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TIME-FREQUENCY FILTER BANKS WITH PERFECT RECONSTRUCTION

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Abstract - We present two methods for the design of perfect-reconstruction filter banks corresponding to a partition of the time-frequency plane into non-overlapping "pass regions". The first method is an extension of the concept of optimal time-frequency projection filters. The second method is an approximate filter bank design with reduced computational cost.

1. INTRODUCTION

The *Wigner distribution* (WD)

$$W_x(t,f) = \int_\tau x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$$

of a signal $x(t)$ is a real-valued function of time t and frequency f which describes the signal's energy distribution over the time-frequency plane [1]. (Integration is from $-\infty$ to ∞.) Specifically, we shall say that a signal is inside (outside) a given time-frequency region R if the essential support of the signal's WD is inside (outside) R.

The problem addressed in this paper is the design of a filter bank corresponding to a *partition of the time-frequency plane* (see *Figure 1*). By "partition of the time-frequency plane", we mean a set of time-frequency regions R_i which are disjoint (non-overlapping) and "complete" in the sense that the union of all R_i yields the entire time-frequency plane. The i-th region R_i is the "time-frequency pass region" of the i-th filter $\mathbf{H}_i$, in the sense that $\mathbf{H}_i$ passes all signals inside the respective time-frequency region R_i and suppresses all signals outside R_i. The overall filter bank is required to possess the "perfect reconstruction property" which postulates that the sum of all output signals $(\mathbf{H}_i x)(t)$ equals the input signal $x(t)$ or, equivalently, the sum of all filter operators $\mathbf{H}_i$ equals the identity operator $\mathbf{I}$,

$$\sum_{(\text{all } i)} \mathbf{H}_i = \mathbf{I} \, .$$

The filters $\mathbf{H}_i$ are assumed to be linear and (generally) time-varying, such that their input-output relations are

$$(\mathbf{H}_i x)(t) = \int_{t'} H_i(t,t')\, x(t')\, dt'$$

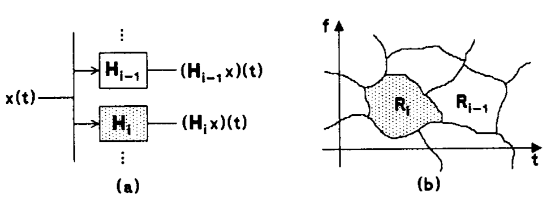

Figure 1. Filter bank (a) and partition of the time-frequency plane (b).

where $H_i(t,t')$ denotes the impulse response of the i-th filter (kernel of the i-th operator $\mathbf{H}_i$). The perfect-reconstruction property can then be formulated as

$$\sum_{(\text{all } i)} H_i(t,t') = I(t,t') = \delta(t-t') \, .$$

The subsequent sections are organized as follows. Section 2 considers the design of a single filter with specified time-frequency pass region. We discuss the method of optimal time-frequency projection filtering and an approximate method which avoids the solution of an eigen problem. These two methods are extended to the design of perfect-reconstruction filter banks in Section 3.

2. TIME-FREQUENCY FILTERING

By way of preparation, we first discuss the design of a single filter $\mathbf{H}$ with given time-frequency pass region R.

Time-frequency projection filters. The first method considered is "time-frequency projection filtering" [2,3] where the filter $\mathbf{H}$ is the orthogonal projection operator $\mathbf{S}$ [4] of a linear signal space $\mathfrak{S}$. The filter design thus reduces to the construction of a linear signal space $\mathfrak{S}$ corresponding to the given pass region R. This correspondence can be phrased mathematically by means of the WD of the signal space $\mathfrak{S}$.

