Phase Matching Algorithms for Wigner-Distribution Signal Synthesis

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Abstract—**Signal processing by means of discrete-time Wigner distribution requires a signal synthesis step. It is known that the result of signal synthesis contains a troublesome phase ambiguity. This paper analyzes the problem of phase ambiguities for both unconstrained and halfband-constrained signal synthesis and discusses various strategies for coping with it. After a review of the well-known phase matching algorithm using a reference signal, we present "autonomous" phase matching algorithms which do not require a reference signal. Next, "on-line" versions of both reference-based and autonomous phase matching algorithms are derived which feature a short-time or causal mode of operation and are thus suited for the on-line processing of signals with arbitrary length. The performance of the new algorithms is finally assessed by computer simulations.**

I. Introduction

The discrete-time Wigner distribution (DTWD) of a discrete-time signal $x(n)$ is defined as [1]

$$W_x(n, \theta) = 2 \sum_m x(n+m)x^*(n-m)e^{-j4\pi\theta m} \quad (1.1)$$

where n is a discrete-time index and θ denotes normalized frequency; summations are infinite unless explicitly specified otherwise. DTWD is $1/2$-periodic with respect to the frequency variable θ; this is in contrast to the signal's spectrum which is a 1-periodic function. Indeed, DTWD $W_x(n, \theta)$ is aliased with respect to θ unless the signal $x(n)$ is a "halfband signal" whose bandwidth is restricted to $1/2$ [1].

A recent paper [2] proposes the application of DTWD to time-frequency signal processing and signal design. An essential step of this application is signal synthesis, i.e., the generation of a signal $x(n)$ from a time-frequency function ("model") $\tilde{W}(n, \theta)$ which is not itself a valid DTWD. Global (unconstrained) signal synthesis [2] is the solution of the minimization problem

$$\epsilon_x^2 \triangleq \|\tilde{W} - W_x\|^2$$

$$= \sum_n \int_{-1/4}^{1/4} \left|\tilde{W}(n, \theta) - W_x(n, \theta)\right|^2 d\theta \rightarrow \min_x. \quad (1.2)$$

The notation implies that the synthesis error ϵ_x is to be minimized over all signals $x(n)$. In contrast, halfband signal synthesis [3] solves (1.2) with the side constraint that the resulting signal $x(n)$ be a halfband signal, such that aliasing in $W_x(n, \theta)$ is prevented. It is shown in [2] that the general solution of global signal synthesis contains a characteristic phase ambiguity: the subsequences of even-indexed and odd-indexed signal samples of the synthesis solution will assume arbitrary and independent phase factors. In the case of halfband signal synthesis, on the other hand, the halfband constraint prohibits the occurrence of independent phase ambiguities of even-indexed and odd-indexed samples, and only a phase ambiguity affecting the entire signal remains. We note that the same phase ambiguities exist if the signal synthesis problem is formulated with smoothed versions of DTWD (e.g., the pseudo-Wigner distribution), rather than the DTWD itself [4]–[6].

This paper analyzes the problem of phase ambiguities and discusses various methods for resolving it. It is organized as follows. In Section II, the occurrence of phase ambiguities is shown to result from a basic invariance property of DTWD. The phase ambiguity of global signal synthesis is represented in terms of an "absolute" and a "relative" phase, and a frequency-domain analysis of relative phase mismatch is given. Section III reviews the standard algorithm for phase matching using a reference signal [2] and presents a modified version suited for halfband signal synthesis. Section IV derives two novel algorithms for "autonomous" phase matching which do not require a reference signal. In Section V, both reference-based and autonomous phase matching algorithms are modified to feature a short-time or causal mode of operation and thus permit on-line processing. Simulation results are finally presented in Section VI.

II. Phase Ambiguities in DTWD Signal Synthesis

As a mathematical basis for discussing phase ambiguities and developing phase matching algorithms, we first introduce three signal subspaces with particular relevance to DTWD.

*A. The "Halfband" Subspace **H***

By definition, the linear signal subspace $\boldsymbol{H}$ consists of all "halfband signals," i.e., signals band limited to a predefined frequency interval ("halfband") $\theta_0 - 1/4 < \theta < \theta_0 + 1/4$ with bandwidth $1/2$:

$$x(n) \in \boldsymbol{H} \quad \Leftrightarrow \quad X(\theta) = 0 \quad \text{for} \quad \theta_0 + 1/4 < \theta < \theta_0 + 3/4. \quad (2.1)$$

Here, θ_0 is the center frequency of the respective halfband (see Fig. 1). Of particular importance are the halfband subspaces with $\theta_0 = 0$ (containing, in particular, real-valued signals oversampled by a factor 2) and $\theta_0 = 1/4$ (containing analytic signals). We note that DTWD will be nonaliased only in the case of a halfband signal [1]. Projecting [7] a signal on a halfband subspace $\boldsymbol{H}$ amounts to ideal halfband filtering: the (or-

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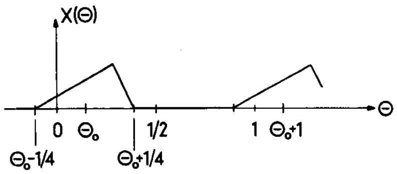

Fig. 1. Spectrum of a halfband signal.

thogonal) projection $x_H(n)$ of an arbitrary signal $x(n)$ on $\boldsymbol{H}$ is given by

$$x_H(n) = \sum_k h(n-k)x(k) \tag{2.2}$$

$$X_H(\theta) = H(\theta)X(\theta) = \begin{cases} X(\theta), & \theta_0 - 1/4 < \theta < \theta_0 + 1/4 \\ 0, & \theta_0 + 1/4 < \theta < \theta_0 + 3/4 \end{cases} \tag{2.3}$$

where

$$h(n) = \tfrac{1}{2}\,\mathrm{sinc}\,(\tfrac{1}{2}n)e^{j2\pi\theta_0 n},$$

$$H(\theta) = \begin{cases} 1, & \theta_0 - 1/4 < \theta < \theta_0 + 1/4 \\ 0, & \theta_0 + 1/4 < \theta < \theta_0 + 3/4 \end{cases} \tag{2.4}$$

are, respectively, the impulse response and transmission function of an ideal halfband filter with cutoff frequencies $\theta_0 \pm 1/4$.

