Duality and Classification of Bilinear Time-Frequency Signal Representations

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***Abstract*—Bilinear time-frequency representations (BTFR's) of signals, like Wigner distribution, ambiguity function, and spectrogram, are both an important signal theoretic concept and a powerful tool for signal analysis and processing. This paper discusses a fundamental duality principle of BTFR's and presents a systematic classification of BTFR's which is consistent with BTFR duality.**

BTFR's are first grouped into two basic domains, namely, the energy density domain (E-domain) with energetic interpretation, and the correlation domain (C-domain) with correlative interpretation. It is shown that these domains are related by a Fourier transform duality: to any BTFR, BTFR relation, or BTFR property of the E-domain, there corresponds a dual BTFR, BTFR relation, or BTFR property of the C-domain, and vice versa.

With this duality principle as a background, a classification of BTFR's is given. This classification is based on two dual shift-invariance properties and a self-dual scale-invariance property of BTFR's. The mathematical description of BTFR's by means of kernel functions is simplified inside the respective BTFR classes. It is shown that BTFR's which are both shift invariant and scale invariant can be represented as superpositions of generalized Wigner distributions (E-domain) or generalized ambiguity functions (C-domain).

I. Introduction

Bilinear time-frequency representations (BTFR's) of signals, like Wigner distribution, ambiguity function, or spectrogram, characterize signals via a joint function of time and frequency. The goal is to combine time-domain and frequency-domain analyses, such that both temporal and spectral characteristics of the signal under investigation are displayed simultaneously. BTFR's have been and are applied in a wide range of different fields, such as speech analysis, pattern recognition, optics, radar and sonar, seismic prospecting, the design of electro-acoustic transducers and surface-acoustic wave filters, the analysis of biological and medical signals, window design, chaotic systems, fault detection, etc.

In view of the practical importance of BTFR's, it is not surprising that BTFR's have been the subject of extensive theoretical study. Some excellent review papers on BTFR's are those of Cohen [1], Mecklenbräuker [2], Claasen and Mecklenbräuker [3], Janse and Kaizer [4], Flandrin [5], Janssen [6], Bastiaans [7], and Boudreaux-Bartels [8]. While a large body of knowledge and insight has thus been accumulated over the past years, it seems that a fundamental duality structure of BTFR's has not been given due attention so far. This can be explained by the fact that most authors concentrate on the "energetic" side of the field, i.e., on BTFR's with energetic interpretation like Wigner distribution (WD) or spectrogram. These BTFR's are mostly studied in the framework provided by Cohen's class of shift-invariant BTFR's [1], [9], [10]. The goal is to find a BTFR that describes the distribution of signal energy over the time-frequency plane and which is optimal with respect to certain criteria. These criteria are mostly prescribed desirable properties such as, for example, the shift-invariance property of Cohen's class, the marginal and finite-support properties, or validity of Moyal's formula. Accordingly, many papers define sets of such properties, derive the associated constraints on the Cohen-class kernel functions, investigate the compatibility or noncompatibility of certain properties, compare various definitions of energetic BTFR's with respect to the properties they satisfy, and argue that WD is optimal in that it meets the largest set of nice properties [1]–[6], [11].

Another group of papers treats BTFR's with correlative interpretation, in particular, various versions of ambiguity function (AF) [12]–[19]. Here, the goal is to obtain a joint time-frequency correlation function.

The fact that there exists a strict duality between the two domains of energetic and correlative BTFR's has only been partly recognized and utilized. As a special case, the Fourier transform duality of WD and AF is well known and is discussed in detail in [16]. A class of correlative BTFR's which is dual to the Cohen class of the energetic domain has recently been introduced in [19] and [20]. More traditionally, the Fourier transforms of Cohen-class BTFR's have been referred to as "characteristic functions" and have been used to study properties of Cohen-class BTFR's [10]. A similar approach is taken in [21] where the Fourier transform of smoothed WD versions is used to analyze the effect of smoothing. Here, the fact is utilized that a convolution operation in the energetic domain maps into a simple multiplication operation in the Fourier domain (i.e., the correlative domain). Thus the correlative domain is used merely for mathematical convenience or ease of interpretation but not as a separate domain of BTFR's with a (dual) interpretation of its own.

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The present paper is a study of BTFR's which explicitly takes into account the Fourier-transform duality of energetic and correlative BTFR's. Indeed, this study is based on those signal parameters on which the interpretation of BTFR's rests, namely, energy densities (instantaneous power and spectral energy density) on the one hand and (time-domain and frequency-domain) autocorrelation functions on the other. The duality of energy densities and autocorrelation functions forms the basis for the general duality of energetic and correlative BTFR's. A classification of BTFR's is then given which is consistent with BTFR duality. This classification is based on fundamental BTFR properties, namely, two dual properties of shift invariance (differently defined in the energetic and correlative BTFR domains), and a self-dual property of scale invariance. These properties are motivated by corresponding shift-invariance and scale-invariance properties of one-dimensional energy densities and correlation functions. In the energetic domain, the class of shift-invariant BTFR's is identical with the well-known Cohen class.

Mathematically, BTFR's are characterized in this paper by a standard description of bilinear signal representations using kernel functions [22]–[24]. It is shown that the restriction to the aforementioned BTFR classes also brings about a significant simplification of this description. This is so because the BTFR properties defining the classes induce characteristic mathematical structures of the BTFR kernel functions. Specifically, the class of shift-invariant BTFR's is characterized by convolution-type (energetic domain) and multiplication-type (correlative domain) kernels. The description of the classes of shift-scale-invariant BTFR's is once again simplified since the kernels here possess a characteristic product form. Thus the classification presented can also be motivated from the standpoint of mathematical structure.

The paper is organized as follows. Section II starts with a review of important BTFR's with energetic or correlative interpretation. For both energetic and correlative BTFR's, marginal properties are considered, and shift-invariance and scale-invariance properties are formulated which are consistent with, and motivated by, respective properties satisfied by the one-dimensional energy density and correlation functions. Based on the Fourier-transform duality of one-dimensional energy densities and correlations, the general concept of dual BTFR's is introduced in Section III, and the duality principle is extended to BTFR relations and BTFR properties. In Section IV, the dual classes of shift-invariant BTFR's are considered, and the simplification of mathematical description provided by the convolution/multiplication structure of the BTFR kernels is pointed out. Section V introduces the dual classes of shift-scale-invariant BTFR's. These are subclasses of the shift-invariant classes in which the kernels are once again simplified due to their "product structure." It is shown that any shift-scale-invariant BTFR is a linear combination of generalized Wigner distributions (energetic domain) or generalized ambiguity functions (correlative domain).

II. E-Domain and C-Domain

Although BTFR's may be quite different with respect to their properties, they can be grouped into two fundamental domains according to their interpretation. 1) The interpretation of the BTFR's of the "energetic" domain (called energy density domain or briefly E-domain in the following) is based on the instantaneous power $p_x(t)$ and the spectral energy density $P_x(f)$ defined by

$$p_x(t) = |x(t)|^2, \qquad P_x(f) = |X(f)|^2.$$

Here, $x(t)$ is the signal under investigation and $X(f)$ is its spectrum (note, however, that $p_x(t)$ and $P_x(f)$ are not a Fourier-transform pair). 2) The BTFR's of the "correlative" domain (subsequently called correlation domain or C-domain) are interpreted in terms of the time-domain and frequency-domain correlation functions

$$r_x(\tau) = \int_t x(t+\tau)\, x^*(t)\, dt$$

$$R_x(\upsilon) = \int_f X(f+\upsilon)\, X^*(f)\, df$$

(integrations are from $-\infty$ to ∞, and all signals are assumed to be square integrable). We note that the energy E_x of a signal $x(t)$ can be derived from the energy densities $p_x(t)$, $P_x(f)$ and correlations $r_x(\tau)$, $R_x(\upsilon)$ according to

$$E_x = \int_t p_x(t)\, dt = \int_f P_x(f)\, df$$

$$E_x = r_x(0) = R_x(0). \tag{2.1}$$

A. E-Domain

BTFR's of the E-domain seek to combine the one-dimensional energy densities $p_x(t)$, $P_x(f)$ into a two-dimensional, joint "time-frequency energy density"[1] $T_x^{(E)}(t, f)$. Some important E-domain BTFR's are the Wigner distribution (WD) [26]

$$\mathrm{WD}_x(t,f) = \int_\tau x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \tag{2.2}$$

and smoothed versions thereof, including the pseudo-WD [26], the smoothed pseudo-WD [27], [28], the exponential distribution [29], and the well-known and widely used spectrogram [3]. An E-domain BTFR which cannot be interpreted as a smoothed version of WD is the Rihaczek distribution (RD) [3]

$$\mathrm{RD}_x(t,f) = \int_\tau x(t+\tau)\, x^*(t)\, e^{-j2\pi f\tau}\, d\tau.$$