The *WD of a linear signal space* $\mathfrak{S}$ is a real-valued time-frequency function describing the space's energy distribution over the time-frequency plane. It can be expressed as [3]

$$W_{\mathfrak{S}}(t,f) = \int_\tau S\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau = \sum_{k=1}^{N_{\mathfrak{S}}} W_{s_k}(t,f) \, , \quad (2.1)$$

where $S(t,t')$ is the kernel of the orthogonal projection operator $\mathbf{S}$ of $\mathfrak{S}$, $\{s_k(t)\}$ is an orthonormal basis of $\mathfrak{S}$, and $N_{\mathfrak{S}}$ is the dimension of $\mathfrak{S}$.

For the purposes of our time-frequency projection filter, the space $\mathfrak{S}$ must be "congruent with the time-frequency pass region R" such that its energy is distributed homogeneously inside the pass region R whereas, ideally, no energy is located outside R. We thus define the optimal space $\mathfrak{S}_R$ corresponding to the pass region R as the space minimizing the *congruence error* $\varepsilon_{\mathfrak{S}}^{(R)}$ defined by [3]

$$\varepsilon_{\mathfrak{S}}^{(R)2} \triangleq \iint_{(t,f)\in R} \left|1-W_{\mathfrak{S}}(t,f)\right|^2 dt\,df + \iint_{(t,f)\notin R} \left|W_{\mathfrak{S}}(t,f)\right|^2 dt\,df \, .$$

The congruence error checks how close $W_{\mathfrak{S}}(t,f)$ is to 1 inside R and to 0 outside R. Introducing the indicator function $\tilde{W}_R(t,f)$ of the pass region R by

$$\tilde{W}_R(t,f) \triangleq \begin{cases} 1, & (t,f)\in R \\ 0, & (t,f)\notin R \end{cases} ,$$

the minimization problem can be compactly written as

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$$\varepsilon_{\mathfrak{S}}^{(R)} = \|\tilde{W}_R - W_{\mathfrak{S}}\| \longrightarrow \min_{\mathfrak{S}} .$$

Note that $\tilde{W}_R(t,f)$ expresses the energy distribution of a hypothetic "ideal space" which, however, does not exist; the optimally congruent space $\mathfrak{S}_R$ minimizing $\varepsilon_{\mathfrak{S}}^{(R)}$ is the best possible approximation to the ideal energy distribution.

Using (2.1) and Parseval's theorem, the congruence error can be rewritten as

$$\varepsilon_{\mathfrak{S}}^{(R)} = \|\tilde{S}_R - S\| \tag{2.2}$$

where $S(t,t')$ is the kernel of the orthogonal projection operator of $\mathfrak{S}$ and $\tilde{S}_R(t,t')$ is defined by

$$\tilde{W}_R(t,f) = \int_\tau \tilde{S}_R\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau , \tag{2.3}$$

i.e., by expressing the region's indicator function $\tilde{W}_R(t,f)$ analogously to the WD of the signal space $\mathfrak{S}$ (cf. (2.1)). Note, however, that $\tilde{S}_R(t,t')$ is *not* the kernel of a projection operator.

It can be shown that an orthonormal basis of the optimal space $\mathfrak{S}_R$ can be constructed as follows:

(i) Calculate the kernel $\tilde{S}_R(t,t')$ by inverting (2.3),

$$\tilde{S}_R(t,t') = \int_f \tilde{W}_R\left(\frac{t+t'}{2}, f\right) e^{j2\pi(t-t')f} df . \tag{2.4}$$

(ii) Solve the eigenvalue/eigenfunction problem

$$\int_{t'} \tilde{S}_R(t,t')\, s_{R,k}(t')\, dt' = \lambda_{R,k}\, s_{R,k}(t) . \tag{2.5}$$

(iii) Those eigenfunctions $s_{R,k}(t)$ whose eigenvalues $\lambda_{R,k}$ are larger than 1/2 form an orthonormal basis of the optimal space $\mathfrak{S}_R$. We denote the number of these "dominant" eigenvalues by N_R; the dimension of $\mathfrak{S}_R$ thus equals N_R.