B. The "Even-Index" Subspace $\boldsymbol{E}$ and the "Odd-Index" Subspace $\boldsymbol{O}$

We define the linear signal subspace $\boldsymbol{E}$ (signal subspace $\boldsymbol{O}$) as the space of all signals for which only the even-indexed (odd-indexed) signal samples are nonzero:

$$x(n) \in \boldsymbol{E} \Leftrightarrow x(2k+1) = 0; \qquad x(n) \in \boldsymbol{O} \Leftrightarrow x(2k) = 0. \tag{2.5}$$

The projection $x_E(n)$ (projection $x_O(n)$) of a signal $x(n)$ on the subspace $\boldsymbol{E}$ (subspace $\boldsymbol{O}$) can be written as

$$x_E(n) = \tfrac{1}{2}[1 + (-1)^n]x(n) = \begin{cases} x(n), & n = 2k \\ 0, & n = 2k+1 \end{cases} \tag{2.6}$$

$$x_O(n) = \tfrac{1}{2}[1 - (-1)^n]x(n) = \begin{cases} 0, & n = 2k \\ x(n), & n = 2k+1. \end{cases} \tag{2.7}$$

Since multiplication by $(-1)^n = \exp(j2\pi\frac{1}{2}n)$ effects a spectral shift by frequency $1/2$, the spectra of $x_E(n)$ and $x_O(n)$ are aliased versions of the original signal spectrum $X(\theta)$:

$$X_E(\theta) = \tfrac{1}{2}[X(\theta) + X(\theta - \tfrac{1}{2})];$$
$$X_O(\theta) = \tfrac{1}{2}[X(\theta) - X(\theta - \tfrac{1}{2})]. \tag{2.8}$$

C. Phase Ambiguity of Global Signal Synthesis

We now discuss the phase ambiguity of the solution of global signal synthesis (1.2). Let $x(n)$ be some signal, and let us derive from $x(n)$ a new, "phase-rotated" signal $\tilde{x}^{(\varphi_e,\varphi_o)}(n)$ by applying two constant phase factors $e^{j\varphi_e}$, $e^{j\varphi_o}$ to the subsequences of even-indexed and odd-indexed signal samples, respectively,

$$\tilde{x}^{(\varphi_e,\varphi_o)}(n) \triangleq e^{j\varphi_e}x_E(n) + e^{j\varphi_o}x_O(n) = \begin{cases} e^{j\varphi_e}x(n), & n = 2k \\ e^{j\varphi_o}x(n), & n = 2k+1. \end{cases} \tag{2.9}$$

Alternatively, the phase-rotated signal (2.9) can also be expressed as

$$\tilde{x}^{(\varphi,\psi)}(n) = e^{j\varphi}\tilde{x}^{(\psi)}(n) \tag{2.10}$$

with

$$\tilde{x}^{(\psi)}(n) = x_E(n) + e^{j\psi}x_O(n) = \begin{cases} x(n), & n = 2k \\ e^{j\psi}x(n), & n = 2k+1 \end{cases} \tag{2.11}$$

The "absolute phase" $\varphi = \varphi_e$ affects the entire signal and is of little importance in many applications. The "relative phase" $\psi = \varphi_o - \varphi_e$, on the other hand, describes a phase rotation of even-indexed and odd-indexed samples relative to each other and generally leads to severe signal distortion. In the frequency domain, this distortion can be interpreted as an aliasing effect: combining (2.10), (2.11), and (2.8), the spectrum of $\tilde{x}^{(\varphi,\psi)}(n)$ is obtained as

$$\tilde{X}^{(\varphi,\psi)}(\theta) = e^{j\varphi}[c_+X(\theta) + c_-X(\theta - \tfrac{1}{2})] \quad \text{with}$$
$$c_\pm = \tfrac{1}{2}(1 \pm e^{j\psi}). \tag{2.12}$$

Now, using the definition (1.1) of DTWD, it can easily be shown that

$$W_{\tilde{x}}(n, \theta) \equiv W_x(n, \theta) \tag{2.13}$$

i.e., DTWD is invariant with respect to arbitrary phase rotation of the subsequences of even-indexed and odd-indexed signal samples. Combining (2.13) and the definition (1.2) of the synthesis error ϵ_x, it is then clear that

$$\epsilon_{\tilde{x}} = \|\tilde{W} - W_{\tilde{x}}\| = \|\tilde{W} - W_x\| = \epsilon_x. \tag{2.14}$$

This shows that the original signal $x(n)$ and its "phased-rotated" version $\tilde{x}(n)$ achieve the same synthesis error. The result of global signal synthesis is hence ambiguous with respect to the phases φ_e and φ_o or, equivalently, φ ("absolute phase ambiguity") and ψ ("relative phase ambiguity"). On practical application of global signal synthesis, these phases will assume arbitrary values. The same is true for global signal synthesis involving a smoothed version of DTWD instead of DTWD itself [4]–[6].

D. Phase Ambiguity of Halfband Signal Synthesis

We next consider halfband signal synthesis which solves the minimization (1.2) subject to the halfband constraint $x(n) \in \boldsymbol{H}$. Now, if $x(n) \in \boldsymbol{H}$ is a solution of halfband signal synthesis, then the phase-rotated signal $\tilde{x}^{(\varphi,\psi)}(n)$ of (2.10) cannot be a halfband signal as well unless $\psi = 0$: indeed, it follows from the aliasing relation (2.12) that a relative phase rotation would destroy the halfband property of a signal. Thus, the result of halfband signal synthesis does not contain a relative phase ambiguity; however, the absolute phase ambiguity remains.

