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TIME-FREQUENCY WEIGHTING AND DISPLACEMENT EFFECTS IN LINEAR, TIME-VARYING SYSTEMS

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Abstract – When a signal passes through a linear, time-varying (LTV) system, various signal components are weighted (attenuated or amplified) and displaced in the time-frequency (TF) plane. A gross description of TF displacement effects is provided by centroids and spreads whose definition is based on the Wigner distribution. The classes of normal, time-invariant, frequency-invariant, and unitary systems are considered, and the class of systems minimizing TF displacement (for fixed TF weighting) is derived. The polar decomposition of LTV systems is shown to provide a separation of weighting and displacement effects.

1. INTRODUCTION

LTV systems. We consider a linear, time-varying (LTV) system **H** with impulse response $H(t,t')$. The system's input-output relations are (all integrals go from $-\infty$ to ∞)

$$(\mathbf{H}x)(t) = \int_{t'} H(t,t')\, x(t')\, dt' , \qquad (\mathfrak{F}\mathbf{H}x)(f) = \int_{f'} \hat{H}(f,f') X(f') df' ,$$

where $\mathfrak{F}$ denotes Fourier transform, $X(f)$ is the spectrum of the input signal $x(t)$, and $\hat{H}(f,f')$ is the bifrequency function [1] of **H**,

$$\hat{H}(f,f') = \iint_{t\,t'} H(t,t')\, e^{-j2\pi(ft-f't')} dt\, dt' .$$

A time-frequency (TF) analysis of LTV systems may be based on the *Wigner distribution* (WD) of a signal $x(t)$ [2]

$$W_x(t,f) = \int_{\tau} x\!\left(t+\tfrac{\tau}{2}\right) x^*\!\left(t-\tfrac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau ,$$

which describes (with some restrictions due to the uncertainty principle) the signal's energy distribution over the TF plane. It is well known [3] that the WD of the input signal $x(t)$ and the WD of the output signal $(\mathbf{H}x)(t)$ are linearly related as

$$W_{\mathbf{H}x}(t,f) = \iint_{t'f'} W_{H,T}(t,f;t',f')\, W_x(t',f')\, dt' df' , \qquad (1.1)$$

where the transformation kernel

$$W_{H,T}(t,f;t',f') = \iint_{\tau\tau'} H\!\left(t+\tfrac{\tau}{2},t'+\tfrac{\tau'}{2}\right) H^*\!\left(t-\tfrac{\tau}{2},t'-\tfrac{\tau'}{2}\right) e^{-j2\pi(f\tau-f'\tau')} d\tau\, d\tau'$$

will be called the *transfer WD of the LTV system H*.

Weighting and displacement effects. Eq. (1.1) is the system's energetic input-output relation in the TF plane. According to (1.1), the transfer WD $W_{H,T}(t,f;t',f')$ describes how the TF energy distribution (WD) of a given input signal $x(t)$ is mapped into the TF energy distribution (WD) of the associated output signal $(\mathbf{H}x)(t)$. We may say that the system **H** picks up energy of the input signal in various places of the TF plane and transfers it to (generally other) time-frequency locations in the output signal. This energy transfer generally involves both a *weighting* (attenuation or amplification) and a *TF displacement*. The transfer WD describes both the weighting effect and the displacement effect. Unfortunately, this description is somewhat cumbersome since the transfer WD is a 4-dimensional function.

A description which is less complete but easier to work with is given by the system's *input WD* $W_{H,I}(t,f)$ and *output WD* $W_{H,O}(t,f)$ defined as marginals of the transfer WD [4],

$$W_{H,I}(t',f') \triangleq \iint_{t\,f} W_{H,T}(t,f;t',f')\, dt\, df$$

$$W_{H,O}(t,f) \triangleq \iint_{t'f'} W_{H,T}(t,f;t',f')\, dt' df' .$$

The input WD $W_{H,I}(t',f')$ describes the system's susceptibility to input energy located around TF points (t',f'), whereas the output WD $W_{H,O}(t,f)$ describes the average energy which is transferred by the system to TF points (t,f). In other words, the input WD shows where energy is picked up (but does not show where this energy goes) and the output WD shows where the energy goes (but does not show where it comes from). These concepts are illustrated in **Fig. 1**.

