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A COMPARATIVE STUDY OF LINEAR AND NONLINEAR TIME–FREQUENCY FILTERS*

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Abstract — Various methods for time-varying filtering with prescribed time–frequency (TF) pass region are compared analytically and numerically. The Weyl correspondence permits a unified TF analysis of apparently unrelated concepts of linear TF filtering, including STFT-based filters, Weyl filters, and TF projection filters. The linear TF filters are compared to nonlinear TF analysis/masking/synthesis methods based on the Wigner distribution (WD) or the smoothed WD. The study of TF signal separation problems permits the objective evaluation of TF filter methods. Linear methods, in general, yield improved performance and require reduced cost as compared to nonlinear methods.

1 INTRODUCTION

Many signals have a time–varying spectral content and are thus best described in a joint time–frequency (TF) domain [1]. In various applications such as seismic signal processing, it is necessary to separate signals with (nearly) disjoint TF supports. Since these signals may overlap both in the time domain and in the frequency domain, a conventional time–domain windowing or frequency–domain windowing (time–invariant bandpass filtering) may not be adequate. However, the disjointness of the signals' TF supports suggests a joint *TF filtering*.

This paper summarizes and compares various linear and nonlinear TF filter methods. Linear methods, like Weyl filters, STFT-based filters, and TF projection filters, are studied in Section 2 in a common framework provided by the (generalized) Weyl symbol. Nonlinear methods (based on a TF analysis/masking/synthesis approach) are reviewed in Section 3. The performances of all methods are compared in Section 4.

The TF filtering problem addressed can be loosely stated as follows: construct a (linear or nonlinear) system that passes all signals located in a "pass region" R of the TF plane but suppresses all signals located outside R. Of course, the concept of "TF plane" always depends on some specific TF signal representation [1]. We shall see that, to a large extent, the diversity of existing TF filter methods is due to the non-uniqueness of a TF representation. Still, in all methods we shall use the indicator function $M(t,f)$ of the TF pass region R,

$$M(t,f) \stackrel{\text{def}}{=} \begin{cases} 1, & (t,f) \in R \\ 0, & (t,f) \notin R \end{cases}$$

which by definition is 1 inside R and 0 outside. The indicator function $M(t,f)$ will be called a *mask* because it is used as such in some of the methods.

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2 LINEAR TIME–FREQUENCY FILTERS

The input-output relation of any linear, time–varying (LTV) filter $\mathbf{H}$ is

$$(\mathbf{H}x)(t) = \int_{t'} h(t,t')\, x(t')\, dt'$$

where $h(t,t')$ is the impulse response of $\mathbf{H}$. Given the TF pass region R (or, equivalently, the mask $M(t,f)$), we have to design the impulse response $h(t,t')$ in a meaningful way.

As a basis for this design and also for a unified analysis of linear TF filters, we use a TF representation of linear systems, the *generalized Weyl symbol* (GWS)[2]. The GWS of an LTV filter $\mathbf{H}$ with impulse response $h(t,t')$ is

$$L_H^{(\alpha)}(t,f) \stackrel{\text{def}}{=} \int_\tau h\left(t+\left(\frac{1}{2}-\alpha\right)\tau, t-\left(\frac{1}{2}+\alpha\right)\tau\right) e^{-j2\pi f\tau} d\tau \quad (1)$$

where α is a real-valued parameter. The GWS can be interpreted as a "time-varying transfer function" of an LTV system. The choice $\alpha = 0$ gives the Weyl symbol [3] $L_H^{(0)}(t,f) = \int_\tau h(t+\tau/2, t-\tau/2)\, e^{-j2\pi f\tau} d\tau$ whereas $\alpha = 1/2$ gives Zadeh's time-varying transfer function [4] $L_H^{(1/2)}(t,f) = \int_\tau h(t, t-\tau)\, e^{-j2\pi f\tau} d\tau$. For any α, the GWS satisfies the following properties:

1. In the special case of a linear time–invariant filter (i.e., $(\mathbf{H}x)(t) = (h * x)(t)$), the GWS reduces to the usual transfer function $H(f)$, $L_H^{(\alpha)}(t,f) = H(f)$.
2. In the dual case of a linear frequency–invariant system (i.e., $(\mathbf{H}x)(t) = h(t)x(t)$), the GWS reduces to the factor (window) $h(t)$, $L_H^{(\alpha)}(t,f) = h(t)$.
3. The quadratic form $(\mathbf{H}x, x)$ can be expressed as the inner product $(\mathbf{H}x, x) = \left(L_H^{(\alpha)}, W_{x,x}^{(\alpha)}\right)$ where

$$W_{x,y}^{(\alpha)}(t,f) \stackrel{\text{def}}{=} \int_\tau x\left(t+\left(\frac{1}{2}-\alpha\right)\tau\right) y^*\left(t-\left(\frac{1}{2}+\alpha\right)\tau\right) e^{-j2\pi f\tau} d\tau \quad (2)$$

denotes the *generalized Wigner distribution* (GWD) of the signals $x(t)$ and $y(t)$ [1]. The GWD reduces to the *Wigner distribution* (WD) for $\alpha = 0$ and to the *Rihaczek distribution* (RD) for $\alpha = 1/2$.

Generalized Weyl Filter. The interpretation of the GWS as a TF transfer function immediately suggests a simple filter design [5, 6]: we construct the filter $\mathbf{H}$ such that its GWS equals our mask $M(t,f)$,

$$L_H^{(\alpha)}(t,f) = M(t,f).$$

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By inversion of (1), the linear filter $\mathbf{H}$ is obtained as

$$h(t,t')=\int_f M\Big(\Big(\frac{1}{2}+\alpha\Big)\,t+\Big(\frac{1}{2}-\alpha\Big)\,t',f\Big)\,e^{j2\pi f(t-t')}df. \quad (3)$$

This filter, which depends on the parameter α, will be called a *generalized Weyl filter*. In the special cases $\alpha = 0$ and $\alpha = 1/2$, we shall speak of a *Weyl filter* and a *Zadeh filter*, respectively.

STFT Filter. A more classical linear filter design is based on the short-time Fourier transform (STFT) [7, 8, 1]. The filtering procedure consists of (i) calculating the STFT of the input signal $x(t)$ using an analysis window $\gamma(t)$, $X^{(\gamma)}(t,f) = \int_{t'} x(t')\gamma^*(t'-t)e^{-j2\pi f t'}dt'$, (ii) multiplying the STFT by our mask $M(t,f)$, $\tilde{X}(t,f) = M(t,f)X^{(\gamma)}(t,f)$, and (iii) synthesizing the output signal from the masked STFT using a synthesis window $g(t)$, $(\mathbf{H}x)(t) = \int_{t'}\int_{f'} \tilde{X}(t',f')g(t-t')e^{j2\pi f' t}dt'df'$. It is easily shown that this analysis/masking/synthesis procedure is a linear filter with impulse response

$$h(t,t'') = \iint_{t'\,f'} M(t',f')g(t-t')\gamma^*(t''-t')e^{j2\pi f'(t-t'')}dt'df'. \quad (4)$$

This filter, which depends on the analysis window $\gamma(t)$ and the synthesis window $g(t)$, will be called *STFT filter* in the following. The influence of the windows becomes apparent in the filter's GWS, which can be shown to be the mask $M(t,f)$ smoothed by the GWD of the windows,

$$L_H^{(\alpha)}(t,f) = M(t,f) ** W_{g,\gamma}^{(\alpha)}(t,f)\,. \quad (5)$$

This expression reflects a TF resolution restriction (similar to the analysis resolution restriction of the spectrogram [1]) since $W_{g,\gamma}^{(\alpha)}(t,f)$ cannot be made arbitrarily concentrated with respect to both t and f.

Multiple–Window STFT Filter. In order to overcome the TF resolution restriction of the STFT filter, we now replace the smoothing function $W_{g,\gamma}^{(\alpha)}(t,f)$ in (5) by an arbitrary, square-integrable TF function $S(t,f)$,

$$L_H^{(\alpha)}(t,f) = M(t,f) ** S(t,f)\,. \quad (6)$$

Note that this defines the filter $\mathbf{H}$. Using the singular value decomposition, $S(t,f)$ can be expanded into a weighted sum of GWDs, $S(t,f) = \sum_{k=0}^{\infty} \sigma_k W_{u_k,v_k}^{(\alpha)}(t,f)$ where $\sigma_k \geq 0$. Insertion into (6) yields

$$L_H^{(\alpha)}(t,f) = \sum_{k=0}^{\infty} \sigma_k \left[M(t,f) ** W_{u_k,v_k}^{(\alpha)}(t,f) \right]\,,$$