With the basis of $\mathfrak{S}_R$ calculated as above, the kernel of the orthogonal projection operator $\mathbf{S}_R$ of the optimal space $\mathfrak{S}_R$ is

$$S_R(t,t') = \sum_{k=1}^{N_R} s_{R,k}(t)\, s_{R,k}^*(t') .$$

This, finally, is the impulse response $H(t,t')$ of the optimal time-frequency projection filter with pass region R.

The eigenvalues $\lambda_{R,k}$ and eigenfunctions $s_{R,k}(t)$ are determined by the time-frequency region R; they will accordingly be called the *eigenvalues/eigensignals of R* [2]. It is easily shown that $\tilde{S}_R(t,t') = \tilde{S}_R^*(t',t)$; hence the eigenvalues $\lambda_{R,k}$ and eigensignals $s_{R,k}(t)$ are real-valued and orthonormal, respectively. The eigenvalues $\lambda_{R,k}$ are a measure of the time-frequency concentration of the respective eigensignals $s_{R,k}(t)$ in R [2].

Experiments indicate that the number N_R of eigenvalues $\lambda_{R,k} \geq 1/2$ (i.e., the dimension of the optimal space $\mathfrak{S}_R$) is very close to the area

$$A_R = \iint_{(t,f)\in R} dt\,df = \int_t\int_f \tilde{W}_R(t,f)\,dt\,df$$

of the pass region R. In fact, we always observed $|N_R - A_R| < 1$. For area A_R well above 1, the eigenvalues $\lambda_{R,k}$ can be split into a "dominant part" where $\lambda_{R,k} \approx 1$ and a "tail" where $\lambda_{R,k} \approx 0$ (including negative eigenvalues of small magnitude). The index $k = N_R$ (where $\lambda_{R,k} \approx 1/2$) can then be viewed as the border between the two parts. If the region's area A_R is considerably smaller than 1, then all eigenvalues must be expected to be smaller than 1/2; the optimally congruent space $\mathfrak{S}_R$ is then the "zero space" with dimension 0 and the optimal projection filter is the "all-suppression" filter, $S_R(t,t') \equiv 0$. This, of course, is a manifestation of the uncertainty principle which prohibits arbitrarily fine time-frequency localization.

For the time-frequency filter banks considered in Section 3, we need a *subspace-constrained* version of the design procedure discussed above [5]. Here, the optimal space $\mathfrak{S}$ is constrained to be a subspace of a given linear signal space $\mathfrak{X}$,

$$\varepsilon_{\mathfrak{S}}^{(R)} = \|\tilde{W}_R - W_{\mathfrak{S}}\| \longrightarrow \min_{\mathfrak{S}\subset\mathfrak{X}} . \tag{2.6}$$

The solution to the constrained problem can be found as usual, with the difference that the eigen problem (2.5) has now to be solved for the "projected kernel" $\tilde{S}_{R,\mathfrak{X}}(t,t')$ which is derived from the kernel $\tilde{S}_R(t,t')$ of (2.4) as

$$\tilde{S}_{R,\mathfrak{X}}(t,t') = \int_u\int_{u'} T(t,u)\, \tilde{S}_R(u,u')\, T^*(t',u')\, du\, du'$$

with $T(t,u)$ being the kernel of the orthogonal projection operator of the space $\mathfrak{X}$. Note that now, in general, the number of eigenvalues above 1/2 no longer equals the area A_R of the pass region R.

Weyl filters. We now consider a simplified filter design which yields an approximation to the optimal projection filter and avoids the eigen problem (2.5).

According to (2.2), the residual congruence error obtained for the optimally congruent space $\mathfrak{S}_R$ can be interpreted as the "distance" between the optimal projection filter $S_R(t,t')$ and the kernel $\tilde{S}_R(t,t')$,

$$\varepsilon_{\mathfrak{S},min}^{(R)} = \varepsilon_{\mathfrak{S}_R}^{(R)} = \|\tilde{S}_R - S_R\| .$$

The residual congruence error will be small since it is actually the minimal congruence error for given region R. Thus $\tilde{S}_R(t,t')$ can be considered as an approximation to the optimal projection filter $S_R(t,t')$. The accuracy of this approximation is directly measured by the residual congruence error $\varepsilon_{\mathfrak{S}_R}^{(R)}$.

