Accuracy of error propagation exemplified with ratios of random variables

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The method of error propagation provides a convenient tool for calculating mean and variance of a measurand from means and variances of primarily measured quantities. However, being based on a (usually first-order) Taylor approximation of the measurement function, it only yields approximate results with unknown accuracy. We develop a method for estimating the accuracy of (Nth-order) error propagation for an arbitrary number of correlated random quantities, and apply our findings to the ratio of two random variables (RVs). A comparison with some analytically solved expressions for certain probability density functions (PDFs) as well as with some computer simulations reveals the excellent quality of our estimates as long as the involved PDFs are not significantly skew. For the ratio of two RVs it turns out that conventional, first-order error propagation is safely applicable (with about 1% accuracy) as long as the denominator's mean is larger than about 12 times its standard deviation. Using second-order error propagation, the approximation for the ratio's mean can be refined, yielding 1% accuracy if the denominator's mean is larger than about four times its standard deviation. [S0034-6748(00)04603-7]

I. INTRODUCTION

The most important (and most frequently used) way to assess the quality of a measurement procedure and to determine the influence of various system parameters on the overall measurement accuracy is the calculation of the root mean square error (i.e., the standard deviation) connected with the final measurement result (the measurand) from means and variances of the primarily measured quantities.[1] From an *a priori* point of view (i.e., without any measurements being yet available) there are basically two approaches to this task:

(1) If the first-order joint probability density function (PDF) of the primarily measured quantities is known, one can, in principle, evaluate all moments of the measurand, either by analytical means[2,3] or by means of computer simulations.[4] This approach is undoubtedly the most exact one, but it has some serious disadvantages:

(i) First, it is rather time-consuming in general, owing either to the analytical complexity of the involved integrals, or to the long simulation times required to generate enough realizations for reliable averaging.

(ii) Second, this method leads (if at all) only to rather complex analytical expressions for the measurand's mean and variance that are impractical to handle, especially if, e.g., system parameters ought to be optimized.

(iii) Third, detailed knowledge of the primarily measured quantities' *joint* PDF (including all correlation effects!) is rarely available. Simple models (such as a jointly Gaussian PDF) are of *no* advantage in general: If the measurand is, e.g., given by the *ratio* of two primarily measured quantities, substituting a jointly Gaussian PDF for the true PDF will lead to the result that finite moments do not exist![5,6]

(2) A standard way of calculating mean and variance of a measurand from means and variances of the primarily measured quantities is the method of error propagation found in most textbooks on statistics and probability,[7–10] recommended by international standards,[1] and frequently applied in all fields of science and engineering.[11–16] The method provides a convenient tool whenever the first two (joint) moments of the primarily measured quantities are known, while their joint PDF is unknown, a case frequently encountered in practice. Evolving from a (usually first-order) Taylor approximation of the measurement function, the method can only yield *approximate* results.[17] These are, however, simple in form, easily derived and interpreted, and sufficiently accurate for many practically relevant cases.

Having introduced error propagation as an approximate method, the next question consequently has to aim at its *validity*. However, there are—to the best of our knowledge—no quantitative analyses in literature on *how well* error propagation approximates the exact moments of random variables (RVs) typically encountered in engineering practice.[18] The only statements we were able to find are very vague, qualitative hints such as ''the measurement function has to be sufficiently smooth to justify the use of a Taylor approximation,''[19] or ''the nonlinearity of the measurement function must not be significant.''[1]

The lack of quantitative criteria of validity makes the application of error propagation problematic in situations where the measurement function is significantly nonlinear (e.g., a ratio), and where the quality of the primarily measured quantities (expressed, e.g., in terms of a signal-to-noise ratio) is bad. Such situations were encountered in the analy-

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sis of Doppler wind lidar measurement systems,[20–23] but are also found in other areas of science and engineering (e.g., in radioactivity measurements[24]).

It is the aim of this article to *quantitatively* analyze the conditions under which error propagation may be employed with acceptable accuracy. After giving a short review of Nth-order error propagation (Sec. II), we introduce a method of estimating its accuracy, enabling us to make quantitative statements on its validity (Sec. III). We then apply our findings to the important case of a *ratio* of two random variables, probably the nonlinear measurement function most frequently encountered in practice (Sec. IV). Finally, we compare the results of our method to some analytically solved problems as well as to some computer simulations (Sec. V).

II. THE METHOD OF ERROR PROPAGATION

As pointed out above, the method of error propagation approximately expresses ensemble average and variance of a function $g(X_1,X_2,\ldots)$ of RVs X_i by means of their ensemble averages $\langle X_i\rangle$, their variances $\sigma^2_{X_i}=\langle(X_i-\langle X_i\rangle)^2\rangle$, and their covariances $c_{X_i,X_j}=\langle X_iX_j\rangle-\sigma_{X_i}\sigma_{X_j}$; it is based on a Taylor series expansion of $g(x_1,x_2,\ldots)$ around $(\langle X_1\rangle,\langle X_2\rangle,\ldots)$. Depending on how many terms of the series are retained, one finds slightly differing formulas. Specializing, for the sake of clearness but without loss of generality, to a function of two RVs X and Y, the following expressions can be given: The equations usually encountered in literature, using only a linear (first-order) Taylor approximation, read

$$\langle g(X,Y)\rangle\approx\langle g_I(X,Y)\rangle=\langle g\rangle_I=g \tag{1}$$

and

$$\sigma^2_{g(X,Y)}\approx\sigma^2_{g_I(X,Y)}=\sigma^2_{g,I}=(\partial_x g)^2\sigma_X^2+(\partial_y g)^2\sigma_Y^2+2\partial_x g\,\partial_y g\,c_{X,Y}. \tag{2}$$

