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WIGNER DISTRIBUTION ANALYSIS OF LINEAR, TIME-VARYING SYSTEMS

F. Hlawatsch

Institut für Nachrichtentechnik und Hochfrequenztechnik
Technische Universität Wien
Gusshausstrasse 25/389, A-1040 Vienna, Austria

Abstract - A time-frequency analysis of linear, time-varying (LTV) systems may be based on two new time-frequency representations of LTV systems called the "input Wigner distribution (WD)" and the "output WD." The two WD definitions coincide in the case of normal systems. The proposed WD representation can also be applied to the optimum design of LTV systems.

1. INTRODUCTION

LTV systems. We consider a linear, time-varying (LTV) system **H** with impulse response $H(t,t')$. The system's input-output relation is (all integrations go from $-\infty$ to ∞)

$$(\mathbf{H}x)(t) = \int_{t'} H(t,t')\, x(t')\, dt' . \tag{1.1}$$

We note, for later use, that the singular value decomposition [1]

$$H(t,t') = \sum_k \sigma_k\, h_{O,k}(t)\, h^*_{I,k}(t') \tag{1.2}$$

exists for square-integrable $H(t,t')$. Here, $\sigma_k \geq 0$ are the singular values and $h_{O,k}(t)$ and $h_{I,k}(t)$ will be called the "output" and "input" singular functions, respectively.

Time-frequency analysis of LTV systems. Since the system **H** is time-varying, a joint time-frequency (TF) description appears to be advantageous. TF representations which are linear in $H(t,t')$ are Zadeh's time-varying transfer function and the Weyl symbol [2-4]. In this paper, we adopt a different (quadratic) approach based on energetic quantities, specifically, the *Wigner distribution* (WD) of a signal $x(t)$ [5]

$$W_x(t,f) = \int_\tau x\!\left(t+\tfrac{\tau}{2}\right) x^*\!\left(t-\tfrac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau .$$

$W_x(t,f)$ describes (with some restrictions due to the uncertainty principle) the signal's energy distribution over the TF plane. In particular, the signal's energy, $E_x = \int |x(t)|^2 dt$, is obtained as

$$E_x = \int_t\!\int_f W_x(t,f)\, dt\, df . \tag{1.3}$$

It is well known [6] that the input-output relation (1.1) induces the following linear relation between the WD of the input signal $x(t)$ and the WD of the output signal $(\mathbf{H}x)(t)$,

$$W_{\mathbf{H}x}(t,f) = \int_{t'}\!\int_{f'} W_{H,T}(t,f;t',f')\, W_x(t',f')\, dt'\, df' . \tag{1.4}$$

The transformation kernel $W_{H,T}(t,f;t',f')$, subsequently called *transfer WD of the LTV system H*, is given by

$$W_{H,T}(t,f;t',f') = \int_\tau\!\int_{\tau'} H\!\left(t+\tfrac{\tau}{2}, t'+\tfrac{\tau'}{2}\right) H^*\!\left(t-\tfrac{\tau}{2}, t'-\tfrac{\tau'}{2}\right) e^{-j2\pi(f\tau - f'\tau')}\, d\tau\, d\tau' .$$

The transfer WD (TWD) describes the mapping from the "input TF plane" ($W_x(t',f')$) to the "output TF plane" ($W_{\mathbf{H}x}(t,f)$) in an energetic framework. The TWD yields a fairly complete TF description of the LTV system **H**; however, being a four-dimensional function, it is cumbersome to work with. Therefore, we will propose a *partial* TF description of **H** using two *two*-dimensional TF representations called "input WD" and "output WD."

Survey of paper. The input WD and output WD are introduced in Sections 2 and 3, respectively. Section 4 considers the information about the system's TF weighting/displacement properties provided by these TF representations. Section 5 studies normal systems for which the two WD definitions coincide, and Section 6 considers a simple example. The application to the TF design of LTV systems is discussed in Section 7.

