Linear Methods for Time-Frequency-ARMA Approximation of Time-Varying Systems*

Michael Jachan (corresponding author), Franz Hlawatsch, Gerald Matz

Institute of Communications and Radio-Frequency Engineering, Vienna University of Technology
Gusshausstrasse 25/E389, A-1040 Vienna, Austria
Phone: +43 1 58801 38914, Fax: +43 1 58801 38999, E-mail: michael.jachan@ieee.org

May 10, 2005

Abstract—Time-frequency autoregressive moving-average (TFARMA) models have recently been introduced as parsimonious parametric models for underspread nonstationary random processes and linear time-varying (LTV) systems. In this paper, we propose linear methods for approximating an underspread LTV system by a TFARMA-type system. The linear equations obtained have Toeplitz/block-Toeplitz structure and thus can be solved efficiently by the Wax-Kailath algorithm.

1 Introduction

We consider causal linear time-varying (LTV) systems $\mathbb{H}$ that operate on discrete-time, finite-length signals $u[n]$ defined on the time interval $[0, N-1]$ according to the input-output relation

$$x[n] \,=\, (\mathbb{H}u)[n] \,=\, \sum_{m=0}^{N/2-1} h[n,m]\, u[(n-m) \bmod N]\,, \qquad n = 0, 1, \cdots, N-1\,,$$

in which $h[n,m]$ denotes the time-varying impulse response of $\mathbb{H}$ and the summation interval is $[0, N/2-1]$ because of causality. We propose to model such LTV systems by *TFARMA systems*, where TFARMA is short for "Time-Frequency Autoregressive Moving Average." TFARMA systems are based on the time-frequency (TF) shift operator $\mathbb{S}_{m,l} = \mathbb{F}^l \mathbb{T}^m$ with the cyclic time-shift operator $\mathbb{T}^m$ defined by $(\mathbb{T}^m x)[n] = x[(n-m) \bmod N]$ and the frequency-shift operator $\mathbb{F}^l$ defined by $(\mathbb{F}^l x)[n] = e^{i\frac{2\pi}{N} ln} x[n]$ (cyclic in l). The time shifts (delays) model the system's memory, while the frequency (Doppler) shifts model the system's time variations. The input-output relation of a TFARMA$(M_A, L_A; M_B, L_B)$ system is given by

$$x[n] \,=\, -\sum_{(m,l)\in\mathcal{A}_1} a_{m,l} (\mathbb{S}_{m,l}\, x)[n] \,+\, \sum_{(m,l)\in\mathcal{B}} b_{m,l} (\mathbb{S}_{m,l}\, u)[n]\,. \tag{1}$$

The first sum is the recursive TFAR component with the TFAR parameters $a_{m,l}$, and the second sum is the nonrecursive TFMA component with the TFMA parameters $b_{m,l}$. The delay-Doppler support regions

*This work was supported by FWF grant P15156-N02.

$\mathcal{A}_1$ and $\mathcal{B}$ are given by $\mathcal{A}_1 \triangleq \{1,\ldots,M_A\} \times \{-L_A,\ldots,L_A\}$ and $\mathcal{B} \triangleq \{0,\ldots,M_B\} \times \{-L_B,\ldots,L_B\}$, with M_A and L_A denoting the TFAR delay and Doppler orders and M_B and L_B denoting the TFMA delay and Doppler orders.

The TFARMA input-output relation (1) can be written as $(\mathbb{A}x)[n] = (\mathbb{B}u)[n]$ or, equivalently, as

$$x[n] \,=\, (\mathbb{H}_{\text{TFARMA}}\, u)[n]\,.$$

Here, the TFARMA$(M_A, L_A; M_B, L_B)$ system $\mathbb{H}_{\text{TFARMA}}$ is given by

$$\mathbb{H}_{\text{TFARMA}} = \mathbb{A}^{-1}\mathbb{B}\,, \qquad \text{with} \quad \mathbb{A} \triangleq \sum_{(m,l)\in\mathcal{A}} a_{m,l}\, \mathbb{S}_{m,l}\,, \qquad \mathbb{B} \triangleq \sum_{(m,l)\in\mathcal{B}} b_{m,l}\, \mathbb{S}_{m,l}\,, \tag{2}$$

where $\mathcal{A} \triangleq \{0,\ldots,M_A\} \times \{-L_A,\ldots,L_A\}$ and $a_{0,l} = \delta[l]$ (the latter equation expresses the monicity of $\mathbb{A}$, which is consistent with the structure of (1)). The maximum lags of the TF shifts used in the TFARMA model (i.e., the lag orders M_A, L_A, M_B, L_B) are typically constrained to be small. Such *underspread* systems [1]—i.e., LTV systems with small time delays and Doppler frequency shifts—are encountered in many applications.

