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CORRELATIVE TIME–FREQUENCY ANALYSIS AND CLASSIFICATION OF NONSTATIONARY RANDOM PROCESSES*

Werner Kozek[1] Franz Hlawatsch[2] Heinrich Kirchauer[2] Uwe Trautwein[3]

[1]NUHAG, Dept. of Mathematics, University of Vienna, A–1090 Vienna, Austria (kozek@tyche.mat.univie.ac.at)
[2]INTHFT, Vienna University of Technology, A–1040 Vienna, Austria (fhlawats@email.tuwien.ac.at)
[3]FG EMT, Fak. EI, Ilmenau University of Technology, D–98684 Ilmenau (tra@e–technik.tu–ilmenau.de)

Abstract—The *expected ambiguity function* (EAF) is shown to provide a generalization of stationary correlation analysis to nonstationary random processes. Important properties of the EAF are discussed, and the EAFs of special processes are considered. Based on the EAF, a fundamental classification (underspread/overspread) of nonstationary processes is introduced and shown to be relevant to time–varying spectral analysis.

1 INTRODUCTION

The correlative analysis of stationary processes using the autocorrelation function (ACF) $\rho_x(\tau) = \mathrm{E}\{x(t)\,x^*(t-\tau)\}$ is of fundamental importance [1]. In particular, the power spectrum $S_x(f)$ is the Fourier transform of the ACF (*Wiener-Khintchine relation*)[1],

$$S_x(f) = \int_\tau \rho_x(\tau)\, e^{-j2\pi f\tau}\, d\tau\,. \qquad (1)$$

For a *nonstationary* process $x(t)$, the ACF

$$r_x(t_1,t_2) = \mathrm{E}\{x(t_1)\,x^*(t_2)\}$$

is a 2-D function [1], and the power spectrum $S_x(f)$ is replaced by a time–varying power spectrum $T_x(t,f)$ such as the *physical spectrum*, the *(generalized) Wigner-Ville spectrum*, or the *evolutionary spectrum* [2, 3, 4]. Nonstationary processes exhibit *spectral correlation* [5] as measured by the spectral ACF $R_x(f_1,f_2) = \mathrm{E}\{X(f_1)\,X^*(f_2)\}$ (assuming existence of the Fourier transform $X(f)$ of $x(t)$).

We now ask if there exists a *joint time–frequency (TF) correlation function* which combines the temporal ACF $r_x(t_1,t_2)$ and the spectral ACF $R_x(f_1,f_2)$ in a meaningful way, and which is related to a meaningful time–varying spectrum by a Fourier transform (generalization of the Wiener–Khintchine relation (1)). In this paper, we show that a satisfactory answer to this question is provided by the *expected ambiguity function* (EAF) recently proposed in [6, 7]. We demonstrate that the effective support region of the EAF provides useful indications about the type of nonstationarity inherent in a process. We then introduce a fundamental *classification* (underspread/overspread) of nonstationary processes. For underspread processes, various time–varying spectra (such as the generalized Wigner–Ville spectra and the evolutionary spectra) are shown to be effectively equivalent. Furthermore, the underspread property is relevant to time–varying spectrum estimation, and finally, the physical spectrum of an underspread process is a complete second-order description of the process.

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[1]Integrals go from $-\infty$ to ∞ unless specified otherwise.

2 THE EXPECTED AMBIGUITY FUNCTION

The *(generalized) ambiguity function* (AF) [8] of a signal $x(t)$ is

$$A_x^{(\alpha)}(\tau,\nu) = \int_t x\Big(t+\Big(\frac{1}{2}-\alpha\Big)\tau\Big)\, x^*\Big(t-\Big(\frac{1}{2}+\alpha\Big)\tau\Big)\, e^{-j2\pi\nu t} dt$$

where α is a realvalued parameter. We define the *expected (generalized) ambiguity function* (EAF) of a nonstationary random process $x(t)$ as the expectation of the AF,

$$EA_x^{(\alpha)}(\tau,\nu) \stackrel{\mathrm{def}}{=} \mathrm{E}\big\{A_x^{(\alpha)}(\tau,\nu)\big\}\,.$$