2. THE INPUT WD

Definition. Conceptually, the input WD $W_{H,I}(t,f)$ at $t=t_0$, $f=f_0$ is the energy of the output signal $(\mathbf{H}x)(t)$ for an input signal $x(t)$ that is perfectly TF-concentrated at the TF point (t_0,f_0). Although the uncertainty principle prohibits the existence of such an input signal $x(t)$, we may *formally* replace $W_x(t',f')$ by the impulse $\delta(t'-t_0)\,\delta(f'-f_0)$ in (1.4) and apply (1.3) to obtain the energy of the (fictitious) output signal. This yields the *input WD of the LTV system H* as

$$W_{H,I}(t_0,f_0) \triangleq \int_t\!\int_f W_{H,T}(t,f;t_0,f_0)\, dt\, df \in \mathbb{R} . \tag{2.1}$$

Note that the input WD (IWD) is a "marginal" of the TWD; it is obtained by integrating the TWD over the entire "output TF plane." Just as the TWD, the IWD is quadratic in $H(t,t')$.

Expressions. The IWD can be expressed in terms of the impulse response $H(t,t')$ as

$$W_{H,I}(t,f) = \int_\tau q_{H,I}\!\left(t+\tfrac{\tau}{2}, t-\tfrac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \tag{2.2a}$$

with

$$q_{H,I}(t,t') = \int_{t''} H^*(t'',t)\, H(t'',t')\, dt'' . \tag{2.2b}$$

$q_{H,I}(t,t')$ is the impulse response of the composite system $\mathbf{H}^+\mathbf{H}$ ($\mathbf{H}^+$ denotes the adjoint of **H** [1] whose impulse response is $H^+(t,t')=H^*(t',t)$). Eq. (2.2) shows that the IWD is the *Weyl symbol* [3,4] of the system $\mathbf{H}^+\mathbf{H}$. Alternatively, an expression in terms of the singular quantities (cf. (1.2)) is

$$W_{H,I}(t,f) = \sum_k \sigma_k^2\, W_{h_{I,k}}(t,f) , \tag{2.3}$$

i.e., the IWD is a linear combination of the WDs of the input singular functions $h_{I,k}(t)$. Note that the output singular functions $h_{O,k}(t)$ do not enter in the IWD.

Energy relations. Even though the definition of the IWD was motivated by an argument that ignored the uncertainty principle, the IWD turns out to be a fundamental and significant TF representation. Specifically, it can easily be shown that *the inner product of the WD of the input signal x(t) and the IWD of the*

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*system **H** equals the energy of the output signal (Hx)(t),*

$$E_{\mathbf{H}x} = \left(W_x, W_{H,I}\right) = \int_t\int_f W_x(t,f)\, W_{H,I}(t,f)\, dt\, df\,. \qquad (2.4)$$

We now reformulate the concept of the IWD in a way that is consistent with the uncertainty principle. The best realizable approximation to an input signal perfectly concentrated at (t_o,f_o) is a Gaussian signal g(t) TF-shifted to the TF point (t_o,f_o),

$$x(t) = g^{(t_o,f_o)}(t) \triangleq g(t-t_o)\, e^{j2\pi f_o t}\,.$$

Again, we are interested in the energy of the output signal. Using (2.4), we easily obtain

$$E_{\mathbf{H}g^{(t_o,f_o)}} = \int_t\int_f W_g(t-t_o, f-f_o)\, W_{H,I}(t,f)\, dt\, df\,, \qquad (2.5)$$

which is essentially the IWD of **H** convolved with the WD of g(t). Repeating for each point (t_o,f_o), $E_{\mathbf{H}g^{(t_o,f_o)}}$ defines a (non-negative) TF representation of **H** which, according to (2.5), is simply a *smoothed version of the IWD*. Eq. (2.5) is valid for any signal g(t) and is analogous to the convolution relation connecting the spectrogram and the WD of a signal [5].

The conclusion to be drawn is that the IWD does not permit a *pointwise* energetic interpretation but that local averages of the IWD do allow such an interpretation. This is perfectly analogous to the WD of a signal [5].

3. THE OUTPUT WD

Definition. Conceptually, the output WD $W_{H,O}(t,f)$ is the WD of the output signal (**H**x)(t) for an input signal x(t) whose energy is distributed perfectly homogeneously over the entire TF plane. Although such an input signal does not exist, we may *formally* replace $W_x(t',f')$ by the constant 1 in (1.4); this yields the *output WD of the LTV system **H*** as

$$W_{H,O}(t,f) \triangleq \int_{t'}\int_{f'} W_{H,T}(t,f;t',f')\, dt'\, df' \in \mathbb{R}\,. \qquad (3.1)$$

The output WD (OWD) is a "marginal" of the TWD obtained by integrating the TWD over the entire "input TF plane." The OWD is again quadratic in H(t,t').