from which the filter's impulse response is obtained as

$$h(t,t') = \sum_{k=0}^{\infty} \sigma_k \, h_k(t,t') \quad (7)$$

where

$$h_k(t,t'') = \iint_{t'\,f'} M(t',f')u_k(t-t')v_k^*(t''-t')e^{j2\pi f'(t-t'')}dt'df'$$

is the impulse response of an STFT filter (4) with analysis and synthesis window given by $v_k(t)$ and $u_k(t)$, respectively. Hence, our filter is a weighted parallel connection of STFT filters with identical masks $M(t,f)$, orthonormal analysis windows $v_k(t)$, and orthonormal synthesis windows $u_k(t)$. This filter type will be called a *multiple-window STFT filter*. If $S(t,f)$ is a typical smoothing kernel, the singular values σ_k converge rapidly to zero, so that only few STFT filters are needed.

We note that the STFT filter and its multi-window extension can be generalized by replacing the STFT by some other linear TF signal representation. In particular, the TF version of the *wavelet transform* presents an interesting alternative to the STFT [9, 1].

Time–Frequency Projection Filter. The filtering task considered (namely, passing signals inside the TF region R and suppressing signals outside R) suggests the assumption of a *projection structure* for the filter $\mathbf{H}$. Thus, we would like to solve $L_H^{(\alpha)}(t,f) = M(t,f)$ under the side constraint that $\mathbf{H}$ be an orthogonal projection operator. This is impossible, however, since our mask $M(t,f)$ is generally not the GWS of a projection operator. Hence, we solve the least-squares problem

$$\mathbf{H} = \arg\min_{\mathbf{H}} \left\| M - L_H^{(\alpha)} \right\| \quad (8)$$

under the projection-operator side constraint. The resulting filter $\mathbf{H}$ will be called a *TF projection filter*. The TF projection filter can be constructed as follows:

1. Generalized Weyl inversion of the mask $M(t,f)$ according to (3),

$$\tilde{h}(t,t')=\int_f M\Big(\Big(\frac{1}{2}+\alpha\Big)t+\Big(\frac{1}{2}-\alpha\Big)t',f\Big)\,e^{j2\pi f(t-t')}df.$$

2. Calculation of the hermitian part $\tilde{\mathbf{H}}_h = \frac{1}{2}(\tilde{\mathbf{H}} + \tilde{\mathbf{H}}^*)$ of the filter $\tilde{\mathbf{H}}$ resulting from Step 1. ($\tilde{\mathbf{H}}^*$ denotes the adjoint of $\tilde{\mathbf{H}}$.)
3. Calculation of the eigenvalues λ_k and eigensignals $u_k(t)$ of the hermitian filter $\tilde{\mathbf{H}}_h$.
4. The TF projection filter $\mathbf{H}$ is given by

$$h(t,t') = \sum_{\lambda_k > 1/2} u_k(t)\, u_k^*(t')\,, \quad (9)$$

where the summation is over all k for which the eigenvalues λ_k are greater than $1/2$.

We note that this optimal TF design of a projection filter is equivalent to the optimal TF synthesis of a linear signal space (the signal space which forms the range of the projection operator) [10, 6, 3].

3 NONLINEAR TIME–FREQUENCY FILTERS

Filtering by WD Analysis/Masking/Synthesis. We have seen in the previous section that a linear TF filtering can be achieved by an analysis/masking/synthesis scheme based on some linear TF signal representation. If we use a quadratic TF representation instead of a linear one, the overall filtering is no longer linear. In particular, using the *Wigner distribution* (WD) $W_x^{(0)}(t,f)$ as underlying (quadratic) TF representation [11], the TF filtering procedure consists of (i) calculating the WD of the input signal $x(t)$, $W_x^{(0)}(t,f) = \int_\tau x(t+\tau/2)\,x^*(t-\tau/2)\,e^{-j2\pi f\tau}d\tau$, (ii) multiplying the WD by our mask,

$\tilde{W}(t,f) = M(t,f)W_x(t,f)$, and (iii) synthesizing the output signal $y(t)$ from the masked WD $\tilde{W}(t,f)$. The last step involves the solution of the least-squares problem

$$y(t) = \arg\min_y \left\| \tilde{W} - W_y \right\|, \qquad (10)$$

which requires the calculation of the dominant eigenvalue and eigenvector of a hermitian matrix. Due to the quadratic WD analysis and the nonlinear synthesis step, the resulting TF filter is highly nonlinear. We note that the analysis/masking/synthesis method is readily extended to other quadratic TF signal representations as long as these are *unitary* [12] (in particular, the GWD of Eq. (2)). Unfortunately, in many cases the method is adversely affected by interference terms which are generally present in any unitary, quadratic TF representation [1, 13].