(For the sake of a compact notation, we introduced the abbreviations

$$g\equiv g(\langle X\rangle,\langle Y\rangle)$$

and

$$\partial_{xy}g\equiv\left.\frac{\partial g(x,y)}{\partial x\,\partial y}\right|_{\substack{x=\langle X\rangle\\ y=\langle Y\rangle}};$$

roman indices denote the order of the Taylor approximation used.) Note that Eqs. (1) and (2) do *not* make use of all available information on X and Y: A second-order Taylor approximation[19] of $g(x,y)$ leads to the more accurate expression for the ensemble average

$$\langle g(X,Y)\rangle\approx\langle g_{\mathrm{II}}(X,Y)\rangle=\langle g\rangle_{\mathrm{II}}=g+\tfrac{1}{2}\partial_{xx}g\,\sigma_X^2+\tfrac{1}{2}\partial_{yy}g\,\sigma_Y^2+\partial_{xy}g\,c_{X,Y} \tag{3}$$

and—if all terms leading to higher moments than the second moment are omitted (thus the second approximate sign in the equations to follow!)—to the variance formula

$$\begin{aligned}\sigma^2_{g(X,Y)}\approx\sigma^2_{g_{\mathrm{II}}(X,Y)}\approx\sigma^2_{g,\mathrm{II}}&=(\partial_x g)^2\sigma_X^2+(\partial_y g)^2\sigma_Y^2+2\partial_x g\,\partial_y g\,c_{X,Y}\\&\quad-\tfrac{1}{4}(\partial_{xx}g\,\sigma_X^2+\partial_{yy}g\,\sigma_Y^2+2\partial_{xy}g\,c_{X,Y})^2.\end{aligned} \tag{4}$$

If the RVs are *uncorrelated*, it is possible to retain parts of up to the fifth term of the Taylor expansion, leading to the refined expressions

$$\begin{aligned}\langle g(X,Y)\rangle\approx\langle g_{\mathrm{IV}}(X,Y)\rangle\approx\langle g\rangle_{\mathrm{IV}}&=g+\tfrac{1}{2}\partial_{xx}g\,\sigma_X^2+\tfrac{1}{2}\partial_{yy}g\,\sigma_Y^2\\&\quad+\tfrac{1}{4}\partial_{xxyy}g\,\sigma_X^2\sigma_Y^2\end{aligned} \tag{5}$$

and

$$\begin{aligned}\sigma^2_{g(X,Y)}\approx\sigma^2_{g_{\mathrm{IV}}(X,Y)}\approx\sigma^2_{g,\mathrm{IV}}&=(\partial_x g)^2\sigma_X^2+(\partial_y g)^2\sigma_Y^2\\&\quad-\tfrac{1}{4}(\partial_{xx}g\,\sigma_X^2+\partial_{yy}g\,\sigma_Y^2)\\&\quad+(\partial_x g\,\partial_{xyy}g+\partial_y g\,\partial_{xxy}g)\sigma_X^2\sigma_Y^2\\&\quad-\tfrac{1}{4}\partial_{xxyy}g\,\sigma_X^2\sigma_Y^2(\partial_{xx}g\,\sigma_X^2\\&\quad+\partial_{yy}g\,\sigma_Y^2+\tfrac{1}{4}\partial_{xxyy}g\,\sigma_X^2\sigma_Y^2).\end{aligned} \tag{6}$$

III. APPROXIMATE LIMITS OF VALIDITY

Obviously, the approximations (1)–(6) are only valid if the use of the respective Taylor approximation $g_N(x,y)$ of order N can be justified in the integral expressions for the first and second moments of $g(X,Y)$. (Clearly, an error criterion like $|g_N(x,y)-g(x,y)|\ll|g(x,y)|$, based on the *local* quality of the Taylor approximation, is not pertinent here.) We will thus proceed to estimate the quality of the method of error propagation by an integral approach: The relative error $\varepsilon_{\mu,N}$ made by using the error propagation formulas for the ensemble average of $g(X,Y)$ and $\varepsilon_{\sigma,N}$ for the standard deviation are given by

$$\varepsilon_{\mu,N}=\left|1-\int\int_{-\infty}^{\infty}g_N(x,y)p(x,y)dx\,dy\bigg/\int\int_{-\infty}^{\infty}g(x,y)p(x,y)dx\,dy\right| \tag{7}$$

and

$$\varepsilon_{\sigma,N}=\left|1-\sqrt{\frac{\int\int_{-\infty}^{\infty}g_N^2(x,y)p(x,y)dx\,dy-[\int\int_{-\infty}^{\infty}g_N(x,y)p(x,y)dx\,dy]^2}{\int\int_{-\infty}^{\infty}g^2(x,y)p(x,y)dx\,dy-[\int\int_{-\infty}^{\infty}g(x,y)p(x,y)dx\,dy]^2}}\right|, \tag{8}$$

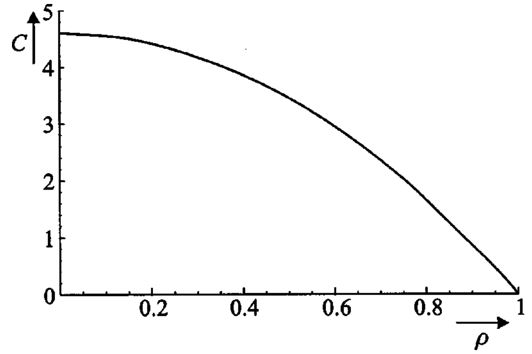

FIG. 1. Parameter of the ellipse $\xi^2-2\rho\xi\eta+\eta^2=C$ that comprises 90% of the probability mass of a jointly Gaussian PDF.