Expressions. The OWD can be expressed in terms of the impulse response H(t,t') as

$$W_{H,O}(t,f) = \int_\tau q_{H,O}\!\left(t+\tfrac{\tau}{2}, t-\tfrac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \qquad (3.2a)$$

with

$$q_{H,O}(t,t') = \int_{t''} H(t,t'')\, H^*(t',t'')\, dt''\,. \qquad (3.2b)$$

$q_{H,O}(t,t')$ is the impulse response of the composite system $\mathbf{H}\mathbf{H}^+$. Due to (3.2), the OWD is the Weyl symbol of the system $\mathbf{H}\mathbf{H}^+$. The expression in terms of the singular quantities is

$$W_{H,O}(t,f) = \sum_k \sigma_k^2\, W_{h_{O,k}}(t,f)\,, \qquad (3.3)$$

which does not contain the input singular functions $h_{I,k}(t)$.

Adjoint system. There obviously exists a strict analogy between the IWD and the OWD. Indeed, it is easily shown that *the IWD (OWD) of **H** is the OWD (IWD) of the adjoint H^+*,

$$W_{H,I}(t,f) = W_{H^+,O}(t,f)\,, \quad W_{H,O}(t,f) = W_{H^+,I}(t,f)\,.$$

"Averaging" Interpretation. The definition of the OWD was motivated assuming a fictitious signal whose WD is 1 over the entire TF plane. We now show that this concept does make sense if it is slightly reformulated. We are interested in the WD of the output signal (**H**x)(t), averaged over all possible input signals x(t). This averaging may be performed either in a deterministic or a stochastic setting [7].

Deterministic averaging. Let $\{e_k(t)\}$ be any orthonormal basis of $L_2(\mathbb{R})$, the space of finite-energy signals. This basis represents an ideally homogeneous energy distribution over the entire TF plane in the sense that $\sum_k W_{e_k}(t,f) \equiv 1$. The sum of WDs of the filtered versions $(\mathbf{H}e_k)(t)$ of all the $e_k(t)$ will thus reflect the average output energy distribution of the system **H**. This sum can be shown to be equal to the system's OWD,

$$\sum_k W_{\mathbf{H}e_k}(t,f) = W_{H,O}(t,f)\,.$$

Stochastic averaging. Let w(t) be wide-sense stationary, zero-mean white noise with power spectral density 1. The WD of w(t) is itself random; we therefore take the expectation $\overline{W}_w(t,f) = E\{W_w(t,f)\}$ which is known as the Wigner-Ville spectrum (WVS) [8]. We obtain $\overline{W}_w(t,f) \equiv 1$, which again expresses an ideally homogeneous energy distribution over the entire TF plane. The WVS of the filtered version (**H**w)(t) of w(t) will thus reflect the average output energy distribution of the system **H**. The WVS of (**H**w)(t) can be shown to be equal to the system's OWD,

$$\overline{W}_{\mathbf{H}w}(t,f) = W_{H,O}(t,f)\,.$$

4. TIME-FREQUENCY WEIGHTING/DISPLACEMENT EFFECTS

Taken together, the IWD and the OWD yield an overall description of the TF weighting and displacement properties of an LTV system. As illustrated in **Fig. 1**, the system picks up energy of the input signal in certain regions of the TF plane and transfers it to (generally different) TF regions in the output signal; this energy transfer (or *displacement*) generally includes a TF-dependent *weighting* (attenuation or amplification) [9]. A detailed description of both the weighting and displacement effects is given by the TWD, $W_{H,T}(t,f;t',f')$.

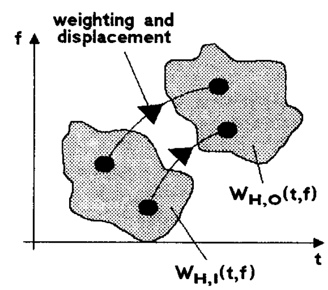

Fig. 1: Schematic illustration of the TF-weighting/displacement effects introduced by an LTV system.