Both WD and RD are special cases of the family of generalized Wigner distributions (GWD's) [3], [6], [30]

[1]We stress, however, that a strict pointwise interpretation of an E-domain BTFR as time-frequency energy density is *a priori* impossible due to the fundamental resolution limitation imposed by the uncertainty principle [25].

$$\mathrm{GWD}_x^{(\alpha)}(t,f) = \int_\tau x[t + (\tfrac{1}{2} + \alpha)\tau]\, x^*[t - (\tfrac{1}{2} - \alpha)\tau] \cdot e^{-j2\pi f\tau}\, d\tau. \quad (2.3)$$

They are obtained from GWD with $\alpha = 0$ and $\alpha = 1/2$, respectively. The family of real-valued GWD's (RGWD's) is defined as the real part of GWD [3]

$$\mathrm{RGWD}_x^{(\alpha)}(t,f) = \tfrac{1}{2}[\mathrm{GWD}_x^{(\alpha)}(t,f) + \mathrm{GWD}_x^{(\alpha)*}(t,f)].$$

This is motivated by the real valuedness of the one-dimensional energy densities $p_x(t)$ and $P_x(f)$. Other E-domain BTFR's which possess a causality (anticausality) property are the distributions of Page [31] and Levin [32]. All these BTFR's are members of Cohen's class [3], [9], [10], which will be considered in Section IV.

B. C-Domain

The concept of C-domain BTFR's is the combination of the one-dimensional correlations $r_x(\tau)$, $R_x(\nu)$ into a two-dimensional, joint "time-frequency correlation" $T_x^{(C)}(\tau, \nu)$. Note that τ and ν denote a time and frequency lag, respectively. Important examples of C-domain BTFR's are the ambiguity function (symmetric definition) [3]

$$\mathrm{AF}_x(\tau, \nu) = \int_t x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi\nu t}\, dt \quad (2.4)$$

and the asymmetric ambiguity function (AAF) [12]

$$\mathrm{AAF}_x(\tau, \nu) = \int_t x(t + \tau)\, x^*(t) e^{-j2\pi\nu t}\, dt$$

which are obtained from the family of generalized ambiguity functions (GAF's)

$$\mathrm{GAF}_x^{(\alpha)}(\tau, \nu) = \int_t x[t + (\tfrac{1}{2} + \alpha)\tau]\, x^*[t - (\tfrac{1}{2} - \alpha)\tau] \cdot e^{-j2\pi\nu t}\, dt \quad (2.5)$$

by inserting, respectively, $\alpha = 0$ and $\alpha = 1/2$ for the parameter α. A modification of GAF, the family of Hermitian GAF's (HGAF's), is obtained by taking the Hermitian part of GAF

$$\mathrm{HGAF}_x^{(\alpha)}(\tau, \nu) = \tfrac{1}{2}[\mathrm{GAF}_x^{(\alpha)}(\tau, \nu) + \mathrm{GAF}_x^{(\alpha)*}(-\tau, -\nu)].$$

This is motivated by the hermiticity of the autocorrelation functions $r_x(\tau)$, $R_x(\nu)$.

C. Marginal Properties

We now review some "desirable properties" which an E-domain or C-domain BTFR could be required to satisfy. The E-domain marginal properties

$$\int_f T_x^{(E)}(t,f)\, df = p_x(t)$$

$$\int_t T_x^{(E)}(t,f)\, dt = P_x(f) \quad (2.6)$$

express a strict relation between an E-domain BTFR $T_x^{(E)}(t,f)$ on the one hand and the one-dimensional energy densities $p_x(t)$, $P_x(f)$ on the other. In the C-domain, it is natural to define the C-domain marginal properties as

$$T_x^{(C)}(0, \nu) = R_x(\nu), \qquad T_x^{(C)}(\tau, 0) = r_x(\tau). \quad (2.7)$$

Due to (2.1), the marginal properties imply

$$\int_t \int_f T_x^{(E)}(t,f)\, dt\, df = E_x, \qquad T_x^{(C)}(0, 0) = E_x. \quad (2.8)$$

D. Shift-Invariance Properties

To a time-frequency shift of the signal $x(t)$

$$\tilde{x}(t) = x(t - t_0) e^{j2\pi f_0 t} \quad (2.9)$$

the one-dimensional energy densities $p_x(t)$, $P_x(f)$ and one-dimensional correlations $r_x(\tau)$, $R_x(\nu)$ react by shifts and modulations, respectively,

$$p_{\tilde{x}}(t) = p_x(t - t_0), \qquad P_{\tilde{x}}(f) = P_x(f - f_0)$$

$$r_{\tilde{x}}(\tau) = r_x(\tau) e^{j2\pi f_0\tau}, \qquad R_{\tilde{x}}(\nu) = R_x(\nu) e^{-j2\pi t_0\nu}.$$

Combining time and frequency domains, it is then natural to define the shift-invariance properties of E-domain and C-domain as

$$T_{\tilde{x}}^{(E)}(t,f) = T_x^{(E)}(t - t_0, f - f_0) \quad (2.10)$$

$$T_{\tilde{x}}^{(C)}(\tau, \nu) = T_x^{(C)}(\tau, \nu) \exp[j2\pi(f_0\tau - t_0\nu)]. \quad (2.11)$$

While shift invariance certainly is a very natural property, there do exist BTFR's which are not shift-invariant (e.g., one can define a spectrogram with time-frequency-varying window). Note, also, that there exist BTFR's which satisfy the marginal properties but are not shift-invariant (see Section IV).

E. Scale-Invariance Properties

To a time-frequency scaling of the signal $x(t)$

$$\tilde{x}(t) = \sqrt{|a|}\, x(at), \qquad \tilde{X}(f) = \frac{1}{\sqrt{|a|}} X\left(\frac{f}{a}\right) \qquad (a \neq 0) \quad (2.12)$$

the energy densities $p_x(t)$, $P_x(f)$ and correlations $r_x(\tau)$, $R_x(\nu)$ react as follows:

$$p_{\tilde{x}}(t) = |a| p_x(at), \qquad P_{\tilde{x}}(f) = \frac{1}{|a|} P_x\left(\frac{f}{a}\right)$$

$$r_{\tilde{x}}(\tau) = r_x(a\tau), \qquad R_{\tilde{x}}(\nu) = R_x\left(\frac{\nu}{a}\right).$$

Again combining time and frequency domains, we define a scale-invariance property (identical in E-domain and C-domain) as

$$T_{\tilde{x}}^{(E)}(t,f) = T_x^{(E)}\left(at, \frac{f}{a}\right) \quad (2.13)$$

$$T_{\tilde{x}}^{(C)}(\tau, \nu) = T_x^{(C)}\left(a\tau, \frac{\nu}{a}\right). \quad (2.14)$$

Of course, there exist a large number of other desirable BTFR properties (an extensive list is given, e.g., in [33]). We concentrate on the above properties since these form the basis for our subsequent development.