In the case of time-invariant systems, techniques for linear AR, MA, and ARMA parameter estimation and system approximation have been proposed and studied by various authors [2–5]. For the time-varying case, TFAR, TFMA, and TFARMA models and corresponding (mostly nonlinear) parameter estimation algorithms have recently been introduced in [6–8]. These methods are related to previously proposed methods for time-varying ARMA modeling and parameter estimation (e.g., [9, 10]).

In this contribution, we propose linear methods for approximating a given LTV system by a TF-ARMA system. Our methods make use of approximations that are justified in the underspread case [1]. They lead to linear Toeplitz/block-Toeplitz equations that can be solved efficiently by the Wax-Kailath algorithm [11]. TFMA and TFAR system approximation are discussed as special cases.

2 TFARMA System Approximation

We wish to approximate a given causal LTV system $\mathbb{H}$ by a TFARMA$(M_A, L_A; M_B, L_B)$ system $\mathbb{H}_{\text{TFARMA}} = \mathbb{A}^{-1}\mathbb{B}$ of given (small) delay orders M_A, M_B and Doppler orders L_A, L_B. Because finding a direct solution to this approximation problem is too difficult, we approximate instead $\mathbb{A}\mathbb{H}$ by $\mathbb{B}$ — the underlying reasoning being that if $\mathbb{H} \approx \mathbb{A}^{-1}\mathbb{B}$, then it can be expected that also $\mathbb{A}\mathbb{H} \approx \mathbb{B}$ and vice versa. We thus express the optimization as[1]

$$(\mathbb{A}_{\text{opt}}, \mathbb{B}_{\text{opt}}) \;\triangleq\; \arg \min_{\substack{\mathbb{A}\in\mathcal{H}_A \\ \mathbb{B}\in\mathcal{H}_B}} \|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2, \tag{3}$$

where $\mathcal{H}_A$ denotes the Hilbert space of all LTV systems of the form $\sum_{(m,l)\in\mathcal{A}} a_{m,l}\, \mathbb{S}_{m,l}$ with given delay-Doppler support region $\mathcal{A} \triangleq \{0,\ldots,M_A\} \times \{-L_A,\ldots,L_A\}$ and $a_{0,l} = \delta[l]$, and $\mathcal{H}_B$ denotes the Hilbert space of all LTV systems of the form $\sum_{(m,l)\in\mathcal{B}} b_{m,l}\, \mathbb{S}_{m,l}$ with given delay-Doppler support region $\mathcal{B} \triangleq \{0,\ldots,M_B\} \times \{-L_B,\ldots,L_B\}$.

[1] Inner products of causal operators are defined as $\langle \mathbb{H}_1, \mathbb{H}_2 \rangle = \sum_{n=0}^{N-1} \sum_{m=0}^{N/2-1} h_1[n,m]\, \overline{h_2[n,m]}$. The (squared) norm is thus given by $\|\mathbb{H}\|^2 = \langle \mathbb{H}, \mathbb{H} \rangle = \sum_{n=0}^{N-1} \sum_{m=0}^{N/2-1} |h[n,m]|^2$.