It follows that the EAF is the Fourier transform of the α–parameterized ACF

$$r_x^{(\alpha)}(t,\tau) \stackrel{\mathrm{def}}{=} r_x\Big(t+\Big(\frac{1}{2}-\alpha\Big)\tau\,,\,t-\Big(\frac{1}{2}+\alpha\Big)\tau\Big)$$

with respect to t,

$$EA_x^{(\alpha)}(\tau,\nu) = \int_t r_x^{(\alpha)}(t,\tau)\, e^{-j2\pi\nu t}\, dt\,. \qquad (2)$$

Since the ACF can be recovered from the EAF by inversion of the Fourier transform (2) followed by a simple substitution to obtain $r_x(t_1,t_2)$ from $r_x^{(\alpha)}(t,\tau)$, the EAF is a *complete second-order description* of the process.

Interpretation as TF Correlation. An intuitively reasonable measure for the statistical correlation between two TF points (t_1,f_1) and (t_2,f_2) is

$$C_x(t_1,f_1;t_2,f_2) \stackrel{\mathrm{def}}{=} \mathrm{E}\big\{\langle x,g_1\rangle\,\langle x,g_2\rangle^*\big\}\,,$$

where $g_1(t)$ and $g_2(t)$ are two normalized "test signals" TF–localized about (t_1,f_1) and (t_2,f_2), respectively (see *Fig. 1(a)*). The inner product $\langle x,g_i\rangle = \int x(t)\,g_i^*(t)\,dt$ measures the "content of $x(t)$ about the TF point (t_i,f_i)." It is easily shown that the TF correlation $C_x(t_1,f_1;t_2,f_2)$ can be derived from the EAF $EA_x^{(\alpha)}(\tau,\nu)$ as

$$C_x(t_1,f_1;t_2,f_2) = \big\langle EA_x^{(\alpha)}, A_{g_1,g_2}^{(\alpha)}\big\rangle = \int_\tau\int_\nu EA_x^{(\alpha)}(\tau,\nu)\, A_{g_1,g_2}^{(\alpha)*}(\tau,\nu)\, d\tau d\nu \qquad (3)$$

where $A_{g_1,g_2}^{(\alpha)}(\tau,\nu)$ is the cross–AF of the test signals $g_1(t)$ and $g_2(t)$. For signals $g_1(t)$ and $g_2(t)$ TF–localized about (t_1,f_1) and (t_2,f_2), respectively, $A_{g_1,g_2}^{(\alpha)}(\tau,\nu)$ is known to be concentrated about the points $(\pm\tau_{12},\pm\nu_{12})$ in the (τ,ν)–plane, where $\tau_{12}=t_1-t_2$ and $\nu_{12}=f_1-f_2$ are the time lag and frequency lag, respectively. If these points $(\pm\tau_{12},\pm\nu_{12})$ are well outside the effective support of the EAF (as shown

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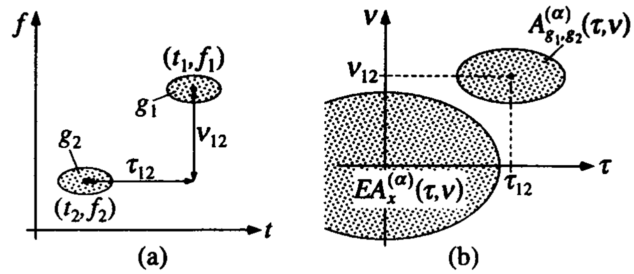

Fig. 1. TF correlation interpretation of the EAF: (a) TF plane, (b) TF lag plane.

in *Fig. 1(b)*), then $\langle EA_x^{(\alpha)}, A_{g_1,g_2}^{(\alpha)}\rangle = 0$ and, due to (3), also $C_x(t_1, f_1; t_2, f_2) = 0$. We have thus shown the following important result: *If the EAF is zero about a given "TF lag point"* (τ_{12}, ν_{12}), *then any two TF points* (t_1, f_1) *and* (t_2, f_2) *with* $t_1 - t_2 = \tau_{12}$ *and* $f_1 - f_2 = \nu_{12}$ *are uncorrelated.*