The discussion of E-domain and C-domain BTFR's and BTFR properties given so far suggests the existence of a parallelism of E-domain and C-domain. In the next section, it will be shown that this parallelism takes the form of a strict duality based on the Fourier transform.

F. The Normal Forms of a BTFR

We conclude this section by introducing four equivalent characterizations of BTFR's using kernel functions. It is well known [22]-[24] that any bilinear signal representation can be written in a standard form which, specializing to E-domain BTFR's, reads

$$T_x^{(E)}(t,f) = \int_{t_1}\int_{t_2} u_T^{(E)}(t,f;t_1,t_2)\, x(t_1) \cdot x^*(t_2)\, dt_1\, dt_2. \tag{2.15}$$

Here, $u_T^{(E)}(t,f;t_1,t_2)$ is a four-dimensional kernel function which characterizes the specific BTFR $T^{(E)}$ and which can be interpreted as the BTFR's "impulse response" [24], [33].

From (2.15), the following four (strictly equivalent) "normal forms" can be derived by means of Fourier and coordinate transforms:

$$T_x^{(E)}(t,f) = \int_{t'}\int_{\tau'} k_{Tq}^{(E)}(t,f;t',\tau')\, q_x(t',\tau')\, dt'\, d\tau' \tag{2.16}$$

$$= \int_{f'}\int_{\upsilon'} k_{TQ}^{(E)}(t,f;f',\upsilon')\, Q_x(f',\upsilon')\, df'\, d\upsilon' \tag{2.17}$$

$$= \int_{t'}\int_{f'} k_{TW}^{(E)}(t,f;t',f')\, \mathrm{WD}_x(t',f')\, dt'\, df' \tag{2.18}$$

$$= \int_{\tau'}\int_{\upsilon'} k_{TA}^{(E)}(t,f;\tau',\upsilon')\, \mathrm{AF}_x(\tau',\upsilon')\, d\tau'\, d\upsilon'. \tag{2.19}$$

Here, $q_x(t,\tau)$ and $Q_x(f,\upsilon)$ are the time-domain and frequency-domain signal product defined as

$$q_x(t,\tau) = x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right)$$

$$Q_x(f,\upsilon) = X\left(f+\frac{\upsilon}{2}\right) X^*\left(f-\frac{\upsilon}{2}\right)$$

and WD and AF are defined by (2.2) and (2.4), respectively. The functions $q_x(t,\tau)$, $Q_x(f,\upsilon)$, $\mathrm{WD}_x(t,f)$, and $\mathrm{AF}_x(\tau,\upsilon)$ are related by Fourier transforms as illustrated in Fig. 1(a) [3]. Inverse Fourier transform relationships

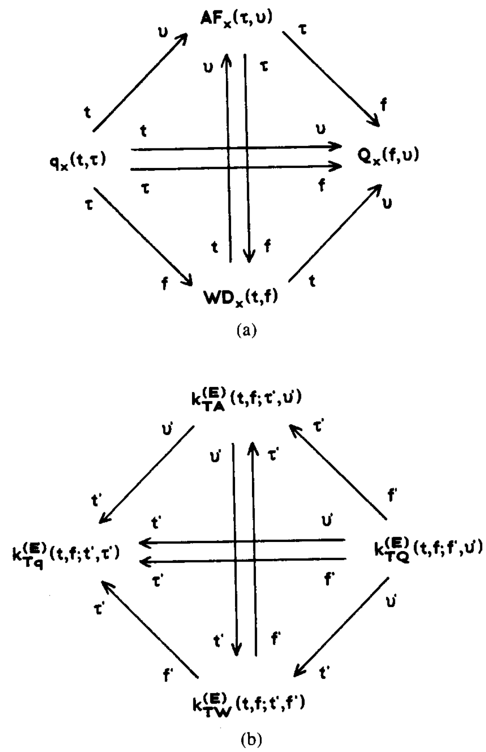

Fig. 1. Fourier transform relations (a) connecting $q_x(t,\tau)$, $Q_x(f,\upsilon)$, $\mathrm{WD}_x(t,f)$, $\mathrm{AF}_x(\tau,\upsilon)$, (b) connecting BTFR kernels $k_{Tq}^{(E)}(t,f;t',\tau')$, $k_{TQ}^{(E)}(t,f;f',\upsilon')$, $k_{TW}^{(E)}(t,f;t',f')$, $k_{TA}^{(E)}(t,f;\tau',\upsilon')$.

(see Fig. 1(b)) apply to the BTFR kernel functions $k_{Tq}^{(E)}(t,f;t',\tau')$, $k_{TQ}^{(E)}(t,f;f',\upsilon')$, $k_{TW}^{(E)}(t,f;t',f')$, and $k_{TA}^{(E)}(t,f;\tau',\upsilon')$, each of which characterizes the BTFR $T_x^{(E)}(t,f)$. For example, in the case of WD ($T^{(E)} = \mathrm{WD}$) the BTFR kernel functions are

$$k_{\mathrm{WD},q}(t,f;t',\tau') = \delta(t-t')e^{-j2\pi f\tau'}$$

$$k_{\mathrm{WD},Q}(t,f;f',\upsilon') = \delta(f-f')e^{j2\pi t\upsilon'}$$

$$k_{\mathrm{WD},W}(t,f;t',f') = \delta(t-t')\,\delta(f-f')$$

$$k_{\mathrm{WD},A}(t,f;\tau',\upsilon') = e^{j2\pi(t\upsilon'-f\tau')}.$$

Analogous normal forms (using BTFR kernel functions $k_{Tq}^{(C)}(\tau,\upsilon;t',\tau')$, etc.) exist for a C-domain BTFR $T_x^{(C)}(\tau,\upsilon)$. Indeed, the normal forms are not based on any time-frequency interpretation of BTFR's but only on their bilinear mathematical structure. While any one of the four normal forms suffices for BTFR characterization, we yet consider all four normal forms since some questions are best studied using a specific normal form.

III. Duality of E-Domain and C-Domain

Clearly, the previous section has suggested the existence of a parallelism of E-domain and C-domain. This parallelism, in fact, is a strict duality based on the Fourier transform; it is consistent with the well-known duality of one-dimensional energy densities $p_x(t)$, $P_x(f)$ and correlations $r_x(\tau)$, $R_x(v)$

$$R_x(v) = \underset{t \to v}{\mathfrak{F}} \, p_x(t), \qquad r_x(\tau) = \underset{f \to \tau}{\mathfrak{F}^{-1}} \, P_x(f). \tag{3.1}$$

Here, $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denote Fourier transformation and inverse Fourier transformation, respectively. Conceptually, an E-domain BTFR $T_x^{(E)}(t, f)$ combines the one-dimensional energy densities $p_x(t)$, $P_x(f)$; similarly, a C-domain BTFR $T_x^{(C)}(\tau, v)$ combines the one-dimensional correlations $r_x(\tau)$, $R_x(v)$. It is thus reasonable to combine the one-dimensional duality relations (3.1) into the following definition of BTFR duality [20]:

Definition: An E-domain BTFR $T_x^{(E)}(t, f)$ and a C-domain BTFR $T_x^{(C)}(\tau, v)$ are said to be dual if they are related as

$$T_x^{(C)}(\tau, v) = \underset{t \to v}{\mathfrak{F}} \, \underset{f \to \tau}{\mathfrak{F}^{-1}} \, T_x^{(E)}(t, f) = \int_t \int_f T_x^{(E)}(t, f) \exp\,[-j2\pi(vt - \tau f)] \, dt \, df. \tag{3.2}$$

Equation (3.2) expresses a unitary one-to-one mapping between dual BTFR's. Representing both $T_x^{(E)}(t, f)$ and $T_x^{(C)}(\tau, v)$ by the normal forms (2.16)-(2.19), it is easily seen that the kernels of dual BTFR's $T_x^{(E)}(t, f)$ and $T_x^{(C)}(\tau, v)$ are likewise related by the double Fourier transform (3.2), e.g.,

$$k_{Tq}^{(C)}(\tau, v; t', \tau') = \underset{t \to v}{\mathfrak{F}} \, \underset{f \to \tau}{\mathfrak{F}^{-1}} \, k_{Tq}^{(E)}(t, f; t', \tau'). \tag{3.3}$$

Due to the duality relation (3.2), we can find a dual C-domain BTFR $T_x^{(C)}(\tau, v)$ to any E-domain BTFR $T_x^{(E)}(t, f)$, and vice versa. Examples of pairs of dual BTFR's are presented in Table I.