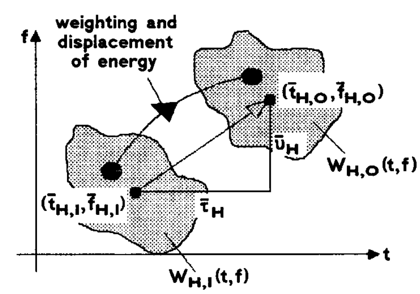

Fig. 1: TF weighting and displacement effects in an LTV system.

We also note that the integral of the input WD or output WD over the entire TF plane is

$$\iint_{t'f'} W_{H,I}(t',f')\, dt' df' = \iint_{t\,f} W_{H,O}(t,f)\, dt\, df = E_H ,$$

where $E_H \triangleq \iint |H(t,t')|^2 dt\, dt'$ will be called the ***energy of the system H***. If the impulse response is square-integrable, $E_H<\infty$, then **H** will be called a *finite-energy system*.

Survey of paper. Section 2 introduces centroids and spreads which yield a gross description of TF displacement effects. Various classes of normal systems are considered in Section 3. The problem of minimizing TF displacement effects for fixed TF weighting is studied in Section 4. Section 5 shows that the polar decomposition of LTV systems decomposes a system into a "weighting part" and a "displacement part."

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2. DISPLACEMENT CENTROIDS AND SPREADS

In this section, we introduce centroids and spreads which yield a gross characterization of TF displacement effects.

Local centroids and spreads. According to (1.1), the transfer WD may be formally viewed as the WD of the output signal if the WD of the input signal is perfectly concentrated at some TF point:

$$W_x(t,f) \to \delta(t-t')\,\delta(f-f') \quad \Rightarrow \quad W_{Hx}(t,f) \to W_{H,T}(t,f;t',f').$$

Of course, this interpretation is purely formal since an input signal with perfectly impulsive WD does not exist. Nonetheless, we define the *local displacement time* $\tau_H(t',f')$ and the *local displacement frequency* $\nu_H(t',f')$ at the "input TF point" (t',f') as

$$\tau_H(t',f') \triangleq \frac{\iint (t-t')\,W_{H,T}(t,f;t',f')\,dt\,df}{\iint W_{H,T}(t,f;t',f')\,dt\,df}$$

$$\nu_H(t',f') \triangleq \frac{\iint (f-f')\,W_{H,T}(t,f;t',f')\,dt\,df}{\iint W_{H,T}(t,f;t',f')\,dt\,df}.$$

Similarly, we define the *local displacement time spread* $T_H(t',f')$ and the *local displacement frequency spread* $F_H(t',f')$ as

$$T_H(t',f') \triangleq \frac{\iint (t-t')^2\,W_{H,T}(t,f;t',f')\,dt\,df}{\iint W_{H,T}(t,f;t',f')\,dt\,df}$$

$$F_H(t',f') \triangleq \frac{\iint (f-f')^2\,W_{H,T}(t,f;t',f')\,dt\,df}{\iint W_{H,T}(t,f;t',f')\,dt\,df}.$$

Formally, these quantities describe the TF displacement of input energy which is perfectly concentrated at the TF point (t',f'). The impossibility of such a point concentration of energy is reflected by the potential negativity of the spread quantities. This is similar to the WD of a signal which does not allow a pointwise interpretation either. Note that the denominator of all centroids and spreads equals the input WD $W_{H,I}(t',f')$.

Global centroids and spreads. Although the local centroids and spreads are purely formal quantities, suitable *averages* are "legal." Let us define the *global displacement time* $\bar{\tau}_H$, *global displacement frequency* $\bar{\nu}_H$, *global displacement time spread* $\bar{T}_H$, and *global displacement frequency spread* $\bar{F}_H$ as

$$\bar{\tau}_H \triangleq \frac{\iint \tau_H(t',f')\,W_{H,I}(t',f')\,dt'\,df'}{\iint W_{H,I}(t',f')\,dt'\,df'}$$

$$\bar{\nu}_H \triangleq \frac{\iint \nu_H(t',f')\,W_{H,I}(t',f')\,dt'\,df'}{\iint W_{H,I}(t',f')\,dt'\,df'}$$

$$\bar{T}_H \triangleq \frac{\iint T_H(t',f')\,W_{H,I}(t',f')\,dt'\,df'}{\iint W_{H,I}(t',f')\,dt'\,df'}$$

$$\bar{F}_H \triangleq \frac{\iint F_H(t',f')\,W_{H,I}(t',f')\,dt'\,df'}{\iint W_{H,I}(t',f')\,dt'\,df'}.$$