The IWD, $W_{H,I}(t,f)$, shows how susceptible the system is to input energy located around a given TF point (t,f). It does not, however, show to which TF locations the energy is transferred. In fact, it averages over all possible output locations (cf. (2.1)). The OWD, $W_{H,O}(t,f)$, shows how much energy is transferred to local neighborhoods around output TF points (t,f). It does not, however, show from which input TF locations this energy is taken. In fact, it averages over all possible input locations (cf. (3.1)).

It is convenient to define the *input region* $R_{H,I}$ (*output region* $R_{H,O}$) of the system **H** as the effective TF support of the IWD (OWD). From the energy theorem (2.4), it follows that the system's output signal will be nonzero only if the input signal's WD is at least partially inside the input region $R_{H,I}$. Similarly, the WD of the system's output signal for arbitrary input signal must be essentially inside the output region $R_{H,O}$.

Thus, it is seen that the IWD and OWD provide important information about the TF characteristics of an LTV system. This information is less complete than that given by the TWD, but it comes in a form that is easier to use.

5. NORMAL SYSTEMS

The TF description discussed previously becomes much simpler if the system **H** is *normal*, i.e., $H^+H = HH^+$ [1]. Here, it follows from (2.2) and (3.2) that *the IWD and OWD coincide*,

$$W_{H,I}(t,f) = W_{H,O}(t,f) \triangleq W_H(t,f).$$

$W_H(t,f)$ will simply be called the *WD of the (normal) LTV system H*. As a consequence, the input region $R_{H,I}$ and output region $R_{H,O}$ coincide as well, $R_{H,I} = R_{H,O} = R_H$, where R_H will be called the *TF pass region of H*. Note that the identity of $R_{H,I}$ and $R_{H,O}$ does not imply that the system does not introduce a TF displacement. However, the displacement is now confined to the TF pass region R_H, i.e., the system **H** picks up input energy inside R_H and transfers it to some output TF locations inside the *same* region R_H. This TF displacement is *not* described by the system's WD (see [9] for further discussion).

In the case of a normal system, the singular value decomposition (1.2) reduces to the eigenvalue decomposition

$$H(t,t') = \sum_k \lambda_k \, h_k(t) \, h_k^*(t')$$

with the complex eigenvalues λ_k and the orthonormal eigenfunctions $h_k(t)$. The expressions (2.3), (3.3) then reduce to

$$W_H(t,f) = \sum_k |\lambda_k|^2 \, W_{h_k}(t,f). \tag{5.1}$$

Note that $W_H(t,f)$ is independent of the eigenvalues' phases.

Classes of normal systems. We now specialize our results to some interesting classes of normal systems.

Time-invariant systems. For a linear, time-invariant system, there is $H(t,t') = g(t-t')$ whence (1.1) becomes a convolution. The WD of **H** here reduces to the squared magnitude of the system's transfer function $G(f)$ (the Fourier transform of $g(t)$),

$$W_H(t,f) \equiv |G(f)|^2, \tag{5.2}$$

and is seen to be *independent of the time t*. As a consequence, the pass region R_H is one or several strips running parallel to the time axis (corresponding to the pass band(s) of the system). Note that the phase of the transfer function $G(f)$ (and thus the group delay describing the time displacement introduced by the system) do not enter in $W_H(t,f)$.

Frequency-invariant systems. For a "frequency-invariant" system, there is $H(t,t') = g(t)\,\delta(t-t')$ so that (1.1) becomes a multiplication, $(Hx)(t) = g(t)\,x(t)$. Here, the WD of **H** is

$$W_H(t,f) \equiv |g(t)|^2,$$

which is *independent of the frequency f* and, of course, dual to (5.2). The pass region R_H is one or several strips running parallel to the frequency axis (corresponding to the effective time support of the factor $g(t)$). Note that the phase of $g(t)$ (and thus the instantaneous frequency describing the frequency displacement introduced by the system) do not enter in $W_H(t,f)$.

Unitary systems. A system is unitary if $H^+H = HH^+ = I$ (the identity operator) or, equivalently, $|\lambda_k| = 1$ for all k. It readily follows that the system's WD is identically 1,

$$W_H(t,f) \equiv 1, \tag{5.3}$$

which expresses the fact that a unitary system does not effect any TF weighting, i.e., no region of the TF plane is attenuated or amplified. The system causes only a TF *displacement* which, however, is not described by the system's WD (5.3). The system's pass region is the entire TF plane, $R_H = \mathbb{R}^2$. A simple example of a unitary system is a time-invariant allpass filter.