TABLE I
EXAMPLES OF DUAL BTFR'S

E-domain	C-domain	Remarks
WD	AF	
RD	AAF	
GWD	GAF	Special cases: WD/AF ($\alpha = 0$) and RD/AAF ($\alpha = 1/2$)
RGWD	HGAF	Special case: WD/AF ($\alpha = 0$)

A. Dual BTFR Relations and Properties

The parallelism constituted by the duality of E-domain and C-domain BTFR's extends to BTFR relations and BTFR properties. The following results are easily derived using the duality definition (3.2).

1) If two E-domain BTFR's $T_x^{(E)}(t,f)$ and $\tilde{T}_x^{(E)}(t,f)$ are linearly related as

$$\tilde{T}_x^{(E)}(t, f) = \int_{t'} \int_{f'} \kappa^{(E)}(t, f; t', f') \, T_x^{(E)}(t', f') \, dt' \, df' \tag{3.4}$$

(with transformation kernel $\kappa^{(E)}(t, f; t', f')$ independent of $x(t)$), then a dual linear BTFR relation exists for the dual C-domain BTFR's $T_x^{(C)}(\tau, v)$ and $\tilde{T}_x^{(C)}(\tau, v)$

$$\tilde{T}_x^{(C)}(\tau, v) = \int_{\tau'} \int_{v'} \kappa^{(C)}(\tau, v; \tau', v') \, T_x^{(C)}(\tau', v') \, d\tau' \, dv' \tag{3.5}$$

where the E-domain and C-domain transformation kernels $\kappa^{(E)}(t, f; t', f')$ and $\kappa^{(C)}(\tau, v; \tau', v')$ are themselves related according to

$$\kappa^{(C)}(\tau, v; \tau', v') = \underset{t \to v}{\mathfrak{F}} \, \underset{f \to \tau}{\mathfrak{F}^{-1}} \, \underset{t' \to v'}{\mathfrak{F}^{-1}} \, \underset{f' \to \tau'}{\mathfrak{F}} \, \kappa^{(E)}(t, f; t', f'). \tag{3.6}$$

We illustrate this type of dual BTFR relations by means of a simple example. In the E-domain, WD and GWD can be shown to be related by

$$\mathrm{GWD}_x^{(\alpha)}(t, f) = \int_{t'} \int_{f'} \frac{1}{|\alpha|} \exp\left[j2\pi \frac{1}{\alpha}(t - t')(f - f')\right] \cdot \mathrm{WD}_x(t', f') \, dt' \, df' \tag{3.7}$$

which corresponds to the transformation kernel

$$\kappa^{(E)}(t, f; t', f') = \frac{1}{|\alpha|} \exp\left[j2\pi \frac{1}{\alpha}(t - t')(f - f')\right]. \tag{3.8}$$

The dual C-domain BTFR's associated with WD and GWD are AF and GAF, respectively, and the dual C-domain transformation kernel can be derived from (3.8) and (3.6) as

$$\kappa^{(C)}(\tau, v; \tau', v') = \delta(\tau - \tau') \, \delta(v - v') e^{j2\pi\alpha\tau v}.$$

Inserting into (3.5), the dual C-domain relation connecting AF and GAF is seen to be

$$\mathrm{GAF}_x^{(\alpha)}(\tau, v) = e^{j2\pi\alpha\tau v} \mathrm{AF}_x(\tau, v). \tag{3.9}$$

We see that the E-domain convolution (3.7) maps into a multiplication in the C-domain.

2) If an E-domain BTFR $T_x^{(E)}(t, f)$ is linearly related with a bilinear, one-dimensional, E-domain signal parameter $a_x^{(E)}(t)$ or $a_x^{(E)}(f)$

$$a_x^{(E)}(\sigma) = \int_{t'} \int_{f'} \kappa^{(E)}(\sigma; t', f') \, T_x^{(E)}(t', f') \, dt' \, df'$$

where σ stands for t or f, then a dual relation holds in the C-domain

$$a_x^{(C)}(\epsilon) = \int_{\tau'} \int_{v'} \kappa^{(C)}(\epsilon; \tau', v') \, T_x^{(C)}(\tau', v') \, d\tau' \, dv'.$$

Here, $\epsilon = v$ for $\sigma = t$ and $\epsilon = \tau$ for $\sigma = f$, $T_x^{(C)}(\tau, v)$ is dual to $T_x^{(E)}(t, f)$ in the sense of (3.2), $a_x^{(C)}(\epsilon)$ is a dual C-domain signal parameter given by

$$a_x^{(C)}(v) = \underset{t \to v}{\mathfrak{F}} \, a_x^{(E)}(t) \qquad \text{or} \qquad a_x^{(C)}(\tau) = \underset{f \to \tau}{\mathfrak{F}^{-1}} \, a_x^{(E)}(f)$$

and the C-domain transformation kernel is

$$\kappa^{(C)}(v; \tau', v') = \underset{t \to v}{\mathfrak{F}} \underset{t' \to v'}{\mathfrak{F}^{-1}} \underset{f' \to \tau'}{\mathfrak{F}} \kappa^{(E)}(t; t', f')$$

or

$$\kappa^{(C)}(\tau; \tau', v') = \underset{f \to \tau}{\mathfrak{F}^{-1}} \underset{t' \to v'}{\mathfrak{F}^{-1}} \underset{f' \to \tau'}{\mathfrak{F}} \kappa^{(E)}(f; t', f').$$

Prominent examples of this type of dual relations are the marginal properties (2.6), (2.7):

$$p_x(t) = \int_f T_x^{(E)}(t, f)\, df \leftrightarrow R_x(v) = T_x^{(C)}(0, v)$$

and

$$P_x(f) = \int_t T_x^{(E)}(t, f)\, dt \leftrightarrow r_x(\tau) = T_x^{(C)}(\tau, 0)$$

where the symbol $\leftrightarrow$ denotes the duality of E-domain and C-domain. We have thus obtained the following result: if an E-domain BTFR satisfies the E-domain marginal properties, then the dual C-domain BTFR satisfies the C-domain marginal properties.

3) If an E-domain BTFR $T_x^{(E)}(t, f)$ is linearly related to a constant bilinear signal parameter a_x,

$$a_x = \int_{t'} \int_{f'} \kappa^{(E)}(t', f')\, T_x^{(E)}(t', f')\, dt'\, df'$$

then a dual C-domain relation involving the dual C-domain BTFR $T_x^{(C)}(\tau, v)$ is

$$a_x = \int_{\tau'} \int_{v'} \kappa^{(C)}(\tau', v')\, T_x^{(C)}(\tau', v')\, d\tau'\, dv'$$

with the dual C-domain transformation kernel given by

$$\kappa^{(C)}(\tau', v') = \underset{t' \to v'}{\mathfrak{F}^{-1}} \underset{f' \to \tau'}{\mathfrak{F}} \kappa^{(E)}(t', f').$$