It can be shown that these quantities may be expressed in terms of the system's impulse response $H(t,t')$ and bifrequency function $\hat{H}(f,f')$ as

$$\bar{\tau}_H = \frac{\iint (t-t')\,|H(t,t')|^2\,dt\,dt'}{\iint |H(t,t')|^2\,dt\,dt'}, \quad \bar{\nu}_H = \frac{\iint (f-f')\,|\hat{H}(f,f')|^2\,df\,df'}{\iint |\hat{H}(f,f')|^2\,df\,df'},$$

$$\bar{T}_H = \frac{\iint (t-t')^2\,|H(t,t')|^2\,dt\,dt'}{\iint |H(t,t')|^2\,dt\,dt'}, \quad \bar{F}_H = \frac{\iint (f-f')^2\,|\hat{H}(f,f')|^2\,df\,df'}{\iint |\hat{H}(f,f')|^2\,df\,df'}. \qquad (2.1)$$

Eq. (2.1) shows that the global spreads are nonnegative and thus "legal." Note that the denominator of all the global quantities equals the system's energy E_H.

Defining the *global input time* $\bar{t}_{H,I}$ and the *global output time* $\bar{t}_{H,O}$ as the temporal centroids of the input WD and output WD, respectively,

$$\bar{t}_{H,I} \triangleq \frac{\iint t'\,W_{H,I}(t',f')\,dt'\,df'}{\iint W_{H,I}(t',f')\,dt'\,df'}, \quad \bar{t}_{H,O} \triangleq \frac{\iint t\,W_{H,O}(t,f)\,dt\,df}{\iint W_{H,O}(t,f)\,dt\,df}, \qquad (2.2)$$

we can show that the global displacement time equals the difference of the global output time and the global input time,

$$\bar{\tau}_H = \bar{t}_{H,O} - \bar{t}_{H,I}, \qquad (2.3)$$

as shown in Fig. 1. An analogous result holds for the global displacement frequency $\bar{\nu}_H$.

3. NORMAL SYSTEMS

If the system $\mathbf{H}$ is *normal* (i.e., $\mathbf{H}^+\mathbf{H} = \mathbf{H}\mathbf{H}^+$ [5]), then the input WD and the output WD coincide [4],

$$W_{H,I}(t,f) = W_{H,O}(t,f) \triangleq W_H(t,f), \qquad (3.1)$$

where $W_H(t,f)$ will simply be called the *WD of the (normal) LTV system* ***H***. The WD of a normal system describes the system's TF weighting but not the TF displacement.

If the global input time and the global output time exist (which, typically, may only be in the case of a finite-energy system), then it follows from (2.2) and (3.1) that they are equal, $\bar{t}_{H,O} = \bar{t}_{H,I} \triangleq \bar{t}_H$, and with (2.3) it follows further that *the global displacement time is zero*, $\bar{\tau}_H = 0$. Similarly, $\bar{\nu}_H = 0$ under analogous conditions. This does not mean that a normal system does not introduce a TF displacement; however, the individual displacements cancel each other *on the average*. We also stress that the above results do *not* apply to normal systems with infinite energy (e.g., unitary systems which are discussed later).

Time-invariant and frequency-invariant systems. An important class of infinite-energy, normal systems is the class of *time-invariant systems* where $H(t,t') = g(t-t')$ such that $(\mathbf{H}x)(t) = (g*x)(t)$ is a convolution. With $G(f)$ denoting the frequency response (Fourier transform of the impulse response $g(t)$), we obtain the weighting characteristic (WD) as [4]

$$W_H(t,f) = |G(f)|^2,$$

and the displacement quantities as

$$\tau_H(t',f') = -\frac{1}{2\pi}\frac{d}{df}\arg\{G(f)\}\Big|_{f'}, \qquad \bar{\tau}_H = \frac{\int t\,|g(t)|^2\,dt}{\int |g(t)|^2\,dt},$$

$$\nu_H(t',f') \equiv \bar{\nu}_H = 0,$$

$$T_H(t',f') = \frac{\int t^2\,W_g(t,f')\,dt}{\int W_g(t,f')\,dt}, \qquad \bar{T}_H = \frac{\int t^2 |g(t)|^2\,dt}{\int |g(t)|^2\,dt},$$

$$F_H(t',f') \equiv \bar{F}_H = 0,$$

$$\bar{t}_H \text{ does not exist}, \qquad \bar{f}_H = \frac{\int f\,|G(f)|^2\,df}{\int |G(f)|^2\,df}.$$

We see that the weighting and displacement characteristics are independent of time, and the frequency displacement quantities (both local and global) are all zero. Dual results are obtained for *"frequency-invariant systems"* where $\hat{H}(f,f') = G(f-f')$ such that $(\mathbf{H}x)(t) = g(t)\,x(t)$ is a multiplication.