III. Reference-Based Phase Matching

A. Global Signal Synthesis

The phase ambiguities of global signal synthesis can be resolved *a posteriori* by means of some separate algorithm for "phase matching." If $x(n)$ is the result of global signal synthesis, then phase matching amounts to forming a phase-rotated signal $\tilde{x}^{(\varphi_e, \varphi_o)}(n)$ as in (2.9) and adjusting the phases φ_e and φ_o according to some optimization criterion. The definition of such a criterion is straightforward when a reference signal $y(n)$ is available [2]: we simply adjust the phases such that the resulting signal is as close to the reference signal as possible

$$\epsilon(\varphi_e, \varphi_o) \triangleq \| y - \tilde{x}^{(\varphi_e, \varphi_o)} \| \rightarrow \min_{\varphi_e, \varphi_o} \tag{3.1}$$

which means that the distance $\epsilon(\varphi_e, \varphi_o)$ is to be minimized with respect to the phases φ_e and φ_o. As shown in [2], the optimal phases are given by

$$\varphi_{e,\text{opt}} = \arg C_{R,e} \quad \text{with}$$
$$C_{R,e} = (y_E, x_E) = \sum_k y(2k)x^*(2k) \tag{3.2}$$

$$\varphi_{o,\text{opt}} = \arg C_{R,o} \quad \text{with}$$
$$C_{R,o} = (y_O, x_O) = \sum_k y(2k+1)x^*(2k+1). \tag{3.3}$$

B. Halfband Signal Synthesis

While halfband signal synthesis avoids the relative phase ambiguity, the absolute phase ambiguity remains. If the absolute phase is of importance and if a reference signal is available, then the absolute phase can be matched to the reference signal as

$$\epsilon(\varphi) \triangleq \| y - e^{j\varphi}x \| \rightarrow \min_{\varphi} \tag{3.4}$$

where $y(n)$ is again the reference signal and $x(n)$ is the result of halfband signal synthesis. The optimal absolute phase is here obtained as (compare (3.2), (3.3))

$$\varphi_{\text{opt}} = \arg C_R \qquad \text{with} \quad C_R = (y, x) = \sum_n y(n)x^*(n). \tag{3.5}$$

IV. Autonomous Phase Matching

Naturally, the reference-based phase matching algorithms reviewed in the previous section cannot be applied in those situations where a meaningful reference signal is not available. In this section, therefore, we present a new class of phase matching algorithms which do not require a reference signal. These "autonomous" algorithms, however, only resolve the relative phase ambiguity; their application is hence restricted to the case of global signal synthesis where a relative phase ambiguity occurs. Indeed, without a reference signal no reasonable criterion for adjusting the absolute phase can be formulated in general. Note, however, that it is the relative phase ambiguity whose resolution is of primary importance.

Let $x(n)$ be the result of global signal synthesis. Since the absolute phase is not to be adjusted, we use the phase-rotated signal version $\tilde{x}^{(\psi)}(n)$ of (2.11) and adjust the relative phase ψ according to some optimality criterion. Two different approaches are discussed in the following.

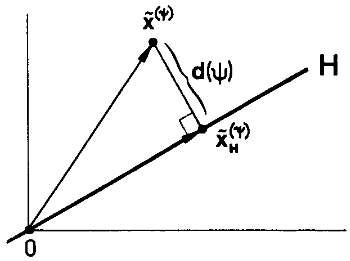

Fig. 2. Distance of signal $\tilde{x}^{(\psi)}(n)$ from halfband subspace $\boldsymbol{H}$.

A. The Halfband Approximation Criterion

In practice, DTWD is applied only to signals which are halfband (or at least nearly halfband) so that no substantial aliasing of DTWD occurs. Let us, therefore, adjust the relative phase ψ such that the resulting signal $\tilde{x}^{(\psi)}(n)$ is as nearly halfband as possible (note that, in general, no choice of the phase ψ can be found such that the signal $\tilde{x}^{(\psi)}(n)$ is exactly halfband). This will be called the *halfband approximation criterion* (HAC). Phrased mathematically, the HAC requires that the distance of $\tilde{x}^{(\psi)}(n)$ from a given halfband subspace $\boldsymbol{H}$ be minimal. This distance is expressed as $d(\psi) = \|\tilde{x}^{(\psi)} - \tilde{x}_H^{(\psi)}\|$, where $\tilde{x}_H^{(\psi)}(n)$ is the projection of $\tilde{x}^{(\psi)}(n)$ on $\boldsymbol{H}$ (see Fig. 2). The HAC thus reads

$$d(\psi) \triangleq \|\tilde{x}^{(\psi)} - \tilde{x}_H^{(\psi)}\| \rightarrow \min_{\psi}. \tag{4.1}$$

Using Parseval's relation and (2.3), the squared norm $d^2(\psi)$ can be expressed in the frequency domain as

$$\begin{aligned} d^2(\psi) &= \|\tilde{x}^{(\psi)} - \tilde{x}_H^{(\psi)}\|^2 = \int_{-1/2}^{1/2} \left|[1 - H(\theta)]\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \\ &= \int_{-1/2}^{1/2} [1 - H(\theta)]\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \\ &= \int_{-1/2}^{1/2} \left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta - \int_{-1/2}^{1/2} H(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta. \end{aligned} \tag{4.2}$$