Examples of this kind of dual relations are the properties (2.8)

$$E_x = \int_t \int_f T_x^{(E)}(t, f)\, dt\, df \leftrightarrow E_x = T_x^{(C)}(0, 0) \quad (3.10)$$

and, as a generalization of (3.10), the following "moment properties":

$$m_x^{(n)} = \int_t \int_f t^n T_x^{(E)}(t, f)\, dt\, df \leftrightarrow m_x^{(n)} = \left(-\frac{1}{2\pi j}\right)^n \frac{\partial^n}{\partial v^n} T_x^{(C)}(\tau, v)\bigg|_{\tau = v = 0}$$

and

$$M_x^{(n)} = \int_t \int_f f^n T_x^{(E)}(t, f)\, dt\, df \leftrightarrow M_x^{(n)} = \left(\frac{1}{2\pi j}\right)^n \frac{\partial^n}{\partial \tau^n} T_x^{(C)}(\tau, v)\bigg|_{\tau = v = 0}$$

where the nth order time and frequency moments are defined as

$$m_x^{(n)} = \int_t t^n |x(t)|^2\, dt, \qquad M_x^{(n)} = \int_f f^n |X(f)|^2\, df.$$

4) If an E-domain BTFR $T_x^{(E)}(t, f)$ reacts to a given linear signal transformation

$$\tilde{x}(t) = \int_{t'} h(t; t')\, x(t')\, dt'$$

by a linear transformation according to

$$T_{\tilde{x}}^{(E)}(t, f) = \int_{t'} \int_{f'} \kappa_h^{(E)}(t, f; t', f')\, T_x^{(E)}(t', f')\, dt'\, df'$$

then the dual C-domain BTFR $T_x^{(C)}(\tau, v)$ reacts to the same signal transformation by a dual transformation

$$T_{\tilde{x}}^{(C)}(\tau, v) = \int_{\tau'} \int_{v'} \kappa_h^{(C)}(\tau, v; \tau', v')\, T_x^{(C)}(\tau', v')\, d\tau'\, dv'. \quad (3.11)$$

The kernel of this dual C-domain transformation is given by (3.6)

$$\kappa_h^{(C)}(\tau, v; \tau', v') = \underset{t \to v}{\mathfrak{F}} \underset{f \to \tau}{\mathfrak{F}^{-1}} \underset{t' \to v'}{\mathfrak{F}^{-1}} \underset{f' \to \tau'}{\mathfrak{F}} \kappa_h^{(E)}(t, f; t', f'). \quad (3.12)$$

As an example, consider the E-domain shift-invariance property (2.9), (2.10). Here, the E-domain transformation kernel is

$$\kappa_h^{(E)}(t, f; t', f') = \delta((t - t_0) - t')\, \delta((f - f_0) - f')$$

and its C-domain dual according to (3.12) is found as

$$\kappa_h^{(C)}(\tau, v; \tau', v') = \delta(\tau - \tau')\, \delta(v - v')\, \exp\,[j2\pi(f_0\tau - t_0 v)].$$

Inserting into (3.11), we obtain as dual C-domain property the C-domain shift-invariance property (2.9), (2.11). E-domain and C-domain shift-invariance properties are thus recognized as dual properties. Specifically, we have obtained the following result: if an E-domain BTFR satisfies the E-domain shift-invariance property, then the dual C-domain BTFR satisfies the C-domain shift-invariance property. An analogous result holds for the (self-dual) scale-invariance properties (2.12), (2.13) and (2.12), (2.14).

The above results show that, just as there are pairs of dual BTFR's, there are also pairs of dual BTFR relations and BTFR properties. This duality structure simplifies BTFR analysis: if we know that, e.g., a given E-domain BTFR $T_x^{(E)}(t, f)$ satisfies a certain relation or property, then we know at once that the dual C-domain BTFR $T_x^{(C)}(\tau, v)$ satisfies the associated dual relation or property. For example, from the E-domain shift invariance of WD there immediately follows the C-domain shift invariance of AF since WD and AF are dual BTFR's and E-domain and C-domain shift invariance are dual BTFR properties.

IV. The Shift-Invariant Classes

If the definition of BTFR classes is based on certain BTFR properties, then the duality of BTFR properties discussed in the previous section naturally leads to a corresponding duality of E-domain and C-domain BTFR classes. Of all properties which may be considered desirable for E-domain and C-domain BTFR's, the dual shift-invariance properties (2.10), (2.11) are perhaps the most important and fundamental. Let us, therefore, define the class $\mathcal{C}_E$ of E-domain shift-invariant BTFR's as the totality of E-domain BTFR's $T_x^{(E)}(t, f)$ satisfying the E-domain shift-invariance property (2.10),

$$\tilde{x}(t) = x(t - t_0)e^{j2\pi f_0 t} \Rightarrow$$
$$T_{\tilde{x}}^{(E)}(t, f) = T_x^{(E)}(t - t_0, f - f_0). \tag{4.1}$$

Similarly, we define the class $\mathcal{C}_C$ of C-domain shift-invariant BTFR's as the totality of C-domain BTFR's $T_x^{(C)}(\tau, \nu)$ satisfying the C-domain shift-invariance property (2.11),

$$\tilde{x}(t) = x(t - t_0)e^{j2\pi f_0 t} \Rightarrow$$
$$T_{\tilde{x}}^{(C)}(\tau, \nu) = T_x^{(C)}(\tau, \nu)e^{j2\pi(f_0\tau - t_0\nu)}.$$

Since the underlying shift-invariance properties are dual, the classes $\mathcal{C}_E$ and $\mathcal{C}_C$ are themselves dual in the following sense: if an E-domain BTFR $T_x^{(E)}(t, f)$ is an element of $\mathcal{C}_E$, then the dual C-domain BTFR $T_x^{(C)}(\tau, \nu)$ is an element of $\mathcal{C}_C$, and vice versa.

A. Description of Shift-Invariant BTFR's

The shift-invariance properties impose characteristic structures on the BTFR kernels or, equivalently, on the normal forms (2.16)–(2.19). Indeed, the left-hand side of the E-domain shift-invariance equation (4.1) can be expressed as

$$T_{\tilde{x}}^{(E)}(t, f) = \int_{t'}\int_{f'} k_{TW}^{(E)}(t, f; t', f')\, \mathrm{WD}_{\tilde{x}}(t', f')\, dt'\, df'$$
$$= \int_{t'}\int_{f'} k_{TW}^{(E)}(t, f; t', f') \cdot \mathrm{WD}_x(t' - t_0, f' - f_0)\, dt'\, df'$$
$$= \int_{t'}\int_{f'} k_{TW}^{(E)}(t, f; t' + t_0, f' + f_0) \cdot \mathrm{WD}_x(t', f')\, dt'\, df'$$

where we have used the third normal form (2.18) and the fact that WD is itself E-domain shift invariant. Similarly, the right-hand side of (4.1) is

$$T_x^{(E)}(t - t_0, f - f_0) = \int_{t'}\int_{f'} k_{TW}^{(E)}(t - t_0, f - f_0; t', f') \cdot \mathrm{WD}_x(t', f')\, dt'\, df'.$$

By comparison, we obtain

$$k_{TW}^{(E)}(t, f; t' + t_0, f' + f_0) = k_{TW}^{(E)}(t - t_0, f - f_0; t', f') \qquad \text{for all } t_0, f_0$$

as a necessary and sufficient condition for E-domain shift invariance of $T_x^{(E)}(t, f)$. This condition is satisfied if and only if the BTFR kernel $k_{TW}^{(E)}(t, f; t', f')$ is convolution type, i.e.,

$$k_{TW}^{(E)}(t, f; t', f') = f_{TW}(t - t', f - f') \tag{4.2}$$

with an arbitrary two-dimensional function $f_{TW}(t, f)$. Inserting into the third normal form (2.18) then gives the important relation

$$T_x^{(E)}(t, f) = \int_{t'}\int_{f'} f_{TW}(t - t', f - f')\, \mathrm{WD}_x(t', f')\, dt'\, df' \tag{4.3}$$

which shows that an E-domain shift-invariant BTFR can be derived from WD by a convolution. What is more, (4.3) shows that the class $\mathcal{C}_E$ of E-domain shift-invariant BTFR's is identical with the well-known Cohen class of E-domain BTFR's [1], [3], [9], [10]. Cohen's class is thus seen to be defined axiomatically by the property of E-domain shift invariance [5], [34].