Unitary systems. *Unitary systems* (for which $H^+H = HH^+ = I$ where I is the identity operator [5]) form another important class of infinite-energy, normal systems. The WD here equals the constant 1 over the entire TF plane [4],

$$W_H(t,f) \equiv 1,$$

which expresses the fact that a unitary system does not introduce a TF weighting (attenuation or amplification) but only a TF displacement. Since $E_H = \infty$, the global displacement time $\bar{\tau}_H$ and global displacement frequency $\bar{\upsilon}_H$ are *not* zero in general. An important subclass of unitary systems corresponds to an affine TF coordinate transform for which (1.1) reduces to

$$W_{Hx}(t,f) = W_x(at+bf-t_o,\, ct+df-f_o)$$

where $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. With $W_{H,T}(t,f;t',f') = \delta(t'-(at+bf-t_o))\,\delta(f'-(ct+df-f_o))$, we obtain the local displacement quantities as

$$\tau_H(t',f') = d(t'-t_o)-b(f'-f_o)-t', \quad \upsilon_H(t',f') = -c(t'-t_o)+a(f'-f_o)-f',$$

$$T_H(t',f') = [\tau_H(t',f')]^2, \qquad F_H(t',f') = [\upsilon_H(t',f')]^2.$$

The global displacement quantities do not exist in general.

4. MINIMIZING TF DISPLACEMENT EFFECTS

In many applications, one would like to design an LTV system with specified TF weighting and *minimum* TF displacement.

Mathematical formulation. An obvious first step in minimizing TF displacement effects is to restrict the system to be normal and finite-energy such that the global displacement time $\bar{\tau}_H$ and the global displacement frequency $\bar{\upsilon}_H$ are both zero. For a normal system, the TF weighting is described by the system's WD, $W_H(t,f)$; since the TF weighting is specified, we must then assume $W_H(t,f)$ to be given. Let $\mathfrak{H}$ denote the class of normal systems with the same (given) WD $W_H(t,f)$. We wish to find the system(s) $H_{opt} \in \mathfrak{H}$ minimizing the global displacement time spread $\bar{T}_H$,

$$H_{opt} \triangleq \underset{H \in \mathfrak{H}}{\operatorname{argmin}}\, \bar{T}_H .$$

Since the system H is assumed finite-energy and normal, the *eigenvalue decomposition* [5]

$$H(t,t') = \sum_k \lambda_k\, h_k(t)\, h_k^*(t') \tag{4.1}$$

exists, and the WD of H can be expressed in terms of the eigenvalues λ_k and eigenfunctions $h_k(t)$ as [4]

$$W_H(t,f) = \sum_k |\lambda_k|^2\, W_{h_k}(t,f).$$

With this, the given $W_H(t,f)$ can be shown to specify the eigenvalue magnitudes $|\lambda_k|$ and the eigenfunctions $h_k(t)$ [6]. However, the eigenvalue phases $\varphi_k = \arg\{\lambda_k\}$ are not specified. Thus, the class $\mathfrak{H}$ comprises all systems with given $|\lambda_k|$ and $h_k(t)$, and our minimization of $\bar{T}_H$ is only with respect to the phases φ_k.

Solution of the minimization problem. With (2.1), we have

$$\bar{T}_H = \frac{1}{E_H}\iint t^2\,|H(t,t')|^2\,dt\,dt' + \frac{1}{E_H}\iint t'^2\,|H(t,t')|^2\,dt\,dt' - \frac{2}{E_H}\iint t\,t'\,|H(t,t')|^2\,dt\,dt'.$$

Inserting (4.1), it easily shown that the first two terms and the denominator of the third term do not depend on the φ_k. Thus, it remains to maximize the numerator of the third term which, after straightforward manipulations, can be written as

$$N \triangleq \iint t\,t'\,|H(t,t')|^2\,dt\,dt' = \sum_k\sum_l A_{kl}\cos(\varphi_k-\varphi_l)$$

where

$$A_{kl} = |\lambda_k||\lambda_l|\left|\int t\, h_k(t)\, h_l^*(t)\,dt\right|^2 \ge 0 .$$