Filtering by Smoothed-WD Analysis/Masking/ Synthesis. To avoid problems caused by interference terms, the WD can be replaced by a *smoothed WD* (SWD) [13]. The synthesis step then calls for the solution of

$$y(t) = \arg\min_y \left\| \tilde{W} - W_y^{(S)} \right\|, \qquad (11)$$

where $W_y^{(S)}(t,f)$ denotes the SWD of the output signal and $\tilde{W}(t,f) = M(t,f)W_x^{(S)}(t,f)$ is the masked SWD of the input signal. Unfortunately, this least-squares problem cannot be reduced to an eigenproblem as in the WD case since the SWD is non-unitary. However, an iterative algorithm for SWD-based signal synthesis is developed in [13]. Since this algorithm is more expensive than the calculation of a dominant eigenvector, the immunity to interference effects inherent in the SWD method comes at the cost of more expensive computation.

4 NUMERICAL RESULTS

In order to illustrate the performance of the various TF filter methods we consider two filtering experiments. Fig. 1 shows the results obtained in a noise suppression problem where the input signal is a Gaussian signal embedded in white noise. Fig. 2 considers a signal separation problem where the input signal is a quadratic FM signal plus a Gaussian signal which is to be suppressed. In both figures, we have plotted the WDs of the output signals of some of the filtering methods discussed in the previous two sections, and the corresponding SNR improvement for all filter methods. The SNR of a signal $x(t)$ is defined as $\|s\|^2/\|x-s\|^2$, where $s(t)$ is the desired signal (the Gaussian signal in the first experiment and the FM signal in the second experiment).

The filtering methods considered are numbered as follows (the numbers will be used for referencing the methods in the figures):

1. WD-based analysis/masking/synthesis, cf. (10);
2. SWD-based analysis/masking/synthesis, cf. (11);
3. STFT filter, cf. (4), with Gaussian window;
4. multiple-window STFT filter, cf. (7), using four windows (Hermite functions of orders 0–3);
5. Zadeh filter, cf. (3) with $\alpha = 1/2$;
6. Weyl filter, cf. (3) with $\alpha = 0$;
7. TF projection filter, cf. (8), (9) with $\alpha = 1/2$ (Rihaczek–Zadeh case);
8. TF projection filter, cf. (8), (9) with $\alpha = 0$ (Wigner–Weyl case).

Note that the first two filter methods are nonlinear while the others are linear.

A comparison of the SNR improvements shown in Fig.1(h) and Fig.2(h) indicates that the performance of the various TF filter methods considerably depends on the specific filter problem. However, two main trends can be identified from the two experiments considered:

- In general, linear methods yield better performance than nonlinear analysis/masking/synthesis methods (cf. Figs. 1(h) and 2(h): Filters No. 1 and No. 2 are nonlinear while Nos. 3–8 are linear).
- Among the linear GWS-based filters, filters based on the Wigner–Weyl case (GWS with $\alpha = 0$, cf. Nos. 6 and 8 in Figs. 1(h) and 2(h)) perform better than filters based on the Rihaczek–Zadeh case (GWS with $\alpha = 1/2$, cf. Nos. 5 and 7 in Figs. 1(h) and 2(h)). This is valid for filter designs with or without a projection side constraint.

We note that these trends are supported by the results we obtained in extensive TF–filter simulations.

5 CONCLUSION

We have compared various methods for time-varying filtering with prescribed time-frequency (TF) pass region. Both linear filters (based on various design strategies) and nonlinear filters (based on an analysis/masking/synthesis scheme using a quadratic TF signal representation) have been considered. The linear filters have been formulated in a unified framework based on the generalized Weyl symbol (GWS). The GWS has also been used for the design of linear TF filters.

Regarding the performance of the various filter methods, two main conclusions have been obtained: (i) linear TF filters tend to perform better than the nonlinear methods considered, and (ii) among the linear filters whose design is based on the GWS (with or without a projection side constraint), the Wigner–Weyl design (GWS with $\alpha = 0$) is better than the Rihaczek–Zadeh design (GWS with $\alpha = 1/2$). Thus, the well-known fact that the WD has better analysis resolution than the Rihaczek distribution carries over to the associated linear filter design.

The relative inferiority of the nonlinear analysis/masking/synthesis methods (as compared to the linear methods) seems to be due to the cross or interference terms of the underlying (quadratic) TF signal representation.