2.1 The Spreading Function

For developing the solution to the above minimization problem, it will be convenient to use the *spreading function* (SF). The SF of an LTV system $\mathbb{H}$ is defined as

$$S_{\mathbb{H}}[m,l] \triangleq \langle \mathbb{H}, \mathbb{S}_{m,l} \rangle = \sum_{n=0}^{N-1} h[n,m]\, e^{-i\frac{2\pi}{N}ln}.$$

It is (to within a factor of $1/N$) the coefficient function in an expansion of $\mathbb{H}$ into TF shift operators $\mathbb{S}_{m,l}$, i.e.,

$$\mathbb{H} = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} \sum_{l=-N/2}^{N/2-1} S_{\mathbb{H}}[m,l]\, \mathbb{S}_{m,l}\,.$$

For the LTV systems $\mathbb{A} = \sum_{(m,l)\in\mathcal{A}} a_{m,l}\, \mathbb{S}_{m,l}$ and $\mathbb{B} = \sum_{(m,l)\in\mathcal{B}} b_{m,l}\, \mathbb{S}_{m,l}$ in (2), we have

$$S_{\mathbb{A}}[m,l] = \begin{cases} N\, a_{m,l}\,, & (m,l)\in\mathcal{A} \\ 0\,, & \text{elsewhere} \end{cases} \qquad \text{and} \qquad S_{\mathbb{B}}[m,l] = \begin{cases} N\, b_{m,l}\,, & (m,l)\in\mathcal{B} \\ 0\,, & \text{elsewhere}\,. \end{cases} \tag{4}$$

2.2 Calculation of $\mathbb{B}$

The minimization problem (3) for $\mathbb{B}$, with fixed $\mathbb{A}$, can be formulated as the problem of finding the best subspace approximation $\mathbb{B} \in \mathcal{H}_B$ to the operator $\mathbb{A}\mathbb{H}$. According to the projection theorem, a necessary and sufficient condition for the optimum $\mathbb{B}$ is that the approximation error $\mathbb{B}-\mathbb{A}\mathbb{H}$ is orthogonal to the subspace $\mathcal{H}_B$, i.e., $\langle \mathbb{B}-\mathbb{A}\mathbb{H}, \mathbb{B}' \rangle = 0$ for all $\mathbb{B}' \in \mathcal{H}_B$. Because the TF shift operators $\{\mathbb{S}_{m,l}\}_{(m,l)\in\mathcal{B}}$ are a basis for $\mathcal{H}_B$, an equivalent condition is that $\langle \mathbb{B}-\mathbb{A}\mathbb{H}, \mathbb{S}_{m,l} \rangle = 0$ or $\langle \mathbb{B}, \mathbb{S}_{m,l} \rangle = \langle \mathbb{A}\mathbb{H}, \mathbb{S}_{m,l} \rangle$ for $(m,l) \in \mathcal{B}$. Using the SF, the last equation can be rewritten as

$$S_{\mathbb{B}}[m,l] = S_{\mathbb{A}\mathbb{H}}[m,l]\,, \qquad (m,l)\in\mathcal{B}\,, \tag{5}$$

or equivalently, using (4), as

$$N\, b_{m,l} = S_{\mathbb{A}\mathbb{H}}[m,l]\,, \qquad (m,l)\in\mathcal{B}\,.$$

The right-hand side can be shown to be the *twisted convolution* [12] of $S_{\mathbb{A}}[m,l]$ and $S_{\mathbb{H}}[m,l]$, i.e.,[2]

$$\begin{aligned} S_{\mathbb{A}\mathbb{H}}[m,l] &= \frac{1}{N} \sum_{(m',l')\in\mathcal{A}} S_{\mathbb{A}}[m',l']\, S_{\mathbb{H}}[m-m',\, l-l']\, e^{-i\frac{2\pi}{N}m'(l-l')} \\ &= \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m',\, l-l']\, e^{-i\frac{2\pi}{N}m'(l-l')}. \end{aligned} \tag{6}$$

Thus, the parameters $b_{m,l}$ characterizing the optimum $\mathbb{B}$ according to (2) are finally obtained as

$$b_{m,l} = \frac{1}{N} \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m',\, l-l']\, e^{-i\frac{2\pi}{N}m'(l-l')}\,, \qquad (m,l)\in\mathcal{B}\,. \tag{7}$$

We now assume that $M_A L_A \ll N$ and $M_B L_B \ll N$, i.e., the LTV systems $\mathbb{A}$ and $\mathbb{B}$ are *underspread* [1]. We then have $e^{-i\frac{2\pi}{N}m'(l-l')} \approx 1$ in the sum in (7), which means that the twisted convolution is

[2] All convolutions and twisted convolutions are cyclic with period N even though this is not indicated by our notation.

approximated by a conventional 2-D convolution and hence (7) is approximated by

$$b_{m,l} \;\approx\; \frac{1}{N} \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m',\, l-l']\,, \qquad (m,l) \in \mathcal{B}\,. \tag{8}$$

2.3 Calculation of $\mathbb{A}$

The previous result (8) allows the approximate calculation of the optimum TFMA parameters $b_{m,l}$ for given TFAR parameters $a_{m,l}$. It remains to calculate the TFAR parameters $a_{m,l}$.