Since the A_{kl} are nonnegative and do not depend on the φ_k, N is maximized for $\cos(\varphi_k-\varphi_l) \equiv 1$, i.e., if all φ_k are identical, $\varphi_k \equiv \varphi_o$ with φ_o arbitrary. Inserting into (4.1), we finally obtain

$$H_{opt}(t,t') = e^{j\varphi_o}\sum_k |\lambda_k|\, h_k(t)\, h_k^*(t') = e^{j\varphi_o}\, H_P(t,t')$$

where H_P is a *positive semidefinite* system (i.e., a self-adjoint system with nonnegative eigenvalues [5]). The optimum design of H_P (for given specifications for $W_H(t,f)$) is discussed in [4].

Exactly the same result is obtained when minimizing the global displacement frequency spread instead of the global displacement time spread. We have thus shown that *the system minimizing TF displacement under the constraint of fixed TF weighting is positive semidefinite up to an arbitrary phase factor.*

Simulation results. **Fig. 2** shows computer simulation results demonstrating the influence of the eigenvalue phases φ_k on the TF displacement of a system [7]. The design method described in [4] was used to design a *positive semidefinite* LTV system with specified circular "TF pass region." This system was then used for TF-filtering a chirp signal. The circular pass region and the WD of the input (chirp) signal are shown in Fig. 2a. The WD of the corresponding output signal is depicted in Fig. 2b and is seen to be properly confined to the pass region.

From the positive semidefinite system, non-definite systems were subsequently derived by re-defining the eigenvalue phases φ_k, with the eigenvalue magnitudes and eigenfunctions left unchanged so that the system's WD remains the same. The output signals obtained with these systems are depicted in Figs. 2c–e for increasing deviation from the positive semidefinite case $\varphi_k \equiv 0$. It is seen that the energy of the output signal, while still confined to the pass region, is increasingly displaced from the original TF support of the chirp signal. This TF displacement naturally results in significant signal distortion.

5. THE POLAR DECOMPOSITION

After this excursion to normal systems, we now return to the general case. For a finite-energy system H, there exists the *singular value decomposition* [5]

$$H(t,t') = \sum_k \sigma_k\, h_{O,k}(t)\, h_{I,k}^*(t'), \tag{5.1}$$

with the singular values $\sigma_k \ge 0$, the "input" singular functions $h_{I,k}(t)$, and the "output" singular functions $h_{O,k}(t)$. Either set of singular functions is an orthonormal basis of $L_2(\mathbb{R})$. We now define the positive semidefinite systems $H_{P,I}$ and $H_{P,O}$ as

$$H_{P,I}(t,t') = \sum_k \sigma_k\, h_{I,k}(t)\, h_{I,k}^*(t'), \quad H_{P,O}(t,t') = \sum_k \sigma_k\, h_{O,k}(t)\, h_{O,k}^*(t'),$$

and the unitary system H_U as

$$H_U(t,t') = \sum_k h_{O,k}(t)\, h_{I,k}^*(t')$$

(the unitarity of H_U is not immediately obvious but can easily be shown). It is then straightforward to check that

$$H = H_U H_{P,I} = H_{P,O} H_U, \tag{5.2}$$

i.e., that the system H may be represented as the cascade of a unitary system (H_U) and a positive semidefinite system ($H_{P,I}$ or $H_{P,O}$). This is known as the *polar decomposition* [5].

Since, as discussed in Sections 3 and 4, a unitary system is a "pure displacement system" (with no TF weighting) and a positive semidefinite system is a "minimum displacement system" (with minimum TF displacement for the respective TF weighting), the polar decomposition (5.2) separates TF weighting and TF displacement effects: the positive part ($H_{P,I}$ or $H_{P,O}$) causes the TF weighting whereas the unitary part (H_U) causes the TF displacement. We note that the system's input WD and output WD equal the WDs of the respective positive parts,

$$W_{H,I}(t,f) = W_{H_{P,I}}(t,f), \qquad W_{H,O}(t,f) = W_{H_{P,O}}(t,f).$$

Normal systems. For a normal system H, the singular value decomposition (5.1) reduces to the eigenvalue decomposition (4.1). The positive parts coincide, $H_{P,I} = H_{P,O} \triangleq H_P$ with