where $p(x,y)$ denotes the joint PDF of X and Y. Since $p(x,y)$ is unknown in general (otherwise we would not have to use the approximate method of error propagation), we try to get *estimates* of the errors $\varepsilon_{\mu,N}$ and $\varepsilon_{\sigma,N}$ by substituting for $p(x,y)$ a function $\tilde{p}(x,y)$ that is

(i) constant within a certain range around $(\langle X\rangle,\langle Y\rangle)$,
(ii) zero outside that range, and
(iii) normalized to unit volume to reflect the unity probability mass of a true PDF.

This substitution can be motivated by the fact that most PDFs typically encountered in engineering practice are concentrated around their means with (if at all) only little skewness, such that roughly 90% of all realizations will be found within about $\pm 1.5\sigma$. It has to be explicitly stated at this point that the described substitution is not intended to be mathematically exact, and will not work for arbitrary PDFs (especially for significantly skew ones, as shown in Sec. V); it provides an *integral estimate* for the error made by applying the method of error propagation. In the discussion in Sec. V it will be seen that this estimate yields highly satisfying results.

Concerning the *boundary* of the range within which $\tilde{p}(x,y)$ differs from zero, our investigations showed that the *ellipse* with constant Gaussian probability density that comprises 90% of the probability mass of the jointly Gaussian PDF, whose first two moments equal those of the unknown PDF, is suited well; this ellipse is given by the equation

$$\frac{(x-\langle X\rangle)^2}{\sigma_X^2}-2\rho\frac{(x-\langle X\rangle)(y-\langle Y\rangle)}{\sigma_X\sigma_Y}+\frac{(y-\langle Y\rangle)^2}{\sigma_Y^2}=C, \tag{9}$$

where $\rho=c_{X,Y}/\sigma_X\sigma_Y$ denotes the correlation coefficient of X and Y. The parameter C is determined from

$$\frac{1}{2\pi\sqrt{1-\rho^2}}\int\int_{\xi^2-2\rho\xi\eta+\eta^2=C}\exp\left[-\frac{(\xi^2-2\rho\xi\eta+\eta^2)}{2(1-\rho^2)}\right] \times d\xi\, d\eta=90\%. \tag{10}$$

Figure 1 gives C as a function of ρ. It can be seen that Eq. (10) is very well approximated by the parabola $C=4.6-4.6\rho^2$. For the uncorrelated case $(\rho=0)$ we find $C=4.6$. (For $\sigma_X=\sigma_Y$ the boundary then becomes a circle of radius $2.1\sigma_X$.)

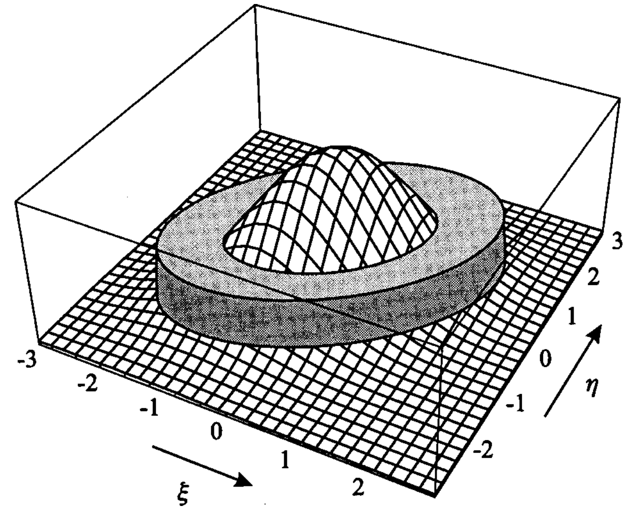

FIG. 2. To estimate the error made by applying error propagation, we substitute for the true PDF an elliptic cylinder, whose boundary is determined by the ellipse comprising 90% of the probability mass of the corresponding jointly Gaussian PDF. (Shown here is the case for $\rho=0.5$.)

Figure 2 visualizes the substitution for $\rho=0.5$ using the abbreviations $\xi=(x-\langle X\rangle)/\sigma_X$ and $\eta=(y-\langle Y\rangle)/\sigma_Y$. It shows the appropriate jointly Gaussian PDF that serves to determine the boundary of the elliptic cylinder $\tilde{p}(x,y)$, given by $\xi^2-\xi\eta+\eta^2=3.45$.

This particular choice for the integration region's shape has several advantages:

(i) it can be treated with reasonable mathematical and computational effort,
(ii) it can easily be extended to more than two RVs by using the multidimensional ellipsoids determined by the corresponding multivariate Gaussian distributions, and
(iii) it is also capable of adequately dealing with correlations.

IV. APPLICATION TO QUOTIENTS

We now proceed to apply our method of estimating the validity of error propagation to the case where $g(X,Y)=X/Y=Z$ is the *ratio* of two random variables, an operation frequently encountered in science and engineering whenever the normalization to a measured quantity is of concern.