The first term in (4.2) is simply the energy of $\tilde{x}^{(\psi)}(n)$ which is independent of ψ. Thus it remains to maximize the second term

$$\begin{aligned} m_H(\psi) &\triangleq \int_{-1/2}^{1/2} H(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \\ &= \int_{\theta_0 - 1/4}^{\theta_0 + 1/4} \left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \rightarrow \max_{\psi}. \end{aligned} \tag{4.3}$$

B. The Spectral Spread Criterion

According to (4.3), the HAC amounts to maximizing the energy of $\tilde{x}^{(\psi)}(n)$ inside the halfband interval $\theta_0 - 1/4 < \theta < \theta_0 + 1/4$. This can be approximated by maximizing the concentration of the spectrum $\tilde{X}^{(\psi)}(\theta)$ relative to the halfband's center frequency θ_0 or, equivalently, by minimizing the spread of $\tilde{X}^{(\psi)}(\theta)$ around θ_0

$$\sigma^2(\psi) \triangleq \frac{\int_{\theta_0 - 1/2}^{\theta_0 + 1/2} \rho(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta}{\int_{\theta_0 - 1/2}^{\theta_0 + 1/2} \left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta} \rightarrow \min_{\psi}. \tag{4.4}$$

This will be called the *spectral spread criterion* (SSC). The conventional definition of spread would use $\rho(\theta) = (\theta - \theta_0)^2$ as spectral weighting function in (4.4); however, to remain inside the framework of discrete-time signals, we here choose the 1-periodic weighting function

$$\rho(\theta) \triangleq [\sin \pi(\theta - \theta_0)]^2 = \tfrac{1}{2}[1 - \cos 2\pi(\theta - \theta_0)] = 1 - S(\theta) \quad (4.5)$$

with

$$S(\theta) = \tfrac{1}{2}[1 + \cos 2\pi(\theta - \theta_0)] \quad (4.6)$$

which is shown in Fig. 3. For minimizing the spread $\sigma^2(\psi)$, we first note that the denominator of the definition (4.4) is the energy of $\tilde{x}^{(\psi)}(n)$ which is independent of the phase ψ. Thus there remains the numerator of (4.4)

$$\int_{\theta_0-1/2}^{\theta_0+1/2} \rho(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta = \int_{-1/2}^{1/2} [1 - S(\theta)]\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta = \int_{-1/2}^{1/2} \left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta - \int_{-1/2}^{1/2} S(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \quad (4.7)$$

to be minimized. The first term of (4.7) again being independent of ψ, there finally remains to maximize the second term

$$m_S(\psi) \triangleq \int_{-1/2}^{1/2} S(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \to \max_{\psi}. \quad (4.8)$$

C. A Unified Framework for Autonomous Phase Matching

Comparing (4.3) and (4.8), we realize that both the HAC and the SSC require the maximization of a frequency-domain moment

$$m_F(\psi) = \int_{-1/2}^{1/2} F(\theta)\left|\tilde{X}^{(\psi)}(\theta)\right|^2 d\theta \to \max_{\psi}. \quad (4.9)$$

The only difference between the two criteria is the specific weighting function $F(\theta)$: in the HAC case, $F(\theta) = H(\theta)$ has rectangular shape; in the SSC case, $F(\theta) = S(\theta)$ is sinusoidal. The weighting functions are similar, though, since they both tend to suppress the signal's spectrum outside the halfband $\theta_0 - 1/4 < \theta < \theta_0 + 1/4$ while emphasizing it inside the halfband (see Fig. 4). With this general property of the weighting function $F(\theta)$, the unified formulation (4.9) represents a generalized criterion for autonomous phase matching.

To solve the general maximization problem (4.9), we insert the frequency-domain version of (2.11) whence $m_F(\psi)$ can be further developed as

$$m_F(\psi) = \int_{-1/2}^{1/2} F(\theta)\left|X_E(\theta)\right|^2 d\theta + \int_{-1/2}^{1/2} F(\theta)\left|X_O(\theta)\right|^2 d\theta + \int_{-1/2}^{1/2} F(\theta) 2 \operatorname{Re}\left\{X_E(\theta)\left[e^{j\psi}X_O(\theta)\right]^*\right\} d\theta. \quad (4.10)$$

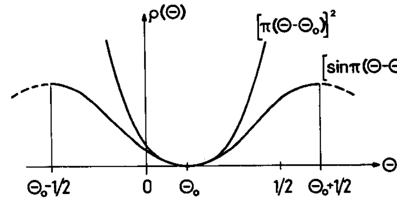

Fig. 3. Weighting functions for the definition of a spectral spread.

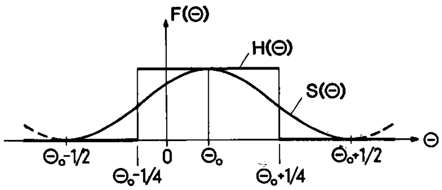

Fig. 4. Weighting functions of HAA and SSA.

Since the first two components of (4.10) do not depend on ψ, there remains to maximize the last component which, assuming $F(\theta)$ to be real-valued, can be written as

$$2 \operatorname{Re}\left\{e^{-j\psi}C_F\right\} = 2\left|C_F\right| \cos\left(-\psi + \arg C_F\right) \quad (4.11)$$

with

$$C_F \triangleq \int_{-1/2}^{1/2} F(\theta)X_E(\theta)X_O^*(\theta)\, d\theta. \quad (4.12)$$

The optimal relative phase, i.e., the phase maximizing (4.11), is thus given by

$$\psi_{\text{opt}} = \arg C_F. \quad (4.13)$$

Using Parseval's relation, the constant C_F can alternatively be expressed in the time domain as

$$C_F = (x_{EF}, x_O) = \sum_k x_{EF}(2k+1)x^*(2k+1) = (x_{EFO}, x_O) \quad (4.14)$$

where $x_{EF}(n)$ stands for the result of filtering $x_E(n)$ with transmission function $F(\theta)$ and $x_{EFO}(n)$ is derived from $x_{EF}(n)$ according to (2.7). The practical computation of C_F naturally depends on the specific weighting funtion $F(\theta)$. Specializing to the weighting functions $H(\theta)$ and $S(\theta)$ defined by the HAC and the SSC, respectively, we obtain the following results.