The simplified filter design then consists of calculating the function $\tilde{S}_R(t,t')$ according to (2.4) and directly using $\tilde{S}_R(t,t')$ as the impulse response of the filter **H**. By this, the cumbersome solution of the eigen problem (2.5) is avoided. Note that the simplified filter **H** is no longer an orthogonal projection operator.

We shall call $\mathbf{H} = \tilde{\mathbf{S}}_R$ the *Weyl filter* corresponding to the time-frequency region R. Indeed, the *Weyl symbol* of a linear operator **H** is defined as [6]

$$L_H(t,f) \triangleq \int_\tau H\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \tag{2.7}$$

where $H(t,t')$ is the impulse response (kernel) of **H**. Since $H(t,t') = \tilde{S}_R(t,t')$ in our case, (2.3) shows that the Weyl symbol of **H** is the pass region's indicator function $\tilde{W}_R(t,f)$,

$$L_{\tilde{\mathbf{S}}_R}(t,f) = \tilde{W}_R(t,f) .$$

Thus the filter **H** is designed such that its Weyl symbol equals $\tilde{W}_R(t,f)$. A more detailed discussion of this design method and the Weyl symbol will be given in a future publication [7].

Simulation results illustrating the performance of the optimal projection filter $\mathbf{S}_R$ and the Weyl filter $\tilde{\mathbf{S}}_R$ are shown in *Figure 2* for two different input signals. The first input signal consists of two Gaussian components. One of the two Gaussians is outside the time-frequency pass region R and is hence suppressed in the output signals of both projection filter and Weyl filter; the other Gaussian is inside the pass region and is accordingly passed with only small distortion. The second input signal is a monocomponent chirp signal extending beyond the time-frequency pass region R. The corresponding output signals are indeed confined to the pass region, with broadening effects along the pass region's boundary. Note that the results obtained with the optimal projection filter and the Weyl filter are quite similar.

3. PERFECT-RECONSTRUCTION FILTER BANKS

We now discuss the application of projection filtering and Weyl filtering to the design of time-frequency filter banks.

Projection filter banks. A conceptually simple filter bank design is obtained by choosing the i-th filter $\mathbf{H}_i$ as the optimal projection filter for the corresponding pass region R_i. Unfortunately, since the individual filters are here designed independently of each other, the resulting filter bank will *not* possess the perfect-reconstruction property $\sum_i \mathbf{H}_i = \mathbf{I}$. We therefore propose a recursive method where the filters $\mathbf{H}_i$ are still orthogonal projections on spaces $\mathfrak{S}_i$ but the design of the i-th space $\mathfrak{S}_i$ is influenced by the spaces $\mathfrak{S}_1$ through $\mathfrak{S}_{i-1}$ designed previously.

Let $\mathfrak{X}$ be the "total signal space" comprising all signals. (Theoretically, $\mathfrak{X}$ can be chosen as $L_2(\mathbb{R})$, the space of all finite-energy signals; in practice, however, signals are discrete-time and with finite length N whence $\mathfrak{X}$ reduces to the N-dimensional space of vectors with length N). We define the space $\mathfrak{T}_i$ as the complement of all spaces designed previously,

$$\mathfrak{T}_i = \mathfrak{X} - (\mathfrak{S}_1 + \mathfrak{S}_2 + .. + \mathfrak{S}_{i-1}) = \mathfrak{T}_{i-1} - \mathfrak{S}_{i-1} .$$

Note that $\mathfrak{T}_i$ is orthogonal to all previously designed spaces $\mathfrak{S}_1$ through $\mathfrak{S}_{i-1}$. The i-th space $\mathfrak{S}_i$ is now constructed as the space which is optimally congruent with the i-th pass region R_i