A. Error propagation formulas

Inserting the partial derivatives of $Z=X/Y$ into Eqs. (1)–(6) directly yields the error propagation formulas for a ratio

$$\langle Z\rangle_{\mathrm{I}}=\frac{\langle X\rangle}{\langle Y\rangle}, \tag{11}$$

$$\sigma_{Z,\mathrm{I}}^2=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}-\frac{2\rho}{\alpha\beta}\right], \tag{12}$$

$$\langle Z\rangle_{\mathrm{II}}=\frac{\langle X\rangle}{\langle Y\rangle}\left[1+\frac{1}{\beta^2}-\frac{\rho}{\alpha\beta}\right], \tag{13}$$

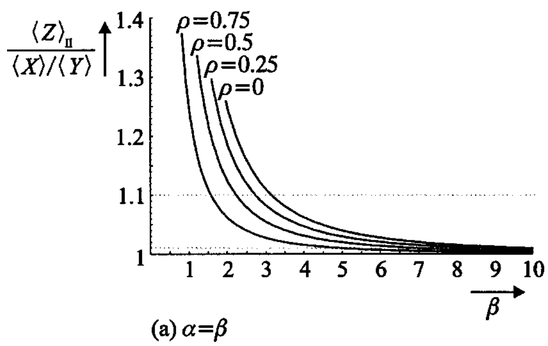

FIG. 3. Evaluation of Eq. (13) for $\alpha=\beta$ (a) and $\alpha=0$ (b) with ρ as a parameter: The second-order results can differ significantly from those obtained by the conventional, first-order formula, Eq. (11).

$$\sigma^2_{Z,\mathrm{II}}=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}-\frac{2\rho}{\alpha\beta}-\frac{1}{\beta^4}\left(\frac{\rho\beta}{\alpha}-1\right)^2\right], \tag{14}$$

$$\langle Z\rangle_{\mathrm{IV}}=\frac{\langle X\rangle}{\langle Y\rangle}\left[1+\frac{1}{\beta^2}\right], \tag{15}$$

$$\sigma^2_{Z,\mathrm{IV}}=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}-\frac{1}{\beta^4}+\frac{3}{\alpha^2\beta^2}\right]. \tag{16}$$

The parameters $\alpha=\langle X\rangle/\sigma_X$ and $\beta=\langle Y\rangle/\sigma_Y$ are identified as the square roots of the RVs' *signal-to-noise ratios* or—expressed in statistical terms—as their inverse coefficients of variation.

An important remark is in order at this point: Comparing Eq. (11) to Eq. (13), we notice an *offset* of $\langle X\rangle/\langle Y\rangle[1/\beta^2-\rho/(\alpha\beta)]$ in the more accurate approximation $\langle Z\rangle_{\mathrm{II}}$ with respect to the conventionally used formula $\langle Z\rangle_{\mathrm{I}}$. As shown in Fig. 3, where Eq. (13) is evaluated as a function of β with ρ as a parameter, this offset can take on significant values: For $\alpha=\beta$ [Fig. 3(a)] it becomes clear that the deviation of $\langle Z\rangle_{\mathrm{II}}$ from $\langle Z\rangle_{\mathrm{I}}$ can well exceed 10% for $\beta<3$ and, for $\rho=0$, does not fall below 1% for $\beta<10$! For $\alpha=0$ (i.e., for $\langle X\rangle=0$) and correlated RVs, we do find significant values for $\langle Z\rangle_{\mathrm{II}}$, whereas the first-order formula Eq. (11), gives $\langle Z\rangle_{\mathrm{I}}\equiv 0$ [Fig. 3(b)].

B. Limits of validity

Having given all error propagation formulas for the ratio of two RVs, we proceed to estimate their limits of validity using the method presented in Sec. III. We will find necessary conditions for the reliable application of error propagation that will be verified by means of some examples in Sec. V.

Inserting the Taylor expansions

$$g_{\mathrm{I}}(x,y)=\frac{\langle X\rangle}{\langle Y\rangle}+\frac{1}{\langle Y\rangle}(x-\langle X\rangle)-\frac{\langle X\rangle}{\langle Y\rangle^2}(y-\langle Y\rangle), \tag{17}$$

$$g_{\mathrm{II}}(x,y)=\frac{\langle X\rangle}{\langle Y\rangle}+\frac{1}{\langle Y\rangle}(x-\langle X\rangle)-\frac{\langle X\rangle}{\langle Y\rangle^2}(y-\langle Y\rangle)+\frac{\langle X\rangle}{\langle Y\rangle^3}(y-\langle Y\rangle)^2-\frac{1}{\langle Y\rangle^2}(x-\langle X\rangle)(y-\langle Y\rangle), \tag{18}$$

for $g(x,y)=x/y$ into the expressions (7) and (8) for the relative errors $\varepsilon_{\mu,N}$ and $\varepsilon_{\sigma,N}$, and using the substitution function $\tilde{p}(x,y)$, we obtain the curves shown in Figs. 4–6: Figures 4(a) and 4(b) give the estimated errors $\varepsilon_{\mu,\mathrm{I}}$, $\varepsilon_{\mu,\mathrm{II}}$, $\varepsilon_{\sigma,\mathrm{I}}$, and $\varepsilon_{\sigma,\mathrm{II}}$ for $\alpha=\beta$ with ρ as a parameter. The result for $\langle Z\rangle$ obtained by conventional, first-order error propagation is predicted to be off its true value by about 10% for $\beta\lesssim 3$ and can still be off by about 1% for $\beta\approx 10$. Second-order error propagation, on the other hand, can be expected to work within 1% accuracy for $\beta\gtrsim 4$, which underlines the importance of the offset term in Eq. (13) as discussed along with Fig. 3. Concerning the estimated error for the variance approximations, Fig. 4(b) shows that first-order error propagation should be nearly as good as its second-order equivalent; the 10% accuracy limit is reached at $\beta\approx 4$, and the 1% limit only at $\beta\approx 12$. This high value for error propagation to be