D. The Halfband Approximation Algorithm

Choosing $F(\theta) = H(\theta)$, we insert (2.4) and (2.8) into (4.12) and obtain $C_F = C_H$ as

$$C_H = \int_{\theta_0-1/4}^{\theta_0+1/4} Z(\theta)\, d\theta \quad \text{with} \quad Z(\theta) \triangleq \tfrac{1}{4}\left[X(\theta) + X\left(\theta - \tfrac{1}{2}\right)\right] \cdot \left[X(\theta) - X\left(\theta - \tfrac{1}{2}\right)\right]^*. \quad (4.15)$$

E. The Spectral Spread Algorithm

For $F(\theta) = S(\theta)$, $C_F = C_S$ is given by

$$C_S = \frac{1}{4} \sum_k \left[e^{-j2\pi\theta_0} x(2k+2) + e^{j2\pi\theta_0} x(2k)\right] x^*(2k+1) \tag{4.16}$$

which is obtained from (4.14) and (4.6) after straightforward manipulation. We note that C_S can be calculated without the need for expensive filtering or computing a Fourier transform.

F. Discussion of Autonomous Phase Matching Algorithms

Among all autonomous phase matching algorithms obtained for various weighting functions $F(\theta)$, the halfband approximation algorithm (HAA) with $F(\theta) = H(\theta)$ is the most satisfying from a theoretical viewpoint since it produces the signal which is "as nearly halfband as possible" and thus causes minimal aliasing in DT WD. On the other hand, the spectral spread algorithm (SSA), obtained with $F(\theta) = S(\theta)$, has the practical advantage of requiring considerably reduced computation.

Since both the HAA and the SSA perform a maximization of frequency-domain moments whose weighting functions have similar overall characteristics, the results of HAA and SSA will be similar in many cases. Indeed, the solutions of HAA and SSA will be strictly identical in the case of a *halfband consistent* model [6]. In this case, there exists a solution $x^{(H)}(n)$ of global signal synthesis which is a halfband signal and equals the result of halfband signal synthesis. However, the global signal synthesis algorithm will generally not yield this specific solution $x^{(H)}(n)$ but some other solution

$$x(n) = x_E^{(H)}(n) + e^{j\psi_0} x_O^{(H)}(n) \tag{4.17}$$

with relative phase mismatch (we suppress the absolute phase ambiguity since it is irrelevant for our discussion). The "halfband solution" $x^{(H)}(n)$ can then be derived from the phase-mismatched solution $x(n)$ by proper phase matching. From the HAC, it immediately follows that the HAA will produce this halfband solution $x^{(H)}(n)$ up to an absolute phase; the minimal distance $d_{\min}(\psi) = d(\psi_{\text{opt}})$ is then zero. This means that, in the case of a halfband consistent model, the result of halfband signal synthesis and the result of global signal synthesis with subsequent phase matching by means of the HAA are identical apart from an absolute phase.

We now show that the very same property holds for the general formulation (4.9) of autonomous phase matching (including SSA). We assume that the weighting function $F(\theta)$ is i) 1-periodic, ii) real-valued, and iii) even with respect to θ_0. Also, we assume iv) that $F(\theta + 1/2) < F(\theta)$ for $\theta_0 < \theta < \theta_0 + 1/4$; this, together with iii), assures that $F(\theta)$ emphasizes (attenuates) the spectrum inside (outside) the halfband. Now, with $x(n)$ of (4.17) being the result of global signal synthesis, we form the phase-corrected signal version

$$\tilde{x}^{(\psi)}(n) = x_E(n) + e^{j\psi} x_O(n) = x_E^{(H)}(n) + e^{j(\psi_0+\psi)} x_O^{(H)}(n). \tag{4.18}$$

Obviously, the desired halfband solution will be obtained for $\psi = -\psi_0$. We now prove that this indeed equals the result $\psi_{\text{opt}} = \arg C_F$ of autonomous phase matching. With $x_E(n) = x_E^{(H)}(n)$ and $x_O(n) = e^{j\psi_0} x_O^{(H)}(n)$, C_F can be expressed as

$$C_F = \int_{-1/2}^{1/2} F(\theta) X_E(\theta) X_O^*(\theta)\, d\theta = e^{-j\psi_0} C_F^{(H)} \tag{4.19}$$

with

$$\begin{aligned} C_F^{(H)} &= \int_{-1/2}^{1/2} F(\theta) X_E^{(H)}(\theta) X_O^{(H)*}(\theta)\, d\theta \\ &= \frac{1}{4} \int_{-1/2}^{1/2} F(\theta) \left|X^{(H)}(\theta)\right|^2 d\theta \\ &\quad - \frac{1}{4} \int_{-1/2}^{1/2} F(\theta) \left|X^{(H)}(\theta - 1/2)\right|^2 d\theta \\ &= \frac{1}{4} \int_{-1/2}^{1/2} \left[F(\theta) - F(\theta + 1/2)\right] \left|X^{(H)}(\theta)\right|^2 d\theta \end{aligned} \tag{4.20}$$

where we have used (2.8) and the fact that $X^{(H)}(\theta) X^{(H)*}(\theta - 1/2) \equiv 0$ since $x^{(H)}(n)$ is a halfband signal. Due to the fact that $X^{(H)}(\theta)$ is confined to the halfband interval $\theta_0 - 1/4 < \theta < \theta_0 + 1/4$ and our assumptions regarding the weighting function $F(\theta)$, it follows from (4.20) that $C_F^{(H)} > 0$ and thus $\arg C_F^{(H)} = 0$. With this and (4.19), we finally obtain

$$\psi_{\text{opt}} = \arg C_F = -\psi_0 + \arg C_F^{(H)} = -\psi_0 \tag{4.21}$$

which completes our proof. Of course, halfband consistency of a model will generally be the exception rather than the rule. Still, the fact that both HAA and SSA are consistent with halfband signal synthesis in the above sense seems to indicate that, in the practically important case of nearly halfband consistent models, the solutions obtained by HAA and SSA will be very similar and, in particular, close to the solution of halfband signal synthesis. This property is confirmed by experiments (see Section VI).

