least $\lceil f(n, p) \rceil$ vertices. It follows that

$$n = \sum_{v \in V(G')} |V(H_v)| \geq pf(n, p) = n + 1.$$

This contradiction shows that, when $n \geq N$, we must have $1 < |V(G')| \leq p - 1$.

Since $p(1) = 5$ and $p(2) = 7$, by Theorem 4.4, the conclusions of Corollary 5.4(i) and (ii) must hold. \square

Final Remark: There exist natural bounds of $\mu'(G)$: if $\kappa'(G) \geq 2k \geq 4$, then $\kappa'(G) \geq \mu'(G) \geq k$. It is not known to which extent this inequality can be improved. In particular, we do not know when $\kappa'(G)$ equals $\mu'(G)$.

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