Optimum calculation. We first use the identity $\|\mathbb{H}\|^2 = \frac{1}{N}\sum_{m=0}^{N/2-1}\sum_{l=-N/2}^{N/2-1}\big|S_{\mathbb{H}}[m,l]\big|^2$ to write the cost function to be minimized as

$$\begin{aligned}
\|\mathbb{B}-\mathbb{A}\mathbb{H}\|^2 \;&=\; \frac{1}{N}\sum_{m=0}^{N/2-1}\sum_{l=-N/2}^{N/2-1}\big|S_{\mathbb{B}}[m,l]-S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2 \\
&=\; \frac{1}{N}\sum_{(m,l)\in\mathcal{B}}\big|S_{\mathbb{B}}[m,l]-S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2 \;+\; \frac{1}{N}\sum_{(m,l)\notin\mathcal{B}}\big|S_{\mathbb{B}}[m,l]-S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2.
\end{aligned}$$

The first term vanishes due to (5), and in the second term $S_{\mathbb{B}}[m,l]=0$ due to (4). Thus, we obtain

$$\|\mathbb{B}-\mathbb{A}\mathbb{H}\|^2 \;=\; \frac{1}{N}\sum_{(m,l)\notin\mathcal{B}}\big|S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2 \;=\; \frac{1}{N}\sum_{(m,l)\notin\mathcal{B}}\bigg|\sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m',\, l-l']\, e^{-i\frac{2\pi}{N}m'(l-l')}\bigg|^2, \tag{9}$$

where (6) has been used. Note that here the underspread approximation $e^{-i\frac{2\pi}{N}m'(l-l')}\approx 1$ cannot be used because the outer sum is over the *outside* of $\mathcal{B}$.

The expression (9) has now to be minimized with respect to the parameters $a_{m,l}$, $(m,l)\in\mathcal{A}_1$ (recall that $a_{0,l}=\delta[l]$). Setting the derivative with respect to $\overline{a_{m_0,l_0}}$, $(m_0,l_0)\in\mathcal{A}_1$ equal to zero [13] yields

$$\sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, k[m,l,m',l'] \;=\; -p[m,l]\,, \qquad (m,l)\in\mathcal{A}_1\,,$$

with

$$\begin{aligned}
k[m,l,m',l'] \;&\triangleq\; \sum_{(\hat{m},\hat{l})\notin\mathcal{B}} \overline{S_{\mathbb{H}}[\hat{m}-m,\, \hat{l}-l]}\, S_{\mathbb{H}}[\hat{m}-m',\, \hat{l}-l']\, e^{i\frac{2\pi}{N}[m(\hat{l}-l)-m'(\hat{l}-l')]}, \\
p[m,l] \;&\triangleq\; \sum_{(\hat{m},\hat{l})\notin\mathcal{B}} \overline{S_{\mathbb{H}}[\hat{m}-m,\, \hat{l}-l]}\, S_{\mathbb{H}}[\hat{m},\hat{l}]\, e^{i\frac{2\pi}{N}m(\hat{l}-l)}.
\end{aligned}$$

This is a system of $M_A(2L_A+1)$ linear equations in the $M_A(2L_A+1)$ unknowns $a_{m,l}$, $(m,l)\in\mathcal{A}_1$.