Projection systems. The orthogonal projection operator P_S associated with a linear signal space S is an (idempotent and self-adjoint) LTV system with eigenvalues $\lambda_k = 1$ for $1 \le k \le N_S$ (N_S is the dimension of S) and $\lambda_k = 0$ for all other k, and eigenfunctions $h_k(t)$ which constitute an orthonormal basis of S [1]. With (5.1), the WD of P_S is

$$W_{P_S}(t,f) = \sum_{k=1}^{N_S} W_{h_k}(t,f),$$

i.e., simply the sum of the WDs of the space's orthonormal basis signals. We note that the WD of the projection P_S is equal to the *WD of the space S* as introduced in [7].

6. EXAMPLE: TIME-VARYING BANDPASS FILTER

An intuitively appealing normal LTV system is the "bandpass filter with time-varying center frequency" depicted in **Fig. 2**.

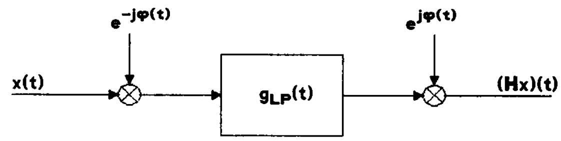

Fig. 2: Bandpass filter with time-varying center frequency.

This system consists of a demodulation, a linear, time-invariant lowpass filter with impulse response $g_{LP}(t)$ (transfer function $G_{LP}(f)$), and a modulation which is the inverse of the initial demodulation. The lowpass filter is assumed zero-phase so that it does not introduce a time delay. The overall system is a bandpass filter with time-varying center frequency $f_c(t) = \varphi'(t)/(2\pi)$, where $\varphi'(t)$ is the derivative of the instantaneous modulation phase $\varphi(t)$. With the impulse response given by $H(t,t') = e^{j\varphi(t)} g_{LP}(t-t') e^{-j\varphi(t')}$, the system's WD is obtained as

$$W_H(t,f) = |G_{LP}(f)|^2 \underset{f}{*} W_{e^{j\varphi}}(t,f), \tag{6.1}$$

where $W_{e^{j\varphi}}(t,f)$ is the WD of the signal $e^{j\varphi(t)}$ and $\underset{f}{*}$ denotes convolution with respect to the frequency variable f. This result is shown for an idealized lowpass filter and a sinusoidal center frequency function $f_c(t)$ in **Fig. 3**; it is consistent with the system's interpretation as a "bandpass filter with time-varying center frequency." If the lowpass filter is not too narrowband, then a (crude) approximation to (6.1) is

$$W_H(t,f) \approx |G_{LP}(f - f_c(t))|^2.$$

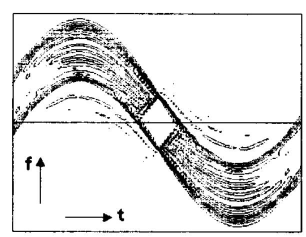

Fig. 3: WD of bandpass filter with time-varying center frequency.

This approximation is exact if and only if $e^{j\varphi(t)}$ is a linear frequency modulation (chirp signal), $f_c(t) = f_0 + at$.

7. OPTIMUM TIME-FREQUENCY DESIGN OF LTV SYSTEMS

Design procedure. The WD of a (normal) LTV system can be used for a TF design of LTV systems. Let $\tilde{W}(t,f) \in \mathbb{R}$ be a TF function expressing the desired TF weighting characteristic of the system **H** to be designed. In general, the "model" $\tilde{W}(t,f)$ will not be a valid WD of a system; therefore, we define the system **H** as the solution to the *synthesis problem* (cf. [7,10,11])

$$\mathbf{H}_{opt} \triangleq \arg\min_{\mathbf{H}} \| \tilde{W} - W_{\mathbf{H}} \| \qquad (7.1)$$

under the side constraint that $\mathbf{H}$ be a positive semidefinite system (i.e., $\mathbf{H}$ is normal with real-valued, nonnegative eigenvalues λ_k). This side constraint serves to minimize unwanted TF displacement effects of the system [9] and also assures that (7.1) has a unique solution. It can be shown that the optimum system is derived by the following procedure:

(i) Transform the model $\tilde{W}(t,f)$ according to

$$\mathfrak{q}(t,t') = \int_f \tilde{W}\left(\frac{t+t'}{2}, f\right) e^{j2\pi(t-t')f}\, df \,.$$

(ii) Solve the eigenproblem

$$\int_{t'} \mathfrak{q}(t,t')\, h_k(t')\, dt' = \mu_k\, h_k(t) \,.$$

Since $\mathfrak{q}(t,t') = \mathfrak{q}^*(t',t)$ for a real-valued model $\tilde{W}(t,f)$, the eigenvalues μ_k and eigenfunctions $h_k(t)$ are real-valued and orthonormal, respectively.

