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A NEW APPROACH TO TIME-FREQUENCY SIGNAL DECOMPOSITION

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Abstract - A recently developed method for separating time-frequency disjoint signal components is based on Wigner distribution (WD) and uses a time-frequency masking procedure followed by a signal synthesis step. A problem with this method is that the positioning of the time-frequency masks is done manually and may be difficult due to WD interference terms. Also, masking is ineffective if a WD interference term overlaps with the WD signal term to be isolated. We here present an approach to time-frequency signal decomposition which uses an extended version of signal synthesis and does not require a masking procedure.

1. INTRODUCTION

Let $x(t)$ be an N-component signal whose components $x_k(t)$ are (approximately) time-frequency disjoint such that their Wigner distributions (WDs) $W_{x_k}(t,f)$ are essentially nonoverlapping,

$$W_{x_k}(t,f)\, W_{x_l}(t,f) \approx 0 \quad \text{for } k \neq l \text{ and for all } t,f$$

as shown for N=2 in **Fig.1**. (Note that two signals may be time-frequency disjoint without being either time-disjoint or frequency-disjoint.) Given the sum $x(t)$, we want to separate the components $x_k(t)$. A recently proposed method [1],[2] first calculates the WD of $x(t)$ which is given by

$$W_x(t,f) = \sum_{k=1}^{N} W_{x_k}(t,f) + \sum_{\substack{k=1\\(k>l)}}^{N} \sum_{l=1}^{N} I_{x_k,x_l}(t,f) \,, \qquad (1.1)$$

where $I_{x_k,x_l}(t,f)$ denotes the *WD interference term* [3] of $x_k(t)$ and $x_l(t)$. The WDs $W_{x_k}(t,f)$ are then isolated by a masking procedure, and the signal components $x_k(t)$ are finally derived from the WDs $W_{x_k}(t,f)$ by means of signal synthesis [1]. There are two problems associated with this approach: first, the masks must be positioned manually since there does not yet exist an automatic masking algorithm; and second, even if the WDs of the components $x_k(t)$ are disjoint as required, some of them may overlap with interference terms; isolating by simple masking is then impossible.- In this paper, we present an alternative approach to the signal decomposition problem which avoids masking. We first discuss some necessary fundamentals.

Signal product and WD. The (cross) WD [4] of two signals $x(t)$ and $y(t)$ can be written as (integrations are assumed infinite)

$$W_{x,y}(t,f) = \int_\tau q_{x,y}\left(t+\tfrac{\tau}{2}, t-\tfrac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \qquad (1.2)$$

with the (cross) signal product

$$q_{x,y}(t_1,t_2) = x(t_1)\, y^*(t_2)$$

which is readily recovered from WD by

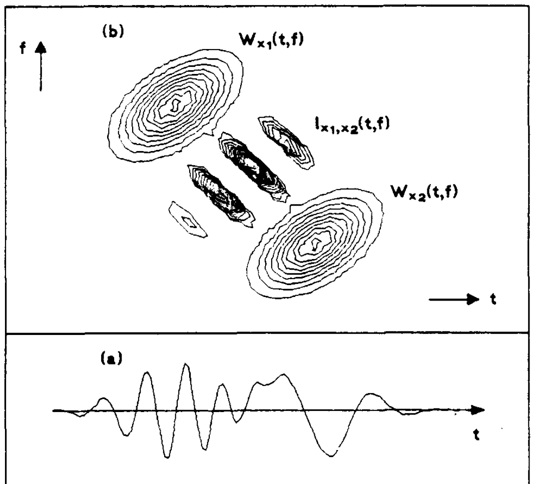

Fig. 1: *(a) Two-component signal $x(t)=x_1(t)+x_2(t)$ with time-frequency disjoint signal components $x_1(t)$, $x_2(t)$; (b) WD of (a). The WDs of $x_1(t)$ and $x_2(t)$ do not overlap.*

$$q_{x,y}(t_1,t_2) = \int_f W_{x,y}\left(\tfrac{t_1+t_2}{2}, f\right) e^{j2\pi(t_1-t_2)f}\, df \,. \qquad (1.3)$$

The *auto* signal product $q_x(t_1,t_2) \triangleq q_{x,x}(t_1,t_2)$ is a hermitian function; as a consequence, the auto WD $W_x(t,f) \triangleq W_{x,x}(t,f)$ is real-valued. The interference term (IT) in (1.1) is given by

$$I_{x,y}(t,f) = W_{x,y}(t,f) + W_{y,x}(t,f) = 2\,\mathrm{Re}\left\{W_{x,y}(t,f)\right\} .$$