Efficient suboptimum calculation. Next, we propose an alternative, suboptimum calculation of $\mathbb{A}$ that is suited for an *underspread* causal system $\mathbb{H}$ and results in linear equations with Toeplitz/block-Toeplitz structure. Instead of minimizing $\|\mathbb{B}-\mathbb{A}\mathbb{H}\|$, we simply postulate

$$\mathbb{A}\mathbb{H} \overset{!}{=} \mathbb{B}$$

and attempt to compute $\mathbb{A}$ accordingly. Rewriting this equation in terms of the SFs of the systems involved and using (4), we obtain

$$\sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m', l-l']\, e^{-i\frac{2\pi}{N}m'(l-l')} \;=\; N\, b_{m,l}\,. \tag{10}$$

Note that while this equation is identical to (7), here it is supposed to hold on the entire causal (m, l) plane—i.e., for $(m, l) \in [0, N/2-1] \times [-N/2, N/2]$—whereas (7) holds only for $(m, l) \in \mathcal{B}$. Because both $\mathbb{A}$ and $\mathbb{H}$ are underspread, we can set $e^{-i\frac{2\pi}{N}m'(l-l')} \approx 1$ in (10), which gives

$$\sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m', l-l'] \;\approx\; N\, b_{m,l}\,. \tag{11}$$

For $m = M_B+1, \ldots, M_B+M_A$, $b_{m,l}$ is zero and thus (11) simplifies as $\sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_{\mathbb{H}}[m-m', l-l'] \approx 0$ or, equivalently, recalling that $a_{0,l} = \delta[l]$,

$$\sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, S_{\mathbb{H}}[m-m', l-l'] \;\approx\; -S_{\mathbb{H}}[m, l]\,, \qquad m = M_B+1, \ldots, M_B+M_A\,,\;\; l = -L_A, \ldots, L_A\,. \tag{12}$$

These are $M_A(2L_A+1)$ linear equations in the $M_A(2L_A+1)$ unknowns $a_{m,l}$. These equations have the structure of "TF Yule-Walker equations" [6]. They can be written as

$$\boldsymbol{S}\boldsymbol{a} = -\boldsymbol{s}\,, \tag{13}$$

with the $M_A(2L_A+1) \times 1$ vectors $\boldsymbol{a} = \big[\boldsymbol{a}_1^T \cdots \boldsymbol{a}_{M_A}^T\big]^T$ and $\boldsymbol{s} = \big[\boldsymbol{s}_{M_B+1}^T \cdots \boldsymbol{s}_{M_B+M_A}^T\big]^T$ containing the $(2L_A+1) \times 1$ vectors $\boldsymbol{a}_m = \big[a_{m,-L_A} \cdots a_{m,L_A}\big]^T$ and $\boldsymbol{s}_m = \big[S_{\mathbb{H}}[m, -L_A] \cdots S_{\mathbb{H}}[m, L_A]\big]^T$, respectively and the $M_A(2L_A+1) \times M_A(2L_A+1)$ Toeplitz/block-Toeplitz matrix[3] $\boldsymbol{S} = \mathrm{toep}\big\{\boldsymbol{S}_{M_A+M_B-1}, \ldots, \boldsymbol{S}_{M_B-M_A+1}\big\}$ containing the $(2L_A+1) \times (2L_A+1)$ Toeplitz blocks $\boldsymbol{S}_m = \mathrm{toep}\big\{S_{\mathbb{H}}[m, 2L_A], \ldots, S_{\mathbb{H}}[m, -2L_A]\big\}$. The *Wax-Kailath algorithm* [11] can be used for solving the Toeplitz/block-Toeplitz equation (13) with a computational complexity of order $\mathcal{O}(M_{\mathrm{A}}^2 L_{\mathrm{A}}^3)$.

2.4 TFMA and TFAR System Approximation

TFMA system approximation is the approximation of $\mathbb{H}$ by $\mathbb{H}_{\mathrm{TFMA}} = \mathbb{B}$. This is a special case of TFARMA system approximation for $a_{m,l} = \delta[m]\delta[l]$. The TFMA parameters are here obtained from (8) as

$$b_{m,l} \;=\; \frac{1}{N}\, S_{\mathbb{H}}[m, l]\,, \qquad (m, l) \in \mathcal{B}\,.$$

Another special case of TFARMA system approximation is TFAR approximation, i.e., approximation of $\mathbb{H}$ by $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{A}^{-1}\mathbb{B}_0$ where $\mathbb{B}_0$ is a degenerate TFMA part with parameters $b_{m,l} = b_{0,l}\delta[m]$. For calculating the TFAR parameters $a_{m,l}$ by means of the suboptimum method in Subsection 2.3, we use (12) with $M_B = 0$, i.e., for $m = 1, \ldots, M_A$ and $l = -L_A, \ldots, L_A$:

$$\sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, S_{\mathbb{H}}[m-m', l-l'] \;\approx\; -S_{\mathbb{H}}[m, l]\,, \qquad m = 1, \ldots, M_A\,,\;\; l = -L_A, \ldots, L_A\,.$$

After solving this Toeplitz/block-Toeplitz equation by means of the Wax-Kailath algorithm, the TFMA coefficients $b_{0,l}$ are obtained by means of (8) evaluated for $m = 0$.