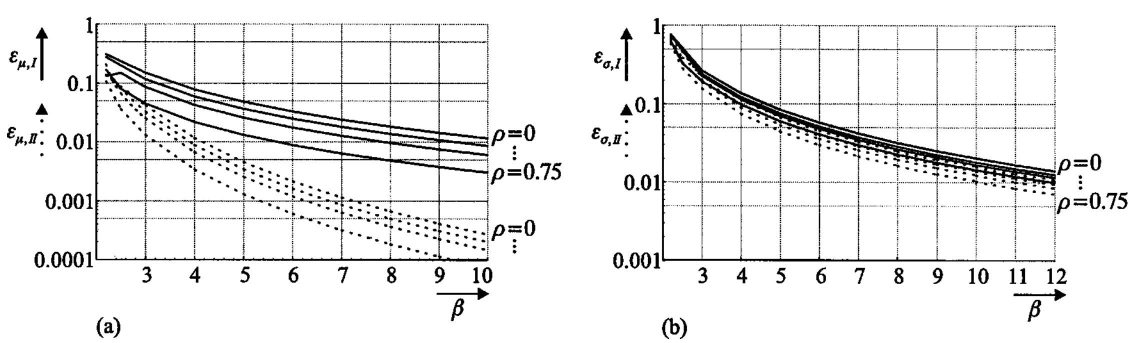

FIG. 4. Estimated errors (a) for the mean, $\varepsilon_{\mu,\mathrm{I}}$ (solid) and $\varepsilon_{\mu,\mathrm{II}}$ (dashed), as well as (b) for the standard deviation, $\varepsilon_{\sigma,\mathrm{I}}$ (solid) and $\varepsilon_{\sigma,\mathrm{II}}$ (dashed), for $\alpha=\beta$. The parameter ρ takes on the values 0, 0.25, 0.5, and 0.75.

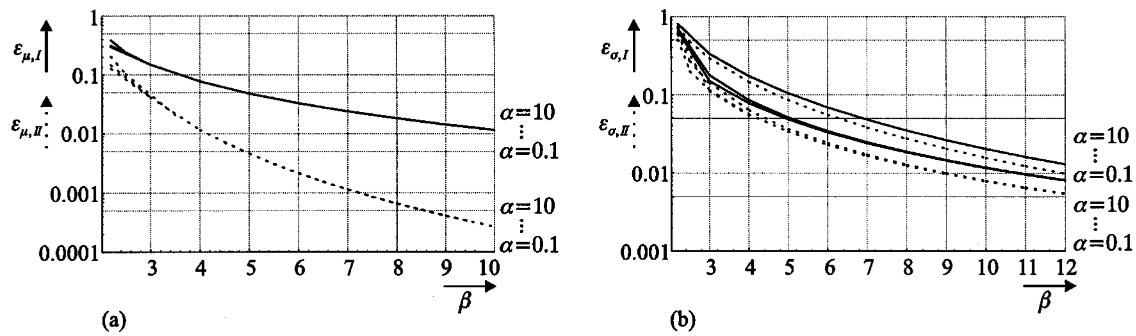

FIG. 5. Estimated errors (a) for the mean, $\varepsilon_{\mu,\mathrm{I}}$ (solid) and $\varepsilon_{\mu,\mathrm{II}}$ (dashed), as well as (b) for the standard deviation, $\varepsilon_{\sigma,\mathrm{I}}$ (solid) and $\varepsilon_{\sigma,\mathrm{II}}$ (dashed), for $\rho=0$. The parameter α takes on the values 0.1, 1, and 10.

applicable with reasonable ($\approx 1\%$) accuracy is a very remarkable result. For example, it implies that for photon or particle counting experiments producing a Poisson distributed variable (whose variance equals its mean), the denominator of a normalizing measurement has to have an average count of at least about $12^2=144$ for conventional error propagation to be safely applicable!

Figures 5(a) and 5(b) give the results for $\rho=0$ with α as a parameter. The good agreement with Fig. 4 shows that the quality of the numerator has only little influence on the error brought by using error propagation in the uncorrelated case. For correlated RVs, however, (Fig. 6 depicts the situation for $\rho=0.5$) the sensitivity of the accuracy of error propagation to changes in α is considerably higher, especially for $\langle Z \rangle$ and small values of α, where conventional error propagation completely fails, as noted along with Fig. 3(b).

Having applied our method of estimating the accuracy of error propagation to quotients of RVs, we will next provide a validation by comparing our estimates to some *exact* results for known PDFs, where Eqs. (7) and (8) can be evaluated analytically, as well as to some results obtained by means of computer simulations.

V. EXAMPLES

For a validation of our method of estimating the quality of error propagation, we analytically determined mean and standard deviation of the following (uncorrelated) RVs (cf. Fig. 7 and see the Appendix for detailed definitions): uniform RVs, two-value discrete RVs with and without skewness, gamma RVs, and power RVs. In addition to these four cases, we generated realizations of *correlated* RVs on a computer via weighted sums of two uncorrelated, uniform RVs, as described in detail in the Appendix.