Conversely, if the TF lag points $(\pm\tau_{12}, \pm\nu_{12})$ are inside the EAF's effective support so that $EA_x^{(\alpha)}(\tau,\nu)$ and $A_{g_1,g_2}^{(\alpha)}(\tau,\nu)$ overlap, this does not necessarily imply that the TF points (t_1, f_1) and (t_2, f_2) are correlated: since both $EA_x^{(\alpha)}(\tau,\nu)$ and $A_{g_1,g_2}^{(\alpha)}(\tau,\nu)$ are typically oscillatory functions, $\langle EA_x^{(\alpha)}, A_{g_1,g_2}^{(\alpha)}\rangle$ may still be zero. Thus, the EAF $EA_x^{(\alpha)}(\tau,\nu)$ indicates the *potential correlation* between TF points separated by time lag τ and frequency lag ν.

Relation with Wigner–Ville Spectrum (Generalized Wiener–Khintchine Relation). The *generalized Wigner-Ville spectrum* $EW_x^{(\alpha)}(t,f)$ [3] is a time-varying power spectrum defined as the expectation of the generalized Wigner distribution $W_x^{(\alpha)}(t,f)$ [9, 10],

$$EW_x^{(\alpha)}(t,f) \stackrel{\text{def}}{=} \mathrm{E}\{W_x^{(\alpha)}(t,f)\} = \int_\tau r_x^{(\alpha)}(t,\tau)\, e^{-j2\pi f\tau}\, d\tau .$$

It can be shown that the generalized Wigner–Ville spectrum is essentially the 2-D Fourier transform of the EAF,

$$EW_x^{(\alpha)}(t,f) = \int_\tau \int_\nu EA_x^{(\alpha)}(\tau,\nu)\, e^{-j2\pi(f\tau - t\nu)} d\tau d\nu . \quad (4)$$

This is a generalization of the Wiener–Khintchine relation (1) to nonstationary processes.

Properties. We next summarize some basic properties of the EAF.

- The EAFs obtained for different choices of α are equal up to a phase factor,
$$EA_x^{(\alpha_2)}(\tau,\nu) = e^{j2\pi(\alpha_1-\alpha_2)\tau\nu}\, EA_x^{(\alpha_1)}(\tau,\nu) .$$
This shows that the EAF's magnitude (which is usually of primary interest) is independent of α.
- The EAF's magnitude satisfies the *symmetry property*
$$\left|EA_x^{(\alpha)}(-\tau,-\nu)\right| = \left|EA_x^{(\alpha)}(\tau,\nu)\right| .$$
- For a *realvalued* process $x(t)$, the EAF's magnitude is symmetric with respect to ν,
$$\left|EA_x^{(\alpha)}(\tau,-\nu)\right| = \left|EA_x^{(\alpha)}(\tau,\nu)\right| \quad \text{for } x(t) \in \mathbb{R} .$$
- The EAF's value at the origin of the (τ,ν)-plane equals the expected energy of the process,
$$EA_x^{(\alpha)}(0,0) = \mathrm{E}\left\{\int_t |x(t)|^2 dt\right\} = \int_t r_x(t,t)\, dt ,$$
and it represents the maximum EAF magnitude,
$$\left|EA_x^{(\alpha)}(\tau,\nu)\right| \le EA_x^{(\alpha)}(0,0) .$$
- The integral of the EAF's squared magnitude is
$$\int_\tau\int_\nu \left|EA_x^{(\alpha)}(\tau,\nu)\right|^2 d\tau d\nu = \int_{t_1}\int_{t_2} |r_x(t_1,t_2)|^2\, dt_1 dt_2 .$$
- TF-shifting a process by time τ_0 and frequency ν_0 such that $\tilde{x}(t) = x(t-\tau_0)\, e^{j2\pi\nu_0 t}$ leaves the EAF invariant up to a phase factor,
$$EA_{\tilde{x}}^{(\alpha)}(\tau,\nu) = e^{j2\pi(\nu_0\tau - \tau_0\nu)}\, EA_x^{(\alpha)}(\tau,\nu) .$$
- On the τ and ν axis, respectively, the EAF is
$$EA_x^{(\alpha)}(\tau,0) = \int_t r_x(t,t-\tau)\, dt$$
$$EA_x^{(\alpha)}(0,\nu) = \int_f R_x(f,f-\nu)\, df .$$