(iii) The eigenfunctions $h_k(t)$ with positive eigenvalues μ_k are the eigenfunctions of the optimum system; the associated system eigenvalues are the square roots of the μ_k. Hence, the optimum positive semidefinite system is (assuming that the index range $k=1,..,N_+$ contains all positive μ_k)

$$H_{opt}(t,t') = \sum_{k=1}^{N_+} \sqrt{\mu_k}\, h_k(t)\, h_k^*(t') \,.$$

We note that this design scheme is an extension of the *time-frequency projection filters* introduced in [7,10]. While TF projection filters are only capable of passing or suppressing signal components, the present design allows the implementation of a largely arbitrary TF weighting characteristic.

Simulation results. The design procedure outlined above is illustrated in **Fig. 4**. The model $\tilde{W}(t,f)$, shown in Fig. 4a, expresses a desire to pass only those signal components which lie inside a quadrangular pass region. This pass region is divided into two subregions with different weighting factors, corresponding to a desired amplification of the signal in the upper subregion. Fig. 4b depicts the WD of the optimum system $\mathbf{H}_{opt}$.

A filtering experiment using the system $\mathbf{H}_{opt}$ is shown in Figs. 4c,d. The input signal (Fig. 4c) consists of two chirp signals of which only one lies inside the pass region. From the output signal shown in Fig. 4d, it is seen that the system passes and partially amplifies this chirp signal as desired; the other chirp signal is duly suppressed.

8. CONCLUSION

The *input WD* and *output WD* allow a partial but convenient time-frequency (TF) analysis of linear, time-varying (LTV) systems. Loosely speaking, the input (output) WD describes the average TF distribution of the energy that is picked up (delivered) by the system at the input (output). These two TF representations coincide in the case of normal systems. The new concept also provides a basis for an optimum TF design of LTV systems, allowing the specification of a system's weighting characteristics in the TF plane.

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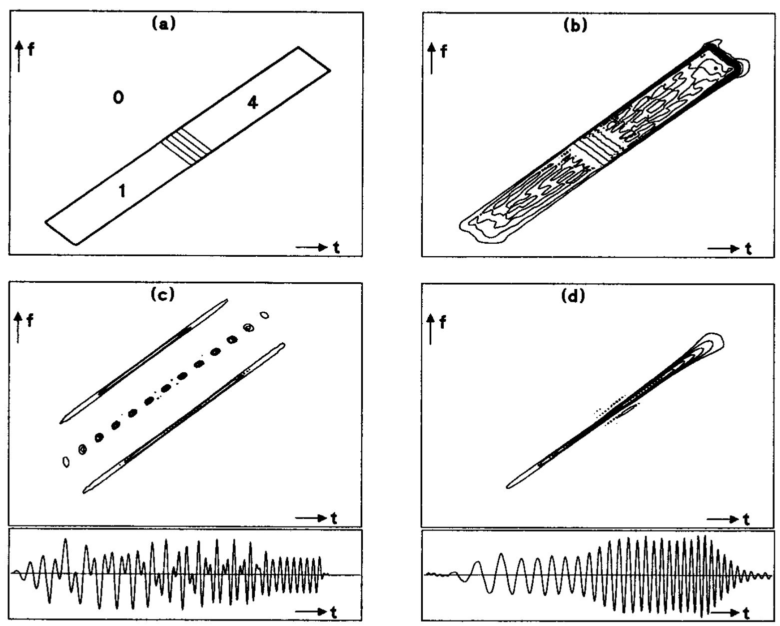

Fig. 4: Optimum TF design of an LTV system and application to signal filtering. (a) TF model $\tilde{W}(t,f)$; (b) WD of optimum system H_{opt}; (c) real part and WD of input signal $x(t)$; (d) real part and WD of output signal $(H_{opt}x)(t)$.