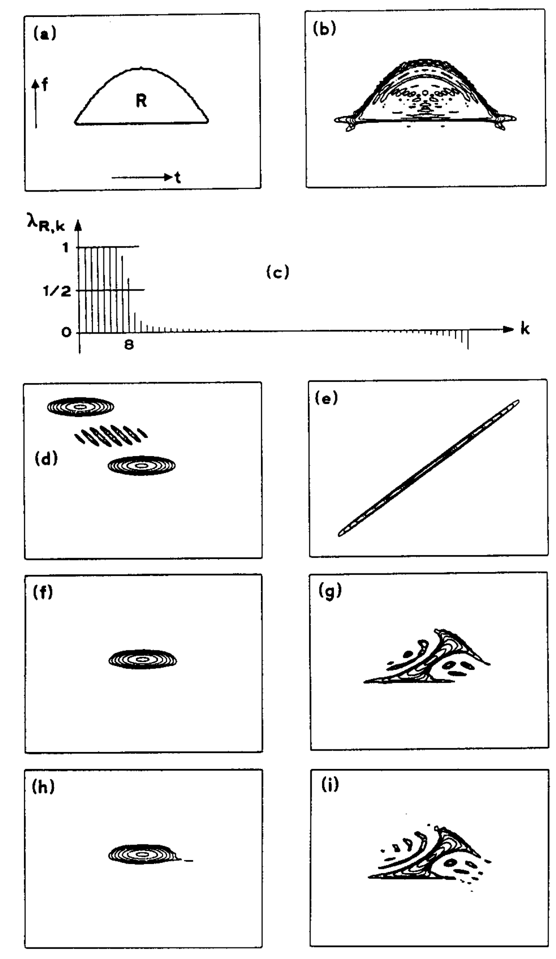

Figure 2. Performance of optimal projection filter (OPF) and Weyl filter (WF). (a) Time-frequency pass region R; (b) WD of optimally congruent signal space $\mathfrak{S}_R$; (c) eigenvalues of region R; (d) input signal No.1; (e) input signal No.2; (f) OPF output for input signal No.1; (g) OPF output for input signal No.2; (h) WF output for input signal No.1; (i) WF output for input signal No.2.

under the subspace constraint $\mathfrak{S}_i \subset \mathfrak{T}_i$ (cf. (2.6)), apart from the following difference: while the optimally congruent space is spanned by all eigensignals whose eigenvalues are above 1/2, we here take the eigensignals corresponding to the N_i largest eigenvalues, where

$$N_i = r(A_{R_i})$$

is the integer number which is closest to the area A_{R_i} of the region R_i ($r(\cdot)$ denotes rounding to the nearest integer). The recursion is started by constructing $\mathfrak{S}_1$ as the space which is optimally congruent with the first pass region R_1, again with the difference that the eigensignals corresponding to the $N_1 = r(A_{R_1})$ largest eigenvalues are taken.

We note the following properties: (i) All spaces $\mathfrak{S}_i$ are orthogonal to each other; (ii) the dimension of each space $\mathfrak{S}_i$ equals (up to a rounding error) the area of the corresponding pass region R_i. (The areas A_{R_i} of the pass regions R_i are assumed to be larger than 1.)