Karhunen–Loève Representation. The ACF can be expanded as

$$r_x(t_1,t_2) = \sum_k \lambda_k\, u_k(t_1)\, u_k^*(t_2) \quad (5)$$

with the nonnegative, realvalued *Karhunen–Loève (KL) eigenvalues* λ_k and the orthonormal *KL eigenfunctions* $u_k(t)$ [1]. It follows from (5) that the EAF is a weighted superposition of the AFs of the KL eigenfunctions,

$$EA_x^{(\alpha)}(\tau,\nu) = \sum_k \lambda_k\, A_{u_k}^{(\alpha)}(\tau,\nu) .$$

Thus, when all KL eigensignals $u_k(t)$ are well concentrated in the TF plane (in which case the AFs of the $u_k(t)$ will be well concentrated about the origin of the (τ,ν)-plane), then also the EAF will be well concentrated about the origin of the (τ,ν)-plane. Conversely, if the KL eigensignals $u_k(t)$ are poorly TF-concentrated, this does not necessarily imply poor concentration of the EAF. We also note the relations $\langle EA_x^{(\alpha)}, A_{u_k}^{(\alpha)}\rangle = \lambda_k$, $EA_x^{(\alpha)}(0,0) = \sum_k \lambda_k$, and $\int_\tau\int_\nu \left|EA_x^{(\alpha)}(\tau,\nu)\right|^2 d\tau d\nu = \sum_k \lambda_k^2$.

3 SPECIAL PROCESSES

It is instructive to study the EAFs of important special types of nonstationary processes (see *Fig. 2*).

Stationary Processes. For a (wide-sense) *stationary* process with ACF $r_x(t_1,t_2) = \rho_x(t_1 - t_2)$, the EAF effectively reduces to the 1-D ACF $\rho_x(\tau)$ since

$$EA_x^{(\alpha)}(\tau,\nu) = \rho_x(\tau)\, \delta(\nu) .$$

Note that the EAF of a stationary process is zero for $\nu \ne 0$, which indicates the absence of spectral correlation.

Nonstationary White Noise. In the dual case of *nonstationary white noise* [1] with ACF $r_x(t_1,t_2) = q_x(t_1)\,\delta(t_1 - t_2)$ ($q_x(t) \ge 0$), the EAF is

$$EA_x^{(\alpha)}(\tau,\nu) = \delta(\tau)\, Q_x(\nu) ,$$

where $Q_x(\nu)$ is the Fourier transform of $q_x(t)$. Note that the spread of $Q_x(\nu)$ grows with increasing temporal variations of the average intensity function $q_x(t)$ (i.e., increasing nonstationarity of the process), and characterizes the amount of spectral correlation. The EAF is zero for $\tau \ne 0$, which indicates the absence of temporal correlation.

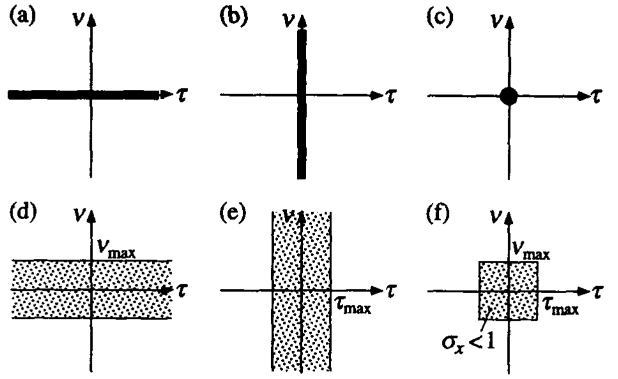

Fig. 2. Effective EAF support for special processes: (a) stationary, (b) nonstationary white, (c) stationary white, (d) quasistationary, (e) finite correlation width, (f) underspread.

Stationary White Noise. In the case of *stationary white noise* with ACF $r_x(t_1, t_2) = q_x\, \delta(t_1 - t_2)$ (the intersection of the two last-mentioned cases), the EAF indicates the absence of temporal or spectral correlation:

$$EA_x^{(\alpha)}(\tau, \nu) = q_x\, \delta(\tau)\, \delta(\nu)\,.$$

Cyclostationary Processes. The ACF of a *cyclostationary process* with period T satisfies $r_x(t_1 + T, t_2 + T) = r_x(t_1, t_2)$. It follows that the EAF is ideally concentrated at equally spaced frequency lags,

$$EA_x^{(\alpha)}(\tau, \nu) = \sum_k c_k^{(\alpha)}(\tau)\, \delta\Big(\nu - \frac{k}{T}\Big)$$

with $c_k^{(\alpha)}(\tau) = \frac{1}{T} \int_0^T r_x^{(\alpha)}(t, \tau) \exp\left\{-j 2\pi \frac{k}{T} t\right\} dt$ [5].