$$H_P(t,t') = \sum_k |\lambda_k|\, h_k(t)\, h_k^*(t'),$$

and the unitary part H_U becomes

$$H_U(t,t') = \sum_k e^{j\varphi_k}\, h_k(t)\, h_k^*(t')$$

with $\varphi_k = \arg\{\lambda_k\}$. These results support the notion that the eigenvalue magnitudes are associated mainly with the TF weighting whereas the eigenvalue phases are associated mainly with the TF displacement. Also, the system's WD equals that of the positive ("weighting") part H_P and is thus independent of the unitary ("displacement") part H_U,

$$W_H(t,f) = W_{H_P}(t,f).$$

Time-invariant systems. For a time-invariant system with frequency response $G(f)$, the polar decomposition (5.2) corresponds to the frequency-response factorization

$$G(f) = G_P(f)\, G_U(f) \quad \text{where} \quad \begin{cases} G_P(f) = |G(f)| \geq 0 \\ G_U(f) = e^{j\varphi(f)} \end{cases}$$

with $\varphi(f) = \arg\{G(f)\}$, i.e., the decomposition into a zero-phase system $G_P(f)$ and an allpass system $G_U(f)$. Clearly, the zero-phase system causes a frequency-dependent weighting (according to the frequency-response magnitude $|G(f)|$) whereas the allpass system causes a frequency-dependent time displacement (according to the group delay defined as the negative derivative of the frequency-response phase $\varphi(f) = \arg\{G(f)\}$). Dual results are obtained for "frequency-invariant" systems (cf. Section 3).

6. CONCLUSION

We have analyzed time-frequency (TF) weighting and displacement effects in linear, time-varying (LTV) systems, using the input and output Wigner distributions as well as displacement centroids and spreads. It has been shown that minimum TF displacement (for given TF weighting) essentially implies a semidefinite system, and that the polar decomposition provides a separation of weighting and displacement effects.

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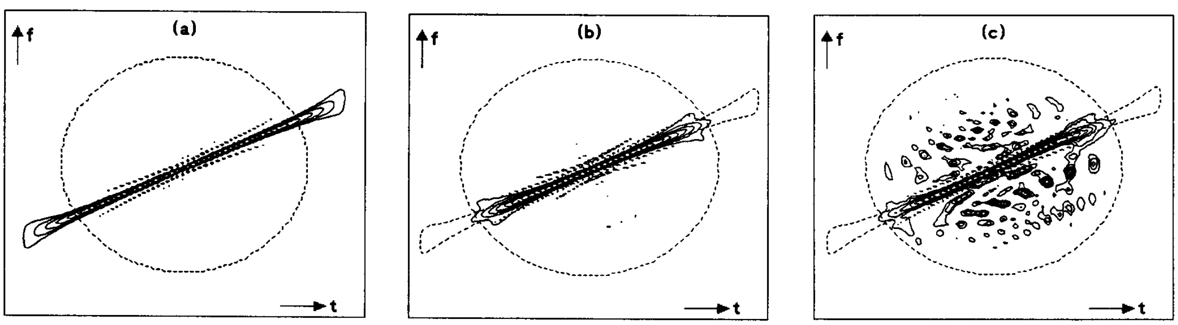

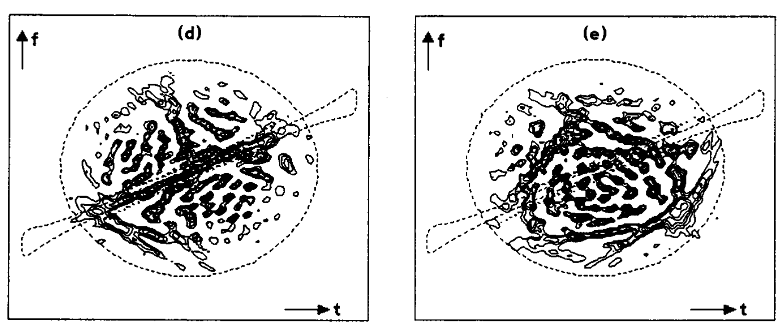

Fig. 2: Influence of the eigenvalue phases φ_k on the TF displacement of a normal system. (a) Pass region of system and WD of input signal, (b) WD of output signal for positive semidefinite system ($\varphi_k \equiv 0$), (c) WD of output signal for φ_k randomly chosen with $|\varphi_k| \leq \pi/4$, (d) WD of output signal for φ_k randomly chosen with $|\varphi_k| \leq \pi/2$, (e) WD of output signal for φ_k chosen as $\varphi_{k+1} = \varphi_k + \pi$, resulting in eigenvalues with alternating signs.