We finally note that some of the linear filter design methods considered can be extended to filtering tasks requiring an arbitrary "TF weighting" (instead of the simple binary pass/suppress weighting considered here) [14]. A second possible extension is to TF filter banks [6].

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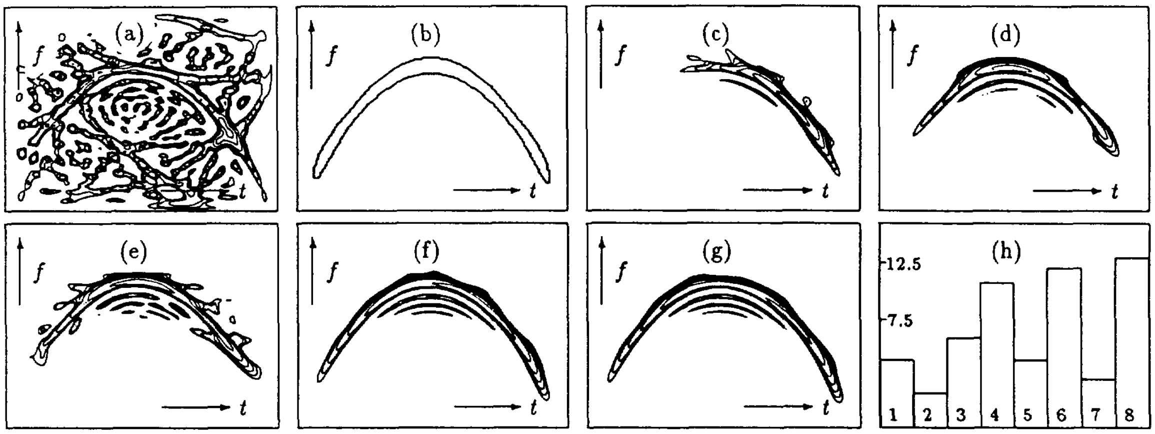

Fig. 1: (a) WD of input signal (noise-corrupted quadratic-FM signal); (b) TF pass region (mask); (c)-(g) WDs of output signals obtained by the following filter methods: (c) WD-based analysis/masking/synthesis (filter No.1); (d) STFT filter (No.3); (e) Zadeh filter (No.5); (f) Weyl filter (No.6); (g) TF projection filter with $\alpha = 0$ (No.8); (h) SNR improvement (in dB) achieved by the various filter methods.

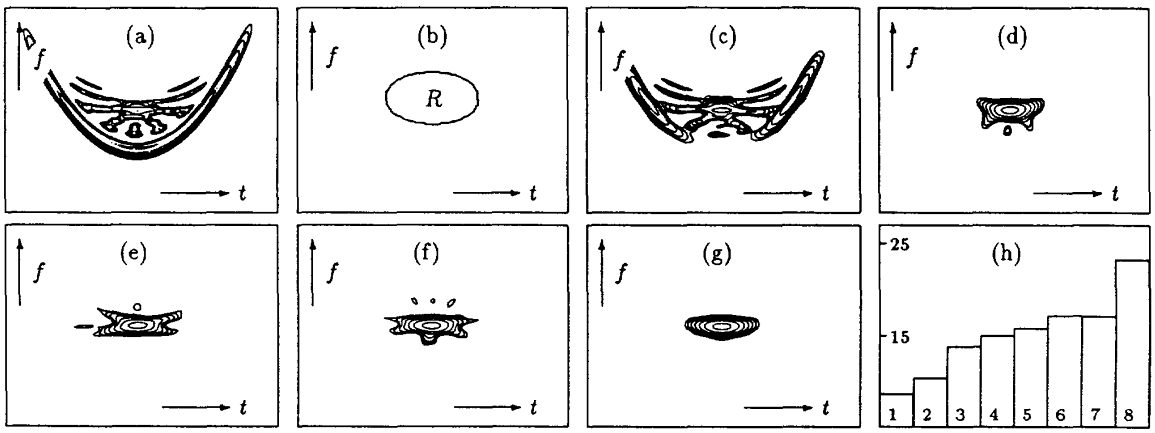

Fig. 2: (a) WD of two-component input signal consisting of a (desired) Gaussian signal and an (undesired) FM signal; (b) TF pass region (mask); (c)–(g) WDs of output signals obtained by the same filter methods as in Fig. 1; (h) SNR improvement (in dB) achieved by the various filter methods.