Processes with Finite Correlation Width. If, for all t, the ACF $r_x^{(1/2)}(t, \tau) = r_x(t, t - \tau)$ is zero for $|\tau| > \tau_{\max}$, then also $EA_x^{(1/2)}(\tau, \nu) = \mathcal{F}_{t \to \nu}\, r_x^{(1/2)}(t, \tau)$ will be zero for $|\tau| > \tau_{\max}$. Since the EAF with arbitrary α equals $EA_x^{(1/2)}(\tau, \nu)$ up to a phase factor, the same will be true also for EAFs with $\alpha \neq 1/2$. Thus, the EAF's spread in the τ–direction indicates the *correlation width* of a process.

Quasistationary Processes. The ACF $r_x^{(1/2)}(t, \tau) = r_x(t, t - \tau)$ of a *quasistationary* process is slowly varying with respect to t [1]. This entails small spread of $EA_x^{(1/2)}(\tau, \nu) = \mathcal{F}_{t \to \nu}\, r_x^{(1/2)}(t, \tau)$ in the ν–direction. Since any arbitrary EAF equals $EA_x^{(1/2)}(\tau, \nu)$ up to a phase, the EAF's spread in the ν–direction is a "spectral correlation width" indicating the "degree of nonstationarity."

4 UNDERSPREAD PROCESSES

Definition. Let $[-\tau_{\max}, \tau_{\max}] \times [-\nu_{\max}, \nu_{\max}]$ be the smallest rectangle (centered at the origin of the (τ, ν)–plane) which contains the effective support of the EAF, i.e., $EA_x^{(\alpha)}(\tau, \nu) \approx 0$ for $|\tau| > \tau_{\max}$ or $|\nu| > \nu_{\max}$. Furthermore, let $\sigma_x = 4\, \tau_{\max}\, \nu_{\max}$ denote the area of this rectangle. We call a process *underspread* if $\sigma_x < 1$ (see Fig. 2(f)) and *overspread* if $\sigma_x > 1$ [6]. This definition is independent of α since the support of $EA_x^{(\alpha)}(\tau, \nu)$ is independent of α.

The property "$EA_x^{(\alpha)}(\tau, \nu) \approx 0$ for $|\tau| > \tau_{\max}$" is a limitation of the temporal correlation width, and "$EA_x^{(\alpha)}(\tau, \nu) \approx 0$ for $|\nu| > \nu_{\max}$" is a limitation of the spectral correlation width or, equivalently, of the degree of nonstationarity. The *underspread* property combines these two properties but allows to exchange one property for the other. For example, a quasistationary process (with small $\nu_{\max}$) and a "nearly white" process (with small $\tau_{\max}$) may both be underspread. Some important consequences of the underspread property are discussed in the following.

Equivalence of Time–Varying Spectra. For underspread processes, many different time–varying spectra are essentially equivalent. First, the deviation between two different *generalized Wigner–Ville spectra* $EW_x^{(\alpha)}(t, f)$ can be bounded (for $\sigma_x < 2$ and $|\alpha_1 - \alpha_2| < 1$) as [11]

$$\left| EW_x^{(\alpha_1)}(t, f) - EW_x^{(\alpha_2)}(t, f) \right| \;<\; C\, \sigma_x^2\, |\alpha_1 - \alpha_2|$$

with $C = \frac{\pi}{2} \sum_k \lambda_k$ where λ_k are the KL eigenvalues of $x(t)$. Thus, if $\sigma_x \ll 1$, then $EW_x^{(\alpha_1)}(t, f) \approx EW_x^{(\alpha_2)}(t, f)$ for all TF points (t, f). A similar bound can be derived [11] for the deviation between a generalized Wigner–Ville spectrum $EW_x^{(\alpha)}(t, f)$ and the *evolutionary spectrum* $ES_x(t, f)$ [3, 4] (defined by self–adjoint factorization of the ACF [11]): For $\sigma_x < 4$ and $|\alpha| < 1/2$, there is