[3] The notation $\boldsymbol{S} = \mathrm{toep}\big\{\boldsymbol{S}_{M_A+M_B-1}, \ldots, \boldsymbol{S}_{M_B-M_A+1}\big\}$ means that the (block) elements of the diagonals of $\boldsymbol{S}$ ordered from SW to NE are given by $\boldsymbol{S}_{M_A+M_B-1}$ (first diagonal), ..., $\boldsymbol{S}_{M_B-M_A+1}$ (last diagonal).

3 Conclusions

We presented linear methods for calculating a TFARMA-type approximation to an underspread linear time-varying system. Exploiting the underspread property to approximate twisted convolutions by conventional 2-D convolutions, we obtained a system of linear equations of the TF Yule-Walker type that has Toeplitz/block-Toeplitz structure and thus can be solved efficiently by the Wax-Kailath algorithm. Simulation results demonstrating the performance of our approximation method will be presented in [14].

References

[1] G. Matz and F. Hlawatsch, "Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory," *J. Math. Phys., Special Issue on Wavelet and Time-Frequency Analysis*, vol. 39, pp. 4041–4071, Aug. 1998.

[2] P. Stoica and R. Moses, *Introduction to Spectral Analysis.* Englewood Cliffs (NJ): Prentice Hall, 1997.

[3] J. Durbin, "Efficient estimation of parameters in moving-average models," *Biometrika*, vol. 46, pp. 306–316, 1959.

[4] D. Graupe, D. J. Krause, and J. B. Moore, "Identification of autoregressive moving-average parameters of time-series," *IEEE Trans. Autom. Contr.*, vol. 20, pp. 104–107, Feb. 1975.

[5] B. Wahlberg, "Estimation of autoregressive moving-average models via high-order autoregressive approximations," *J. Time Series Analysis*, vol. 10, no. 3, pp. 283–299, 1989.

[6] M. Jachan, G. Matz, and F. Hlawatsch, "Time-frequency-autoregressive random processes: Modeling and fast parameter estimation," in *Proc. IEEE ICASSP-2003*, vol. VI, (Hong Kong), pp. 125–128, April 2003.

[7] M. Jachan, G. Matz, and F. Hlawatsch, "Time-frequency-moving-average random processes: Principles and cepstral methods for parameter estimation," in *Proc. IEEE ICASSP-2004*, vol. II, (Montreal, Canada), pp. 757–760, May 2004.

[8] M. Jachan, G. Matz, and F. Hlawatsch, "TFARMA models: Order estimation and stabilization," in *Proc. IEEE ICASSP-2005*, (Philadelphia, PA), pp. 301–304, March 2005.

[9] Y. Grenier, "Time-dependent ARMA modeling of nonstationary signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 31, pp. 899–911, Aug. 1983.

[10] M. G. Hall, A. V. Oppenheim, and A. S. Willsky, "Time-varying parametric modeling of speech," *Signal Processing*, vol. 5, pp. 267–285, May 1983.

[11] M. Wax and T. Kailath, "Efficient inversion of Toeplitz-block Toeplitz matrix," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 31, pp. 1218–1221, Oct. 1983.

[12] G. B. Folland, *Harmonic Analysis in Phase Space*, vol. 122 of *Annals of Mathematics Studies.* Princeton (NJ): Princeton University Press, 1989.

[13] R.Remmert, *Funktionentheorie I.* Heidelberg: Springer, 1989.

[14] M. Jachan, F. Hlawatsch, and G. Matz, "Linear methods for TFARMA parameter estimation and system approximation," in *Proc. 13th IEEE Workshop on Statistical Signal Processing*, (Bordeaux, France), July 2005.