Invoking the Fourier transform relations (see Fig. 1(b)) connecting the four BTFR kernels, the remaining three BTFR kernels can be derived from (4.2), and we finally obtain the normal forms of an E-domain shift-invariant BTFR as

$$T_x^{(E)}(t, f)$$
$$= \int_{t'}\int_{\tau'} f_{Tq}(t - t', \tau')e^{-j2\pi f\tau'} q_x(t', \tau')\, dt'\, d\tau'$$
$$= \int_{f'}\int_{\nu'} f_{TQ}(f - f', \nu')e^{j2\pi t\nu'} Q_x(f', \nu')\, df'\, d\nu'$$
$$= \int_{t'}\int_{f'} f_{TW}(t - t', f - f')\, \mathrm{WD}_x(t', f')\, dt'\, df'$$
$$= \int_{\tau'}\int_{\nu'} f_{TA}(\tau', \nu') \exp\,[j2\pi(t\nu' - f\tau')] \cdot \mathrm{AF}_x(\tau', \nu')\, d\tau'\, d\nu'$$

where the four two-dimensional kernel functions $f_{Tq}(t, \tau)$, $f_{TQ}(f, \nu)$, $f_{TW}(t, f)$, and $f_{TA}(\tau, \nu)$ are related by Fourier transforms according to Fig. 2 (cf. Fig. 1(a)). Any one of these kernel functions provides a complete characterization of the shift-invariant BTFR $T_x^{(E)}(t, f)$. The kernel $f_{TA}(\tau, \nu)$ is usually considered in the literature on the

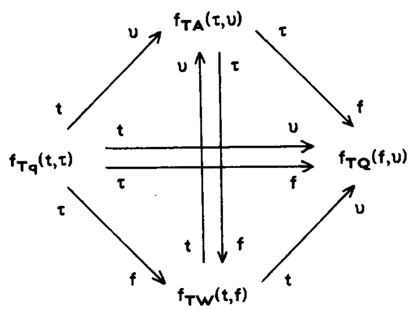

Fig. 2. Fourier transform relations connecting kernels $f_{Tq}(t, \tau)$, $f_{TQ}(f, \upsilon)$, $f_{TW}(t, f)$, $f_{TA}(\tau, \upsilon)$ of shift-invariant BTFR's.

Cohen class; in particular, the validity of desirable BTFR properties has been expressed by means of constraints on $f_{TA}(\tau, \upsilon)$. For example, it is well known that $T_x^{(E)}(t, f)$ satisfies the E-domain marginal properties (2.6) if and only if $f_{TA}(0, \upsilon) = f_{TA}(\tau, 0) = 1$ (note, however, that there exist BTFR's which satisfy the E-domain marginal properties but are not E-domain shift-invariant, i.e., are not contained in Cohen's class [20]).

The normal forms of a C-domain shift-invariant BTFR $T_x^{(C)}(\tau, \upsilon)$ can be derived in a similar way or, alternatively, by using the duality relations (see (3.3)) for the BTFR kernels. We obtain

$$\begin{aligned} T_x^{(C)}(\tau, \upsilon) &= f_{TA}(\tau, \upsilon) \int_{t'} e^{-j2\pi\upsilon t'} q_x(t', \tau)\, dt' \\ &= f_{TA}(\tau, \upsilon) \int_{f'} e^{j2\pi\tau f'} Q_x(f', \upsilon)\, df' \\ &= f_{TA}(\tau, \upsilon) \int_{t'} \int_{f'} \exp\,[j2\pi(\tau f' - \upsilon t')] \\ &\quad \cdot \mathrm{WD}_x(t', f')\, dt'\, df' \\ &= f_{TA}(\tau, \upsilon)\, \mathrm{AF}_x(\tau, \upsilon). \end{aligned} \tag{4.4}$$

The first three expressions are simply restatements of the fourth. We see, in particular, that a C-domain shift-invariant BTFR is derived from AF by a multiplication. The class $\mathcal{C}_C$ of C-domain shift-invariant BTFR's has been considered previously in [19] and [20].[2]

It should be noted that any of the four kernels $f_{Tq}(t, \tau)$, $f_{TQ}(f, \upsilon)$, $f_{TW}(t, f)$, $f_{TA}(\tau, \upsilon)$ characterizes both the E-domain shift-invariant BTFR $T_x^{(E)}(t, f)$ and its C-domain dual $T_x^{(C)}(\tau, \upsilon)$. If $T_x^{(E)}(t, f) \in \mathcal{C}_E$ and $T_x^{(C)}(\tau, \upsilon) \in \mathcal{C}_C$ and if $T_x^{(E)}(t, f) \leftrightarrow T_x^{(C)}(\tau, \upsilon)$, then

$$T_x^{(E)}(t, f) = \int_{t'} \int_{f'} f_{TW}(t - t', f - f')\, \mathrm{WD}_x(t', f')\, dt'\, df'$$

$$T_x^{(C)}(\tau, \upsilon) = f_{TA}(\tau, \upsilon)\, \mathrm{AF}_x(\tau, \upsilon)$$

where $f_{TW}(t, f)$ and $f_{TA}(\tau, \upsilon)$ are related by a double Fourier transform (see Fig. 2) exactly as are dual BTFR's (cf. (3.2))

$$f_{TA}(\tau, \upsilon) = \underset{t \to \upsilon}{\mathfrak{F}}\ \underset{f \to \tau}{\mathfrak{F}^{-1}} f_{TW}(t, f).$$

In particular, if $T_x^{(E)}(t, f)$ is a smoothed version of WD like pseudo-WD or spectrogram, then $f_{TW}(t, f)$ will have low-pass character, i.e., its Fourier transform $f_{TA}(\tau, \upsilon)$ will generally tend to zero for large values of $|\tau|$ and/or $|\upsilon|$ [21], [29].

B. BTFR Relations

If two E-domain shift-invariant BTFR's $T_x^{(E)}(t, f)$ and $\tilde{T}_x^{(E)}(t, f)$ are linearly related as in (3.4), then it is easily shown that this relation must itself be a convolution

$$\begin{aligned} \tilde{T}_x^{(E)}(t, f) = \int_{t'} \int_{f'} \varphi^{(E)}(t - t', f - f') \\ \cdot T_x^{(E)}(t', f')\, dt'\, df'. \end{aligned} \tag{4.5}$$

In the C-domain, the dual relation (3.5) is a multiplication:

$$\tilde{T}_x^{(C)}(\tau, \upsilon) = \varphi^{(C)}(\tau, \upsilon)\, T_x^{(C)}(\tau, \upsilon). \tag{4.6}$$

The E-domain and C-domain transformation kernels $\varphi^{(E)}(t, f)$ and $\varphi^{(C)}(\tau, \upsilon)$ are again related by the usual double Fourier transform

$$\varphi^{(C)}(\tau, \upsilon) = \underset{t \to \upsilon}{\mathfrak{F}}\ \underset{f \to \tau}{\mathfrak{F}^{-1}} \varphi^{(E)}(t, f).$$

We also note that (4.5) and (4.6) entail the kernel relations

$$f_{\tilde{T}W}(t, f) = \int_{t'} \int_{f'} \varphi^{(E)}(t - t', f - f')\, f_{TW}(t', f')\, dt'\, df'$$

$$f_{\tilde{T}A}(\tau, \upsilon) = \varphi^{(C)}(\tau, \upsilon)\, f_{TA}(\tau, \upsilon).$$

As an example, we reconsider the relations connecting WD and GWD in the E-domain and the dual BTFR's AF and GAF in the C-domain. With (3.7) and (3.9), the transformation kernels are seen to be

$$\varphi^{(E)}(t, f) = \frac{1}{|\alpha|} e^{j2\pi(1/\alpha)tf} \leftrightarrow \varphi^{(C)}(\tau, \upsilon) = e^{j2\pi\alpha\tau\upsilon}.$$

V. The Shift-Scale-Invariant Classes

In the previous section, it has been shown that the dual shift-invariance properties entail a characteristic convolution/multiplication structure of BTFR kernels and BTFR relations. We shall now see that the additional assumption of scale invariance (2.13), (2.14) leads to a further characteristic structural constraint. We define the class $\mathfrak{D}_E$ of E-domain shift-scale-invariant BTFR's as the totality of E-domain shift-invariant BTFR's satisfying the scale-invariance property (2.13)

$$\tilde{x}(t) = \sqrt{|a|}\, x(at) \Rightarrow T_{\tilde{x}}^{(E)}(t, f) = T_x^{(E)}\left(at, \frac{f}{a}\right).$$

[2] In [19], this class has been given the name "generalized ambiguity function," which, in the present paper, is used for the C-domain BTFR family $\mathrm{GAF}_x^{(\alpha)}(\tau, \upsilon)$ defined in (2.5).