The results for the uncorrelated case and $\alpha=\beta$ are shown in Figs. 8(a) and 8(b), where the thick black lines reproduce our estimates for $\varepsilon_{\mu,\mathrm{I}}$ and $\varepsilon_{\sigma,\mathrm{I}}$ (solid), as well as for $\varepsilon_{\mu,\mathrm{II}}$ and $\varepsilon_{\sigma,\mathrm{II}}$ (dashed) from Fig. 4. The shaded regions indicate the areas within which the *exact* evaluations of $\varepsilon_{\mu,\mathrm{I}}$, $\varepsilon_{\mu,\mathrm{II}}$, $\varepsilon_{\sigma,\mathrm{I}}$, and $\varepsilon_{\sigma,\mathrm{II}}$, as well as the results of our computer simulations lie. It can be seen that the estimate for $\varepsilon_{\mu,\mathrm{I}}$ is very well reproduced by the exact results, whereas the estimate of $\varepsilon_{\mu,\mathrm{II}}$ is slightly pessimistic for unskewed RVs. The three thin dashed curves termed ‘‘skew’’ in Fig. 8(a) fall outside that range; they correspond to RVs, whose PDF exhibits significant asymmetry (skewness): The uppermost curve was obtained for power RVs with skewness between -0.57 and -2, depending on the respective value of β, and the intermediate and lower curves give the results for two-value discrete RVs with skewness -1.5 and 1.5, respectively. As expected, our estimation becomes less accurate in this situation, owing to the choice of an *unskewed* substitution function $\tilde{p}(x,y)$, Sec. III. It is interesting to note that our estimates are more accurate for positive skewness than for negative skewness; the results for gamma RVs (with moderate positive skewness, cf. the Appendix) do not even lie outside the shaded region.

Figure 8(b) presents the results for the standard deviation: Again, our estimates are seen to accurately fit the exact error curves for the unskew cases lying within the shaded area, whereas they fail for significantly skew RVs (thin lines). [As in Fig. 8(a) the upper curve represents power

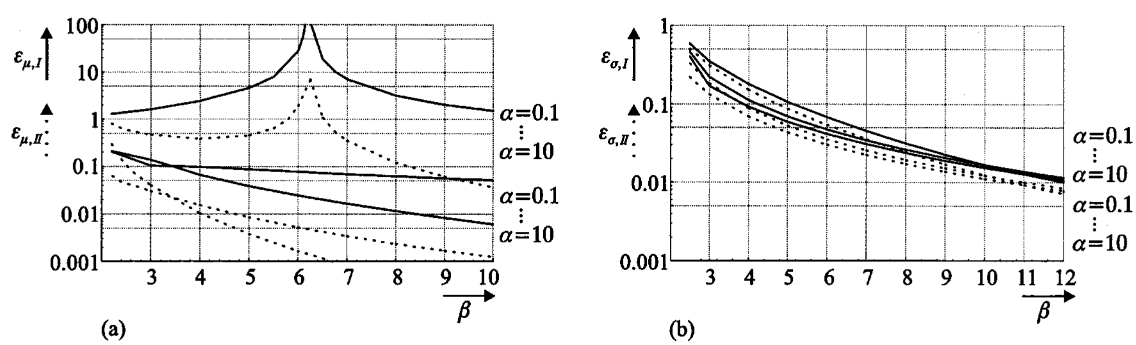

FIG. 6. Estimated errors (a) for the mean, $\varepsilon_{\mu,\mathrm{I}}$ (solid) and $\varepsilon_{\mu,\mathrm{II}}$ (dashed), as well as (b) for the standard deviation, $\varepsilon_{\sigma,\mathrm{I}}$ (solid) and $\varepsilon_{\sigma,\mathrm{II}}$ (dashed), for $\rho=0.5$. The parameter α takes on the values 0.1, 1, and 10.

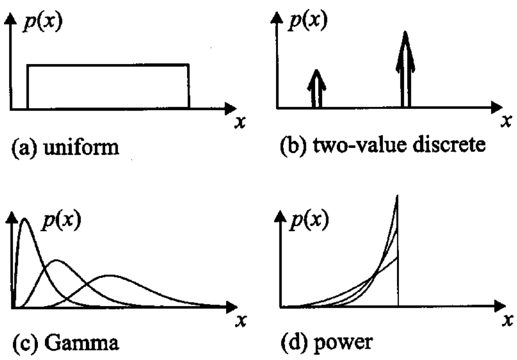

FIG. 7. To validate our estimates for the validity of error propagation, we used the PDFs shown here. The Gamma PDF and power PDF is given for three different parameters (cf. the Appendix).

RVs, and the intermediate and lower curves correspond to two-value discrete RVs with skewness -1.5 and 1.5, respectively.] Note that—unlike expected at a first glance—second-order error propagation sometimes produces slightly *worse* results for the standard deviation than the first-order formulas do. The reason for this feature (which applies for skew and unskew RVs alike) lies in the omission of many important terms leading to higher moments than the first two in Eq. (4), as indicated there by the second approximate sign.

We also investigated the quality of the fourth-order error propagation formula for the standard deviation of uncorrelated RVs, Eq. (16): Despite the fact that even more terms of the Taylor approximation had to be omitted there [cf. Eq. (6)], this expression was seen to be more accurate by up to a factor of 5 compared to the first-order formulas.

Finally, Figs. 9(a) and 9(b) give a comparison for *correlated* RVs with $\rho=0.5$ and with α as a parameter. Again, the thick lines represent our estimates and are redrawn from Fig. 6(a). The good agreement between the estimates and the simulated values (thin lines) is evident for the ensemble averages (a). The same is true for the standard deviation (b), where all curves (estimated and simulated) are found to lie within the shaded region.

APPENDIX

In this Appendix we define the RVs used for validating the estimated limits of validity of error propagation.