An interesting relation exists between reference-based phase matching on the one hand and autonomous phase matching on the other. According to (3.3), reference-based phase matching calculates the phase of odd-indexed signal samples as $\varphi_{o,\text{opt}} = \arg(y_O, x_O)$. On the other hand (see (4.13) and (4.14)), autonomous phase matching calculates the relative phase ψ (which, up to the absolute phase, can be interpreted as the phase of odd-indexed signal samples φ_o) as $\psi_{\text{opt}} = \arg(x_{EFO}, x_O)$. We thus see that, as far as the relative phase is concerned, autonomous phase matching can be interpreted as reference-based phase matching with reference signal $y(n) = x_{EF}(n)$.

V. On-Line Algorithms

All phase matching algorithms discussed so far use the entire synthesis result $x(n)$ to derive the optimal phase or phases for phase matching. This mode of operation is suited for DTWD signal synthesis where all signal samples are synthesized simultaneously. On the other hand, there exist signal synthesis algorithms for pseudo Wigner distribution where successive signal samples or signal blocks are synthesized one after the other [5], [6]; this mode of operation allows the on-line processing of signals with arbitrary length. There is thus a need for phase matching algorithms which are compatible with this on-line mode of operation.

A. Reference-Based Phase Matching

We first derive an on-line version of the reference-based algorithm (3.2), (3.3) used in the context of global signal synthesis. Suppose that $x(k)$ is the result of global signal synthesis and $y(k)$ is a reference signal. At time n, we want to calculate local, time-varying estimates $\hat{\varphi}_e(n)$, $\hat{\varphi}_o(n)$ of the optimal phases $\varphi_{e,\mathrm{opt}}$, $\varphi_{o,\mathrm{opt}}$ defined by (3.2) and (3.3). With these local estimates, the nth sample of the phase-matched signal is then formed according to

$$\hat{x}(n) \triangleq e^{j\hat{\varphi}_e(n)} x_E(n) + e^{j\hat{\varphi}_o(n)} x_O(n). \tag{5.1}$$

Note that, in general, different phases will be used for different samples $\hat{x}(n)$. The phase $\hat{\varphi}_e(n)$ is only relevant for even n since $x_E(n) = 0$ for odd n. Similarly, $\hat{\varphi}_o(n)$ is only relevant for odd n.

To be compatible with on-line processing, we assume that, at time n, the signals $x(k)$ and $y(k)$ are available only for $k \leq n + N$, where $N \geq 0$ is a fixed parameter. It is then natural to define the local estimates $\hat{\varphi}_e(n)$, $\hat{\varphi}_o(n)$ at a given time instant n as the solution of the local minimization problem

$$\begin{aligned} \epsilon_n^2(\varphi_e, \varphi_o) &= \left\| y - \tilde{x}^{(\varphi_e, \varphi_o)} \right\|_{w_n}^2 \\ &\triangleq \sum_k w(k - n) \left| y(k) - \tilde{x}^{(\varphi_e, \varphi_o)}(k) \right|^2 \to \min_{\varphi_e, \varphi_o} \end{aligned} \tag{5.2}$$

where $w(k)$ is some nonnegative window satisfying $w(k) = 0$ for $k > N$. We note that (5.2) is just a local or windowed version of the error norm (3.1), and that, due to windowing, only samples of $x(k)$ and $y(k)$ with $k \leq n + N$ are contained in this local error. It can be shown that the solution to the minimization problem (5.2) is

$$\begin{aligned} \hat{\varphi}_e(n) &= \arg \hat{C}_{R,e}(n), \quad \hat{C}_{R,e}(n) = (y_E, x_E)_{w_n} \\ &= \sum_k w(2k - n) y(2k) x^*(2k) \end{aligned} \tag{5.3}$$

$$\begin{aligned} \hat{\varphi}_o(n) &= \arg \hat{C}_{R,o}(n), \qquad \hat{C}_{R,o}(n) = (y_O, x_O)_{w_n} \\ &= \sum_k w(2k + 1 - n) y(2k + 1) x^*(2k + 1). \end{aligned} \tag{5.4}$$

Note that the above result for on-line phase matching is analogous to the result for off-line phase matching as given by (3.2), (3.3); the only difference is the local windowing contained in the inner products (5.3) and (5.4).

An on-line version of the reference-based algorithm (3.5) used for matching the absolute phase can be derived in an analogous way; we here obtain

$$\begin{aligned} \hat{\varphi}(n) &= \arg \hat{C}_R(n), \qquad \hat{C}_R(n) = (y, x)_{w_n} \\ &= \sum_k w(k - n) y(k) x^*(k). \end{aligned} \tag{5.5}$$

B. Autonomous Phase Matching

For on-line autonomous phase matching where only the relative phase is adjusted, we define the phase-matched signal as

$$\hat{x}(n) \triangleq x_E(n) + e^{j\hat{\psi}(n)} x_O(n) \tag{5.6}$$

where $\hat{\psi}(n)$ is a local estimate (at time n) of the optimal relative phase ψ_{opt}. An on-line version of the general algorithm (4.13), (4.14) for autonomous phase matching is then obtained heuristically by using a windowed version of the inner product (4.14); this yields the relative phase estimate

$$\begin{aligned} \hat{\psi}(n) &= \arg \hat{C}_F(n), \qquad \hat{C}_F(n) = (x_{EF}, x_O)_{w_n} \\ &= \sum_k w(2k + 1 - n) x_{EF}(2k + 1) x^*(2k + 1). \end{aligned} \tag{5.7}$$