Similarly, we define the class $\mathfrak{D}_C$ of C-domain shift-scale-invariant BTFR's as the totality of C-domain shift invariant BTFR's satisfying the scale-invariance property (2.14)

$$\tilde{x}(t) = \sqrt{|a|}x(at) \Rightarrow T_{\tilde{x}}^{(C)}(\tau, \upsilon) = T_x^{(C)}\left(a\tau, \frac{\upsilon}{a}\right). \tag{5.1}$$

We see that $\mathfrak{D}_E$ and $\mathfrak{D}_C$ are subclasses of $\mathcal{C}_E$ and $\mathcal{C}_C$, respectively. Just as $\mathcal{C}_E$ and $\mathcal{C}_C$, $\mathfrak{D}_E$ and $\mathfrak{D}_C$ are dual classes: if $T_x^{(E)}(t, f)$ is an element of $\mathfrak{D}_E$, then the dual C-domain BTFR $T_x^{(C)}(\tau, \upsilon)$ is an element of $\mathfrak{D}_C$, and vice versa.

A. Description of Shift-Scale-Invariant BTFR's

Using (4.4) and the fact that AF is itself scale invariant, the left-hand side of the C-domain scale-invariance relation (5.1) can be written as

$$T_{\tilde{x}}^{(C)}(\tau, \upsilon) = f_{TA}(\tau, \upsilon)\ \mathrm{AF}_x\left(a\tau, \frac{\upsilon}{a}\right).$$

Similarly, the right-hand side of (5.1) is

$$T_x^{(C)}\left(a\tau, \frac{\upsilon}{a}\right) = f_{TA}\left(a\tau, \frac{\upsilon}{a}\right) \mathrm{AF}_x\left(a\tau, \frac{\upsilon}{a}\right).$$

By comparison, we obtain

$$f_{TA}(\tau, \upsilon) = f_{TA}\left(a\tau, \frac{\upsilon}{a}\right) \qquad \text{for all } a \neq 0$$

as a necessary and sufficient condition for scale invariance. This condition is satisfied if and only if $f_{TA}(\tau, \upsilon)$ is of the product type

$$f_{TA}(\tau, \upsilon) = G_T(\tau\upsilon) \tag{5.2}$$

with an arbitrary one-dimensional function $G_T(\xi)$. Using the Fourier-transform relationships of Fig. 2, the remaining three kernels $f_{Tq}(t, \tau)$, $f_{TQ}(f, \upsilon)$, and $f_{TW}(t, f)$ can be derived from (5.2); we obtain

$$f_{Tq}(t, \tau) = \int_\alpha g_T(\alpha)\ \delta(t - \alpha\tau)\ d\alpha = \frac{1}{|\tau|}\ g_T\left(\frac{t}{\tau}\right)$$

$$f_{TQ}(f, \upsilon) = \int_\alpha g_T(\alpha)\ \delta(f + \alpha\upsilon)\ d\alpha = \frac{1}{|\upsilon|}\ g_T\left(-\frac{f}{\upsilon}\right)$$

$$f_{TW}(t, f) = \int_\alpha g_T(\alpha)\ \frac{1}{|\alpha|}\ e^{-j2\pi(1/\alpha)tf}\ d\alpha \tag{5.3}$$

$$f_{TA}(\tau, \upsilon) = \int_\alpha g_T(\alpha) e^{-j2\pi\alpha\tau\upsilon}\ d\alpha = G_T(\tau\upsilon) \tag{5.4}$$

where

$$G_T(\xi) = \underset{\alpha \to \xi}{\mathfrak{F}}\ g_T(\alpha).$$

It follows from (5.3) that $f_{TW}(t, f)$, too, is a product-type function, i.e., only depends on the product of t and f.

A prototype example of shift-scale-invariant BTFR's is given by the dual families of GWD (2.3) and GAF (2.5) which, for a specific parameter $\alpha = \alpha_0$, are characterized by

$$f_{\mathrm{GWD},W}^{(\alpha_0)}(t, f) = \frac{1}{|\alpha_0|}\ e^{j2\pi(1/\alpha_0)tf}$$

or

$$f_{\mathrm{GWD},A}^{(\alpha_0)}(\tau, \upsilon) = e^{j2\pi\alpha_0\tau\upsilon}. \tag{5.5}$$

(Note that we name the kernel functions f, g, and G after the E-domain BTFR even though they are characteristic of the dual C-domain BTFR as well.) Comparing with (5.3) and (5.4), we obtain

$$g_{\mathrm{GWD}}^{(\alpha_0)}(\alpha) = \delta(\alpha + \alpha_0), \qquad G_{\mathrm{GWD}}^{(\alpha_0)}(\xi) = e^{j2\pi\alpha_0\xi}.$$

The dual families of RGWD and HGAF, too, are members of $\mathfrak{D}_E$ and $\mathfrak{D}_C$, respectively; they are characterized by

$$g_{\mathrm{RGWD}}^{(\alpha_0)}(\alpha) = \tfrac{1}{2}[\delta(\alpha + \alpha_0) + \delta(\alpha - \alpha_0)]$$

$$G_{\mathrm{RGWD}}^{(\alpha_0)}(\xi) = \tfrac{1}{2}[e^{j2\pi\alpha_0\xi} + e^{-j2\pi\alpha_0\xi}] = \cos(2\pi\alpha_0\xi).$$

We note that conventional smoothed versions of WD, like pseudo-WD or spectrogram, are not scale invariant. However, the exponential distribution (ED) [29] is scale invariant; it is characterized by

$$g_{\mathrm{ED}}(\alpha) = \sqrt{\pi\sigma}\ \exp(-\pi^2\sigma\alpha^2),$$

$$G_{\mathrm{ED}}(\xi) = \exp(-\xi^2/\sigma) \qquad (\sigma > 0).$$

The particular structure of shift-scale-invariant BTFR's can be given another interpretation. Let $T_x^{(E)}(t, f)$ and $T_x^{(C)}(\tau, \upsilon)$ be a pair of shift-scale-invariant BTFR's characterized by the kernel $f_{TA}(\tau, \upsilon)$. With (5.5), (5.4) can be written as

$$f_{TA}(\tau, \upsilon) = \int_\alpha g_T(-\alpha)\ f_{\mathrm{GWD},A}^{(\alpha)}(\tau, \upsilon)\ d\alpha$$

whence it follows that $T_x^{(E)}(t, f)$ and $T_x^{(C)}(\tau, \upsilon)$ can be expressed as

$$T_x^{(E)}(t, f) = \int_\alpha g_T(-\alpha)\ \mathrm{GWD}_x^{(\alpha)}(t, f)\ d\alpha \leftrightarrow$$

$$T_x^{(C)}(\tau, \upsilon) = \int_\alpha g_T(-\alpha)\ \mathrm{GAF}_x^{(\alpha)}(\tau, \upsilon)\ d\alpha.$$

We have thus obtained the following result: any E-domain (C-domain) shift-scale-invariant BTFR is a linear combination of GWD's (GAF's). In both cases, the weighting function is $g_T(-\alpha)$. We conclude that GWD and GAF are indeed the basic elements of $\mathfrak{D}_E$ and $\mathfrak{D}_C$, respectively.