To show that the filter bank obtained with this recursive design method satisfies the perfect-reconstruction property, we consider the sum of all filters $\mathbf{H}_i$ (projection operators $\mathbf{S}_i$) up to index n,

$$\mathbf{H}^{(n)} \triangleq \sum_{i=1}^{n} \mathbf{H}_i = \sum_{i=1}^{n} \mathbf{S}_i .$$

This filter performs an orthogonal projection on the union space $\mathfrak{S}^{(n)} = \mathfrak{S}_1 + \mathfrak{S}_2 + .. + \mathfrak{S}_n$, where the dimension $N^{(n)} = N_1 + N_2 + .. + N_n$ of the space $\mathfrak{S}^{(n)}$ equals (up to a possible "rounding lag") the area $A^{(n)} = A_{R_1} + A_{R_2} + .. + A_{R_n}$ of the union $R^{(n)} = R_1 \cup R_2 \cup .. \cup R_n$ of all pass regions R_1 through R_n. For $n \to \infty$, the union region $R^{(n)}$ will become the entire time-frequency plane, the area $A^{(n)}$ of the union region $R^{(n)}$ and the dimension $N^{(n)}$ of the union space $\mathfrak{S}^{(n)}$ will both tend to infinity, and the union space $\mathfrak{S}^{(n)}$ itself will become the total signal space $\mathfrak{X}$. Thus the sum filter $\mathbf{H}^{(\infty)}$ is the projection operator of the total signal space $\mathfrak{X}$ which, of course, is the identity operator $\mathbf{I}$. (This reasoning is not mathematically strict for an infinite-dimensional total signal space $\mathfrak{X}$. However, in practice $\mathfrak{X}$ is finite-dimensional, and then the rounding lag and the completeness of the union space can easily be controlled).

Weyl filter banks. A perfect-reconstruction filter bank based on the Weyl filter concept is obtained simply by designing the filters $\mathbf{H}_i$ as the Weyl filters $\tilde{\mathbf{S}}_{R_i}$ of the corresponding pass regions R_i. To show that the perfect-reconstruction property is here satisfied even though the individual filter designs are independent of each other, we consider the sum of all filters,

$$\mathbf{H}^{(\infty)} = \sum_{i=1}^{\infty} \mathbf{H}_i .$$

Since the Weyl symbol of the i-th filter is the indicator function $\tilde{W}_{R_i}(t,f)$ of the i-th pass region R_i and the Weyl symbol is derived from the filter's impulse response by the linear transform (2.7), the Weyl symbol of the sum filter $\mathbf{H}^{(\infty)}$ is the sum of all indicator functions $\tilde{W}_{R_i}(t,f)$. But the sum of all indicator functions is everywhere 1 since the pass regions were assumed in Section 1 to be disjoint and "complete" in the time-frequency plane. Therefore, the Weyl symbol of the sum filter is

$$L_{\mathbf{H}^{(\infty)}}(t,f) = \sum_{i=1}^{\infty} \tilde{W}_{R_i}(t,f) \equiv 1 .$$

Inversion of (2.7) according to (2.4) then yields $H^{(\infty)}(t,t') = \delta(t-t')$ which shows that the sum filter $\mathbf{H}^{(\infty)}$ is indeed the identity operator $\mathbf{I}$.

Figure 3 shows simulation results obtained with the projection filter bank and the Weyl filter bank. The input signal is three-component, with all signal components well inside a corresponding pass region. Both filter banks are seen to perform a separation of the signal components.

The perfect-reconstruction property is demonstrated in *Figure 4*. The input signal is a chirp extending over all three pass regions. While the individual output signals are properly confined to the corresponding pass regions (again with broadening effects along the pass regions' boundaries), the composite output signals are strictly identical with the input signal.

4. CONCLUSION

Two methods for the design of a linear, time-varying filter with given time-frequency pass region have been presented, and the extension of these methods to the design of perfect-reconstruction time-frequency filter banks has been discussed.

The *optimal projection filter* is an orthogonal projection on the signal space which is optimally congruent with the time-frequency pass region. The congruence of a space with a time-frequency region is defined via the space's Wigner distribution. A recursive design procedure is used for extending optimal projection filtering to perfect-reconstruction filter banks.

The *Weyl filter* approximates the optimal projection filter. The extension of Weyl filtering to perfect-reconstruction filter banks is straightforward.