Here, $x_{EF}(n)$ is again the result of filtering $x_E(n)$ with transmission function $F(\theta)$; in practice, this filter has generally to be approximated by a recursive or short-time (FIR) version in order to be compatible with on-line processing. Comparing with (5.4), we see that on-line autonomous phase matching can again be interpreted as on-line reference-based phase matching with reference signal $y(n) = x_{EF}(n)$. For $F(\theta) = S(\theta)$, we obtain a computationally inexpensive on-line version of the SSA, where $\hat{C}_F(n) = \hat{C}_S(n)$ is given by

$$\begin{aligned} \hat{C}_S(n) &\triangleq (x_{ES}, x_O)_{w_n} \\ &= \tfrac{1}{4} \sum_k w(2k + 1 - n) \left[e^{-j2\pi\theta_0} x(2k + 2) \right. \\ &\qquad \left. + e^{j2\pi\theta_0} x(2k) \right] x^*(2k + 1). \end{aligned} \tag{5.8}$$

C. Window Types, Recursive Calculation, and Convergence

Depending on the type of the window $w(k)$, there exist two versions of the local phase matching algorithms presented above. First, if the window has finite length $2N + 1$, $w(k) = 0$ for $|k| > N$, then the phase estimates at time n are derived from a local signal interval $n - N \leq k \leq n + N$ centered at time n. There then results a "short-time" algorithm for phase matching. On the other hand, the window may extend to $-\infty$. Here, the exponential window

$$w(k) = e^{\alpha k} u(-k) = \begin{cases} 0, & k > 0 \\ e^{\alpha k}, & k \leq 0 \end{cases} \quad \text{with } \alpha \geq 0 \tag{5.9}$$

where $u(k)$ denotes the unit step function, is particularly efficient since the inner products defining the time-varying phase estimates may then be calculated recursively; for example, the inner product $\hat{C}_R(n)$ of (5.5) can be written as

$$\hat{C}_R(n) = e^{-\alpha} \hat{C}_R(n - 1) + y(n) x^*(n) \tag{5.10}$$

with similar recursions existing for all other local algorithms. For $\alpha = 0$, the window $w(k)$ equals the time-inverted unit step function $u(-k)$. In this case, the windowed inner products defining the time-varying phases for the on-line case formally converge to the nonwindowed inner products of the off-line case as n approaches ∞: indeed, comparing, e.g., $\hat{C}_R(n)$ (with $w(k) = u(-k)$) with C_R of (3.5)

$$\hat{C}_R(n) = \sum_{k=-\infty}^{n} y(k) x^*(k), \qquad C_R = \sum_{k=-\infty}^{\infty} y(k) x^*(k) \tag{5.11}$$

it is seen that the phase estimate $\hat{\varphi}(n) = \arg \hat{C}_R(n)$ and the optimal phase $\varphi_{\mathrm{opt}} = \arg C_R$ become equal in the limit $n \to \infty$.

VI. Simulation Results

We finally present some simulation results for the autonomous phase matching algorithms HAA and SSA, including online versions. Global signal synthesis was performed from a time-frequency model $\tilde{W}(n, \theta)$ defined for $1 \leq n \leq 256$ and $0 \leq \theta < 1/2$ (note that this choice of frequency interval fixes the halfband center frequency as $\theta_0 = 1/4$). The model is shown in Fig. 5; we stress that it is not halfband consistent. The spectrum of the result of global signal synthesis without phase matching is shown in Fig. 6(a). This spectrum by far extends beyond the halfband $0 \leq \theta < 1/2$ to which the model is restricted; in fact, the out-of-band components are seen to be even

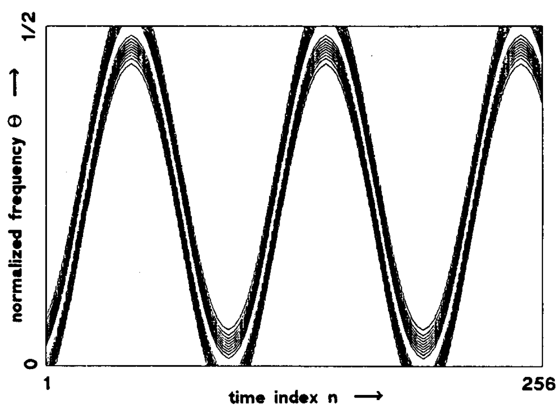

Fig. 5. Time-frequency model function.

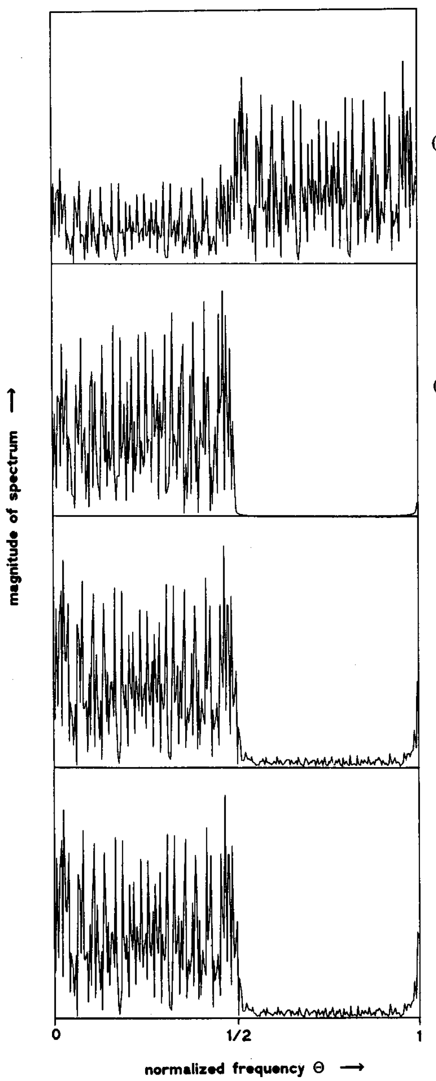

Fig. 6. Simulation results: (a) global signal synthesis without phase matching; (b) halfband signal synthesis; (c) global signal synthesis followed by HAA phase matching; (d) global signal synthesis followed by SSA phase matching.