$$\left| ES_x(t, f) - EW_x^{(\alpha)}(t, f) \right| \;<\; D\, \sigma_x^2$$

where $D = \frac{\pi}{2} \left(\sigma_x\, a/32 + |\alpha|\, b\right)$ with $a = \left(\sum_k \sqrt{\lambda_k}\right)^2$ and $b = \sum_k \lambda_k$. Thus, if $\sigma_x \ll 1$, then $ES_x(t, f) \approx EW_x^{(\alpha)}(t, f)$.

Fig. 3 shows the similarity of the Wigner–Ville spectrum $EW_x^{(0)}(t, f)$, the Rihaczek spectrum $EW_x^{(1/2)}(t, f)$, and the evolutionary spectrum $ES_x(t, f)$ in the case of a specific underspread process.

Estimation of Time–Varying Spectra. The EAF of an underspread process is restricted to a rectangle $[-\tau_{\max}, \tau_{\max}] \times [-\nu_{\max}, \nu_{\max}]$ with area $\sigma_x < 1$. Due to (4), the generalized Wigner–Ville spectrum $EW_x^{(\alpha)}(t, f)$ is a 2-D lowpass function that is uniquely characterized by its samples taken on a rectangular grid with time period $T = 1/(2\nu_{\max})$ and frequency period $F = 1/(2\tau_{\max})$ [12]. The underspread property then implies $TF = 1/(4\, \tau_{\max}\, \nu_{\max}) = 1/\sigma_x > 1$. This property is of relevance to the problem of estimating the generalized Wigner–Ville spectrum $EW_x^{(\alpha)}(t, f)$ from a single realization of the process $x(t)$. Since $EW_x^{(\alpha)}(t, f)$ is completely specified by its samples $w_{kl} = EW_x^{(\alpha)}(kT, lF)$, the spectrum estimation problem reduces to the problem of estimating the samples w_{kl}. Let us assume that the process $x(t)$ is bandlimited with bandwidth B_x, and can thus be represented by its samples $x_n = x(nT_s)$ with $T_s = 1/(2\, B_x)$. If a realization of $x(t)$ is observed over a time interval of length $T_{\text{obs}} = N\, T_s$, then our data set comprises N samples x_n. On the other hand, the number M of parameters to be estimated equals the number of samples w_{kl} falling into the rectangular TF region with time length T_{obs} and frequency length $2B_x$,

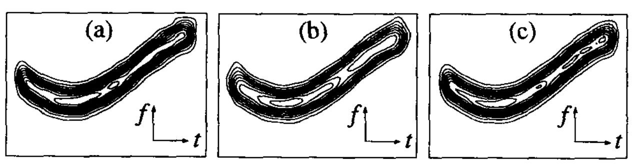

Fig. 3. Time–varying spectra of an underspread process: (a) Wigner–Ville spectrum, (b) Rihaczek spectrum, (c) evolutionary spectrum. The time length is 128 samples.

$$M = \frac{2\,T_{\rm obs}\,B_x}{TF} = \frac{N}{TF} = \sigma_x\,N\,.$$

For robust estimation, the number of parameters to be estimated must be smaller than the size of our data set, i.e. $M < N$ which implies $\sigma_x < 1$. Thus, the underspread property $\sigma_x < 1$ is a necessary condition for robust estimation of the generalized Wigner–Ville spectrum.

Sufficiency of the Physical Spectrum. The *physical spectrum* [3]

$$PS_x^{(w)}(t,f) = \mathrm{E}\left\{ \left| \int_{t'} x(t')\, w^*(t'-t)\, e^{-j2\pi f t'} dt' \right|^2 \right\}$$

is defined as the expectation of the spectrogram of $x(t)$ with analysis window $w(t)$. It can be shown that

$$PS_x^{(w)}(t,f) = \int_{t'}\int_{f'} EW_x^{(\alpha)}(t',f')\; W_w^{(\alpha)*}(t'-t,f'-f)\; dt'df'. \tag{6}$$