While the performance of the Weyl filter (filter bank) is quite similar to that of the optimal projection filter (filter bank), the computational cost required for the design of Weyl filters (filter banks) is greatly reduced as compared to the optimal projection design.

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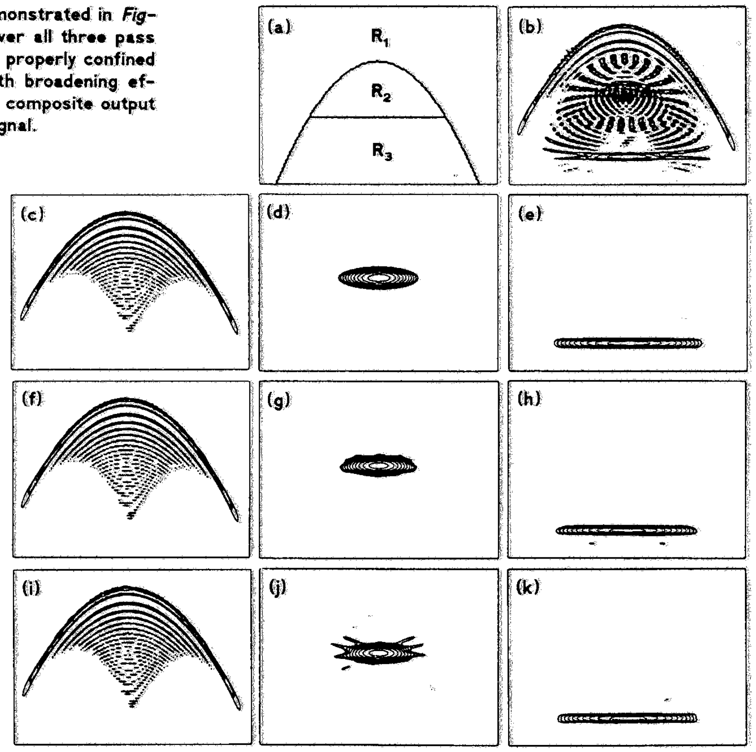

Figure 3. Performance of projection filter bank (PFB) and Weyl filter bank (WFB). (a) Time-frequency partition; (b) input signal; (c)-(e) components of input signal; (f)-(h) PFB output signals; (i)-(k) WFB output signals.

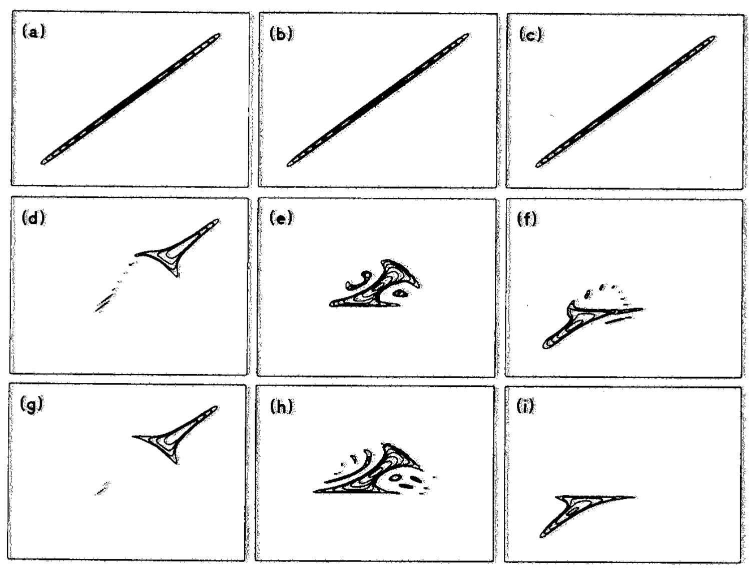

Figure 4. Perfect reconstruction of projection filter bank (PFB) and Weyl filter bank (WFB). (a) Input signal; (b) sum of PFB output signals; (c) sum of WFB output signals; (d)-(f) PFB output signals; (g)-(i) WFB output signals.