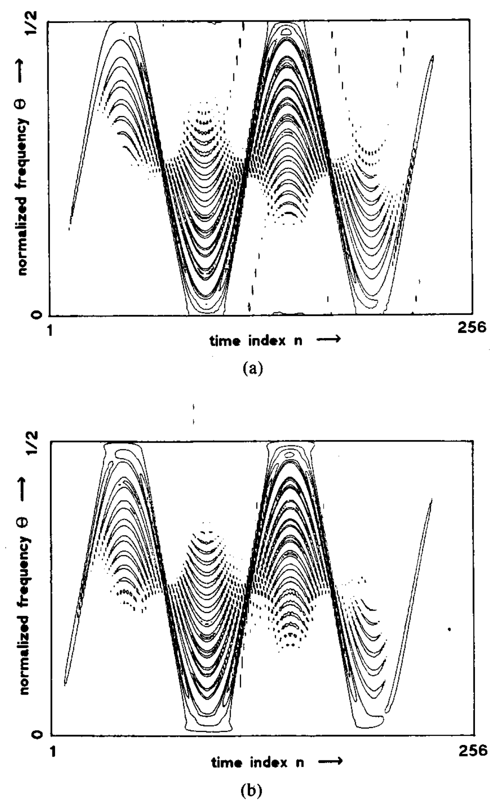

Fig. 7. Simulation results: pseudo-Wigner distribution of results of (a) global signal synthesis and (b) halfband signal synthesis.

stronger than the in-band components. This, of course, is caused by aliasing due to relative phase mismatch (compare (2.12)). The result of halfband signal synthesis is given in Fig. 6(b); it is seen to be properly confined to the halfband $0 \leq \theta < 1/2$. Figs. 6(c) and (d) then present the spectra obtained by global signal synthesis and subsequent phase matching by means of HAA and SSA, respectively. These results are seen to be very similar even though the model is not halfband consistent. It is evident that both autonomous phase matching algorithms succeed fairly well in suppressing the out-of-band components. Figs. 7(a) and (b) show the pseudo-Wigner distribution [1] of the results of global signal synthesis and halfband signal synthesis, respectively. A pseudo-Wigner distribution with a Hamming window of length 99 was used instead of DTWD itself in order to reduce inner interference [8]. Note that the pseudo-Wigner distribution is invariant with respect to phase matching; thus Fig. 7(a) corresponds to the signals of Figs. 6(a), (c) and (d). Finally, Fig. 8 shows the time evolution of the relative phase estimates $\hat{\psi}(n)$ for the on-line versions of HAA and SSA with window $w(k) = u(-k)$. For increasing time index n, the phase estimates are seen to converge to the optimal phases as discussed in Section V.

VII. Conclusion

The two different methods for signal synthesis in the case of discrete-time Wigner distribution (or smoothed versions thereof) give rise to different phase ambiguities of the synthesized signal. While the result of halfband signal synthesis contains only

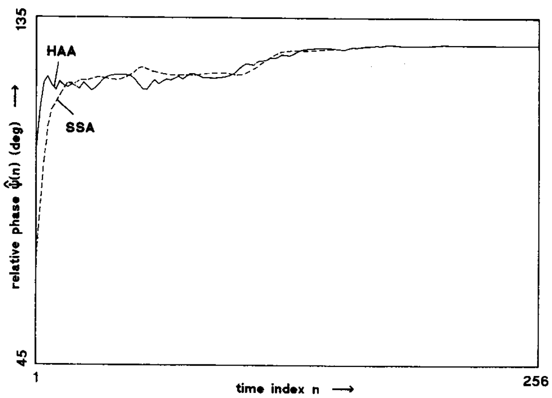

Fig. 8. Simulation results: time evolution of relative phase estimate $\hat{\psi}(n)$ for on-line versions of HAA and SSA.

an "absolute" phase ambiguity which produces a phase rotation of the entire signal and can often be tolerated, the result of global signal synthesis contains, in addition, a "relative" phase ambiguity which produces a phase rotation of even-indexed and odd-indexed signal samples relative to each other and thus gives rise to severe signal distortion. Both phase ambiguities can be resolved by "reference-based" phase matching algorithms, provided that a meaningful reference signal is available. If a reference signal is unavailable, the relative phase ambiguity of global signal synthesis can still be resolved by means of "autonomous" phase matching algorithms. Two optimality criteria for autonomous phase matching have been considered, both of them being motivated by DTWD's aliasing property. These criteria have provided a unified framework in which certain properties of autonomous phase matching have been studied. "On-line" versions of both reference-based and autonomous phase matching algorithms have then been developed; these are compatible with the on-line mode of operation featured by certain synthesis algorithms for pseudo-Wigner distribution. The performance of the phase matching algorithms has finally been assessed by simulation results.

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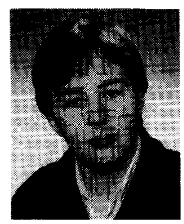

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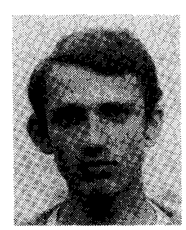

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