First and second moments of the quotient X/Y of two uncorrelated RVs X and Y with PDFs $p_X(x)$ and $p_Y(y)$ are generally determined by

$$\langle Z\rangle=\int\int_{-\infty}^{\infty}\frac{x}{y}p_X(x)p_Y(y)dx\,dy \tag{A1}$$

and

$$\langle Z^2\rangle=\int\int_{-\infty}^{\infty}\frac{x^2}{y^2}p_X(x)p_Y(y)dx\,dy, \tag{A2}$$

yielding the variance $\sigma_Z^2=\langle Z^2\rangle-\langle Z\rangle^2$.

Uncorrelated uniform RVs

From Eqs. (A1) and (A2), first and second moments of the quotient of two uncorrelated RVs, uniformly distributed over $[\langle X\rangle-\sqrt{3}\sigma_X,\langle X\rangle+\sqrt{3}\sigma_X]$ and $[\langle Y\rangle-\sqrt{3}\sigma_Y,\langle Y\rangle+\sqrt{3}\sigma_Y]$ are readily calculated as

$$\langle Z\rangle=\frac{\langle X\rangle}{\langle Y\rangle}\frac{\beta}{2\sqrt{3}}\ln\left|\frac{\beta+\sqrt{3}}{\beta-\sqrt{3}}\right| \tag{A3}$$

and

$$\sigma_Z^2=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\frac{1+1/\alpha^2}{1-3/\beta^2}-\langle Z\rangle^2, \tag{A4}$$

provided that the PDF of the denominator does not extend to zero (i.e., that $\beta>\sqrt{3}$). In these expressions we used the abbreviations $\alpha=\langle X\rangle/\sigma_X$ and $\beta=\langle Y\rangle/\sigma_Y$.

Uncorrelated two-value discrete RVs

The PDF of a two-value discrete RV is given by

$$p_X(x)=a\,\delta(x-[A-\Delta A])+(1-a)\delta(x-A),\quad 0\leqslant 1, \tag{A5}$$

where $\delta(\cdot)$ denotes Dirac's delta functional. For this type of RV, the inverse coefficient of variation α and the coefficient of skewness $s_X=\langle(X-\mu_X)^3\rangle/\sigma_X^3$ (a measure for the PDF's asymmetry) evaluate to

$$\alpha=(A/\Delta A-a)/\sqrt{a(1-a)}, \tag{A6}$$

$$s_X=(2a-1)/\sqrt{a(1-a)}. \tag{A7}$$

For a desired skewness one can thus determine a, and from a and α one finds the parameter $A/\Delta A$. As expected, for a

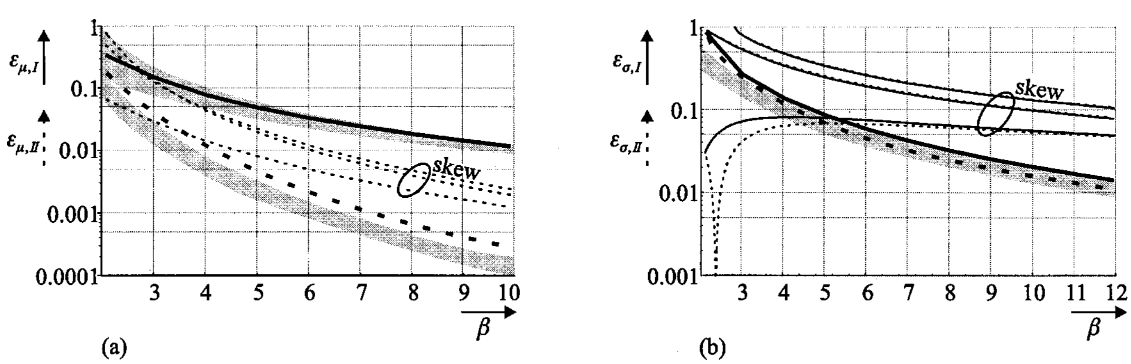

FIG. 8. Comparison of our estimates (thick lines) for $\varepsilon_{\mu,\mathrm{I}}$, $\varepsilon_{\mu,\mathrm{II}}$, $\varepsilon_{\sigma,\mathrm{I}}$, and $\varepsilon_{\sigma,\mathrm{II}}$ with the exact solutions for $\alpha=\beta$ and $\rho=0$. Apart from three significantly skew cases (thin lines), all exact solutions lie within the shaded regions.

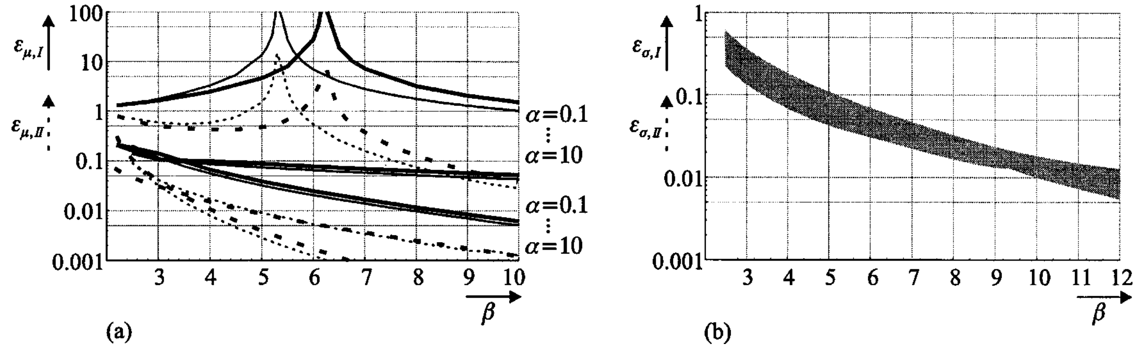

FIG. 9. Comparison of our estimates for $\varepsilon_{\mu,\mathrm{I}}$, $\varepsilon_{\mu,\mathrm{II}}$, $\varepsilon_{\sigma,\mathrm{I}}$, and $\varepsilon_{\sigma,\mathrm{II}}$ with results of computer simulations for $\rho=0.5$ and $\alpha=0.1$, 1, and 10. Our estimates for the mean [thick lines in (a)] agree well with the simulation results [thin lines in (a)]. The estimated and simulated curves for the standard deviation lie all within the shaded area in (b).