Apart from shift and scale invariance, a number of other desirable BTFR properties are satisfied by BTFR's of $\mathfrak{D}_E$ and $\mathfrak{D}_C$ provided that the weighting function $g_T(\alpha)$ satisfies some few constraints [33]. Specifically, let us assume that $g_T(\alpha)$ is real, even, normalized such that $\int g_T(\alpha)\, d\alpha = 1$, and zero for $|\alpha| > 1/2$. It can then be shown that $T_x^{(E)}(t, f)$ and $T_x^{(C)}(\tau, v)$ are real-valued and Hermitian, respectively; furthermore, they will satisfy the marginal, moment, finite-support, and instantaneous-frequency/group-delay properties [3], and they will be invariant with respect to Fourier transformation of the signal (i.e., if $\tilde{x}(t) = (\mathfrak{F}x)(t)$, then $T_{\tilde{x}}^{(E)}(t, f) = T_x^{(E)}(-f, t)$ and $T_{\tilde{x}}^{(C)}(\tau, v) = T_x^{(C)}(-v, \tau)$).

B. BTFR Relations

The transformation kernels $\varphi^{(E)}(t, f)$ and $\varphi^{(C)}(\tau, v)$ describing the relations of shift-invariant BTFR's (see (4.5), (4.6)) can be shown to have the following product forms if the BTFR's are also scale invariant:

$$\varphi^{(E)}(t, f) = \int_\alpha \gamma(\alpha) \frac{1}{|\alpha|} e^{-j2\pi(1/\alpha)tf}\, d\alpha$$

$$\varphi^{(C)}(\tau, v) = \int_\alpha \gamma(\alpha) e^{-j2\pi\alpha\tau v}\, d\alpha = \Gamma(\tau v) \qquad (5.6)$$

with

$$\Gamma(\xi) = \mathfrak{F}_{\alpha \to \xi}\, \gamma(\alpha).$$

According to (4.6) and (5.6), a linear relation of two BTFR's of $\mathfrak{D}_C$ has the simple form

$$\tilde{T}_x^{(C)}(\tau, v) = \Gamma(\tau v)\, T_x^{(C)}(\tau, v).$$

We finally note that a linear BTFR relation in $\mathfrak{D}_E$ or $\mathfrak{D}_C$ entails the kernel relations

$$g_{\tilde{T}}(\alpha) = \int_{\alpha'} \gamma(\alpha - \alpha')\, g_T(\alpha')\, d\alpha'$$

and

$$G_{\tilde{T}}(\xi) = \Gamma(\xi)\, G_T(\xi).$$

VI. Conclusion

Fig. 3 illustrates the classification scheme of bilinear time-frequency signal representations (BTFR's) developed in the previous sections. In the following, we briefly summarize the different steps of this classification.

1) A first and fundamental distinction is between E-domain and C-domain BTFR's. E-domain BTFR's are interpreted in terms of time-domain and frequency-domain energy densities $p_x(t)$ and $P_x(f)$ while C-domain BTFR's are interpreted in terms of time-domain and frequency-domain correlations $r_x(\tau)$ and $R_x(v)$. The affiliation of a BTFR to the E-domain or the C-domain is a matter of interpretation and thus not specified by a mathematically strict criterion. There exists, however, a mathematically strict Fourier transform duality of E-domain BTFR's and C-domain BTFR's. This duality is consistent with the Fourier transform duality $p_x(t) \leftrightarrow R_x(v)$, $P_x(f) \leftrightarrow r_x(\tau)$ of energy densities and correlations, and it extends to BTFR relations, BTFR properties, and BTFR classes.

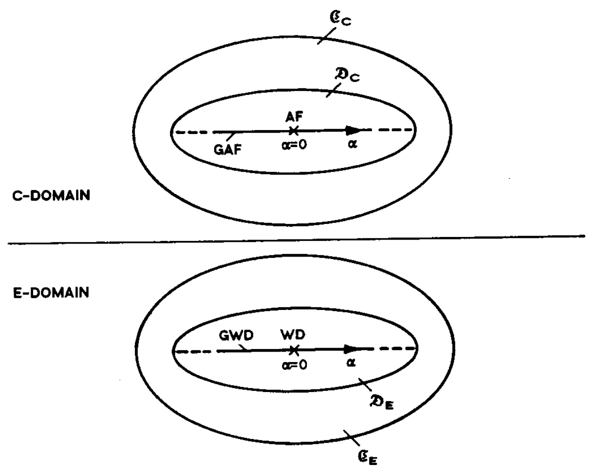

Fig. 3. A BTFR classification scheme.

2) Inside the E-domain and C-domain, the dual classes $\mathfrak{C}_E$ and $\mathfrak{C}_C$ of E-domain and C-domain shift-invariant BTFR's are defined by two dual shift-invariance properties. These shift-invariance properties are consistent with the shift-invariance properties of energy densities $p_x(t)$, $P_x(f)$ and correlations $r_x(\tau)$, $R_x(v)$, respectively. The class $\mathfrak{C}_E$ of E-domain shift-invariant BTFR's is identical with the well-known Cohen class. The mathematical description of shift-invariant BTFR's is simplified as compared to the general case: while the most general description of a BTFR is in terms of a four-dimensional kernel function, a shift-invariant BTFR is characterized by a two-dimensional kernel function. If two E-domain (C-domain) BTFR's are linearly related, then this relation is a convolution (multiplication).

3) Inside the classes $\mathfrak{C}_E$ and $\mathfrak{C}_C$ of shift-invariant BTFR's, the dual subclasses $\mathfrak{D}_E \subset \mathfrak{C}_E$ and $\mathfrak{D}_C \subset \mathfrak{C}_C$ of E-domain and C-domain shift-scale-invariant BTFR's are defined by a self-dual scale-invariance property which is again consistent with the scale-invariance properties of energy densities $p_x(t)$, $P_x(f)$ and correlations $r_x(\tau)$, $R_x(v)$. The description of shift-scale-invariant BTFR's is once again simplified: due to a characteristic product structure of the BTFR kernels, a one-dimensional kernel function now suffices for BTFR characterization. Any shift-scale-invariant BTFR is a superposition of generalized Wigner distributions (E-domain) or generalized ambiguity functions (C-domain).

4) Inside the shift-scale-invariant classes $\mathfrak{D}_E$ and $\mathfrak{D}_C$, the two dual BTFR families generalized Wigner distribution (GWD) and generalized ambiguity function (GAF) are distinguished by the simple (exponential) form of their kernels, the large number of desirable properties they satisfy, and by the fact that any shift-scale-invariant BTFR is a superposition of GWD's or GAF's. GWD and GAF are characterized by a real-valued parameter α.

5) Inside the dual BTFR families GWD and GAF, the dual BTFR's Wigner distribution (WD) and ambiguity function (AF), which are obtained with $\alpha = 0$, are finally

uniquely defined by the single requirement of real valuedness (E-domain) or Hermiticity (C-domain).

It should be noted that the classification scheme presented (with the exception of the basic distinction between E-domain and C-domain) is only one of many possible such schemes. Obviously, any pair of dual BTFR properties can be used for the definition of two dual BTFR classes of E-domain and C-domain. Still, the classes $\mathcal{C}_E$, $\mathcal{C}_C$ and $\mathfrak{D}_E$, $\mathfrak{D}_C$ of shift-invariant and shift-scale-invariant BTFR's are fundamental for two reasons: i) they are axiomatically defined by basic geometrical invariance properties of BTFR's, and ii) they correspond to characteristic mathematical structures of BTFR's and, by that, to simplifications of BTFR description. Our classification is thus well motivated from the viewpoint of mathematical structure and mathematical description. There exists, however, a class of bilinear time-frequency representations which does not fit into the classification considered in this paper. This class is derived from the class of bilinear time-scale representations by formally introducing frequency as the inverse of a scale parameter [35]–[37]. Here, the concept of "frequency" is somewhat different from the usual frequency concept based on Fourier analysis.

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