In general, the smoothing described by (6) makes it impossible to recover $EW_x^{(\alpha)}(t,f)$ (and, in turn, the ACF $r_x(t_1,t_2)$) from $PS_x^{(w)}(t,f)$, which means that the physical spectrum is not a complete second-order description of the process. Indeed, recovering $EW_x^{(\alpha)}(t,f)$ requires a deconvolution to invert (6). Taking the 2-D Fourier transform of (6) yields $\widetilde{PS}_x^{(w)}(\tau,\nu) = EA_x^{(\alpha)}(\tau,\nu)\;A_w^{(\alpha)*}(\tau,\nu)$ where $\widetilde{PS}_x^{(w)}(\tau,\nu) = \mathcal{F}_{t\to\nu}\mathcal{F}^{-1}_{f\to\tau}\{PS_x^{(w)}(t,f)\}$. The deconvolution then corresponds to performing the division $EA_x^{(\alpha)}(\tau,\nu) = \widetilde{PS}_x^{(w)}(\tau,\nu)\,/\,A_w^{(\alpha)*}(\tau,\nu)$ which is ill-conditioned if $A_w^{(\alpha)}(\tau,\nu) \approx 0$ for TF lag points $(\tau,\nu) \in \mathcal{S}$, where $\mathcal{S}$ denotes the effective support of $EA_x^{(\alpha)}(\tau,\nu)$.

In the case of an *underspread* process, the EAF's support $\mathcal{S}$ is contained in a rectangle with area $\sigma_x < 1$. Here, one can always find windows $w(t)$ whose AF is sufficiently bounded away from zero for $(\tau,\nu) \in \mathcal{S}$ (see *Fig. 4*),

$$\left|A_w^{(\alpha)}(\tau,\nu)\right| \;\geq\; \epsilon > 0 \qquad \text{for all } (\tau,\nu) \in \mathcal{S}\,.$$

The EAF can then be recovered from $\widetilde{PS}_x^{(w)}(\tau,\nu)$ by performing the division $EA_x^{(\alpha)}(\tau,\nu) = \widetilde{PS}_x^{(w)}(\tau,\nu)\,/\,A_w^{(\alpha)*}(\tau,\nu)$ for (τ,ν) where $|A_w^{(\alpha)}(\tau,\nu)| \geq \epsilon$, and setting $EA_x^{(\alpha)}(\tau,\nu) = 0$ elsewhere. From the EAF, the ACF can finally be derived as explained in Section 2. Thus, the physical spectrum of an *underspread* process is a complete second-order description of the process, provided that the analysis window $w(t)$ used in $PS_x^{(w)}(t,f)$ is matched to the process in the sense that the EAF's effective support is contained in the effective support of the window's AF. For example, a long window will be suited to a quasistationary process whereas a short window will be suited to a "nearly white" process. Note that the underspread property guarantees the existence of "matched" windows. Techniques for optimum window matching can be found in [6, 7, 11].

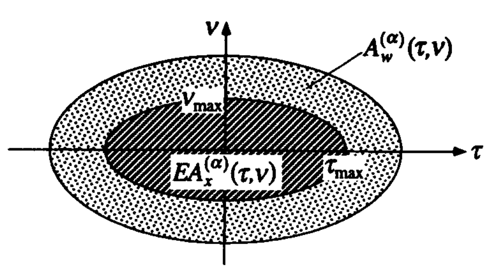

Fig. 4. The effective support of $EA_x^{(\alpha)}(\tau,\nu)$ is contained in the effective support of $A_w^{(\alpha)}(\tau,\nu)$.

5 CONCLUSIONS

The *expected ambiguity function* (EAF) is a useful time–frequency correlation function of nonstationary processes which indicates both the temporal and spectral correlation widths. The EAF allows the definition of the class of *underspread* processes for which various time–varying spectra (the generalized Wigner–Ville spectra and the evolutionary spectrum) are essentially equivalent. The underspread property is a necessary condition for robust estimation of the generalized Wigner–Ville spectrum. For underspread processes, the physical spectrum (with suitable analysis window) is a complete second–order description.

The underspread property is important in many other respects as well, such as nonstationary Wiener filters [13], the Gabor expansion [7], and the short–time Fourier transform [6]. A class of time–varying spectrum estimators for underspread processes is studied in [14].

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