symmetric PDF ($s_X=0$) we obtain $a=0.5$, i.e., equal weights of the two Dirac functionals; the parameter $A/\Delta A$ then evaluates to $0.5(\alpha+1)$.

The moments of the quotient of two Dirac RVs can be evaluated analytically, yielding

$$\langle Z\rangle=\frac{\langle X\rangle}{\langle Y\rangle}\left[1+\frac{b(1-b)}{(B/\Delta B-1)B/\Delta B}\right] \tag{A8}$$

and

$$\sigma_Z^2=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\left[1+\frac{1}{\alpha^2}\right] \times\left[\frac{(B/\Delta B)^2+(1-b)(1-2B/\Delta B)}{(B/\Delta B)^2(B/\Delta B-1)^2}(B/\Delta B-b)^2\right] -\langle Z\rangle^2, \tag{A9}$$

where b and $B/\Delta B$ have the same meaning for the Y as a and $A/\Delta A$ have for X.

Uncorrelated gamma RVs

A gamma RV is given by the PDF[2]

$$p_X(x)=\frac{1}{\Gamma(\alpha^2)}x^{\alpha^2-1}e^{-x},\quad x\geqslant 0, \tag{A10}$$

where $\Gamma(\cdot)$ denotes Euler's gamma function. Shown in Figure 7(c) are the PDFs for $\alpha=\sqrt{2}$, $\sqrt{5}$, and $\sqrt{10}$. Gamma variables are slightly skew with a coefficient of skewness[7] of $s_X=2/\alpha$.

It is shown in Ref. 2 that the ratio of two uncorrelated gamma RVs have beta distribution

$$p_Z(z)=\frac{1}{B(\alpha^2,\beta^2)}\frac{z^{\alpha^2-1}}{(1+z)^{\alpha^2+\beta^2}},\quad z\geqslant 0, \tag{A11}$$

with the beta function $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$. The first two moments of Eq. (A11) exist for $\beta>\sqrt{2}$. With the help of Ref. 25, the ensemble average and variance of Z can then be expressed analytically as

$$\langle Z\rangle=B(\alpha^2+1,\beta^2-1)/B(\alpha^2,\beta^2) \tag{A12}$$

and

$$\sigma_Z^2=B(\alpha^2+2,\beta^2-2)/B(\alpha^2,\beta^2) -[B(\alpha^2+1,\beta^2-1)/B(\alpha^2,\beta^2)]^2. \tag{A13}$$

Uncorrelated power RVs

A power RV is given by the PDF[2]

$$p_X(x)=(\sqrt{1+\alpha^2}-1)x^{\sqrt{1+\alpha^2}-2},\quad 0<x<1. \tag{A14}$$

Figure 7(d) shows this PDF for $\alpha=4$, 6, and 8. The asymmetry of this type of PDF is immediately evident; its coefficient of skewness lies between -0.57 and -2 for $\alpha>2\sqrt{2}$.

The mean and variance of the quotient of two power RVs can readily be evaluated analytically; the results read

$$\langle Z\rangle=\frac{\langle X\rangle}{\langle Y\rangle}\frac{(\sqrt{1+\beta^2}-1)^2}{\sqrt{1+\beta^2}(\sqrt{1+\beta^2}-2)} \tag{A15}$$

and

$$\sigma_Z^2=\frac{\langle X\rangle^2}{\langle Y\rangle^2}\left[1+\frac{1}{\alpha^2}\right]\left[\frac{(\sqrt{1+\beta^2}-1)^3}{(1+\beta^2)(\sqrt{1+\beta^2}-3)}\right]-\langle Z\rangle^2, \tag{A16}$$

valid for $\beta>2\sqrt{2}$.

Simulation of correlated RVs

To validate our limits of validity for *correlated* RVs as well, we generated M realizations of two independent RVs U and V, uniformly distributed over $[-0.5,0.5]$. We then linearly transformed them according to

$$X=a+U, \tag{A17}$$

$$Y=c+bU+V, \tag{A18}$$

where the parameters a, b, and c were determined according to

$$a=\alpha/(2\sqrt{3}), \tag{A19}$$

$$b=\rho/\sqrt{1-\rho^2}, \tag{A20}$$

$$c=\beta/(2\sqrt{3}\sqrt{1-\rho^2}). \tag{A21}$$

After calculating the quotient Z_i for each realization, we used the unbiased estimators

$$\hat{\mu}_Z = 1/M \sum_{i=1}^{M} Z_i, \quad \text{(A22)}$$

$$\hat{\sigma}_Z^2 = 1/(M-1) \sum_{i=1}^{M} (Z_i - \hat{\mu}_Z)^2 \quad \text{(A23)}$$

for $\langle Z \rangle$ and σ_Z^2. Depending on α and β we needed up to $M = 5 \times 10^8$ realizations for sufficiently accurate results.

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