Time-frequency disjoint signals. We now state some properties of time-frequency disjoint signals $x(t)$, $y(t)$ satisfying $W_x(t,f)W_y(t,f)=0$. First, the IT of $x(t)$, $y(t)$ is oscillatory and its time-frequency integral is zero [3]. Second, time-frequency disjoint signals are also *orthogonal* which follows from Moyal's formula $(W_x,W_y)=|(x,y)|^2$. For the third property, we define the *time-frequency moment* $\rho_{x,y}$ of $x(t)$ and $y(t)$ as [4]

$$\rho_{x,y} \triangleq \int_t\int_f \left[\left(\tfrac{t}{T}\right)^2 + (Tf)^2\right] W_{x,y}(t,f)\, dt\, df = \frac{1}{T^2}\, m_{x,y} + T^2 M_{x,y} \qquad (1.4)$$

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where $m_{x,y} \triangleq \int_t t^2 x(t) y^*(t)\,dt$, $M_{x,y} \triangleq \int_f f^2 X(f) Y^*(f)\,df$,

and T is an arbitrary time parameter which is assumed fixed. Note that $\rho_{x,y}=\rho^*_{y,x}$ and $\rho_x=\rho_{x,x}\geq 0$. We now show that *$\rho_{x,y}=0$ if x(t) and y(t) are time-frequency disjoint*, i.e., $\rho_{x,y}=0$ is a necessary (though not sufficient) condition for time-frequency disjointness of x(t) and y(t).- Let x(t) and y(t) be time-frequency disjoint as shown in **Fig.2.a**. In general, as in Fig.2.a, the signals are neither time-disjoint nor frequency-disjoint. However, it is easily shown that we can always perform a coordinate transform (essentially a rotation) of the time-frequency plane such that $x(t)\mapsto\tilde{x}(t)$ and $y(t)\mapsto\tilde{y}(t)$ where the new signals $\tilde{x}(t)$ and $\tilde{y}(t)$ are now e.g. *time-disjoint* (see **Fig.2.b**) and the time-frequency moment is unchanged,

$$\rho_{\tilde{x},\tilde{y}} = \rho_{x,y} . \qquad (1.5)$$

Since $\tilde{x}(t)$ and $\tilde{y}(t)$ are time-disjoint, there is $\tilde{x}(t)\tilde{y}^*(t)=0$ and hence $m_{\tilde{x},\tilde{y}}=0$. Applying Parseval's theorem, the frequency moment is

$$M_{\tilde{x},\tilde{y}} = \int_f f^2 \tilde{X}(f)\tilde{Y}^*(f)\,df = \frac{1}{4\pi}\int_t \tilde{x}'(t)\tilde{y}'^*(t)\,dt$$

which is equally zero since there is also $\tilde{x}'(t)\tilde{y}'^*(t)=0$. With (1.4), it follows that $\rho_{\tilde{x},\tilde{y}}=0$ and, using (1.5), it is finally shown that $\rho_{x,y}=0$.

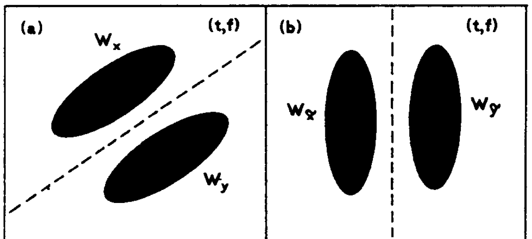

***Fig. 2:** Time-frequency coordinate transform to convert time-frequency disjoint signals (a) into time-disjoint signals (b).*

2. EXTENDED SIGNAL SYNTHESIS

A central part of our signal decomposition scheme is an extension of the signal synthesis problem [1]

$$\| \tilde{W} - W_x \| \rightarrow \min_x \qquad (2.1)$$

where $\tilde{W}(t,f)$ is a given real-valued time-frequency function which, in general, is not a valid WD outcome of any signal; this function will be termed *time-frequency model*. From $\tilde{W}(t,f)$, we can derive a *hermitian* function $\tilde{q}(t_1,t_2)$ as prescribed by the inversion formula (1.3). If $\tilde{q}(t_1,t_2)$ is square-integrable, it allows the spectral decomposition

$$\tilde{q}(t_1,t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) = \sum_{k=1}^{\infty} \lambda_k q_{u_k}(t_1,t_2) \qquad (2.2)$$

with real-valued λ_k and *orthonormal* $u_k(t)$ which are the eigenvalues/eigenfunctions of $\tilde{q}(t_1,t_2)$,

$$\int_{t'} \tilde{q}(t,t')\, u_k(t')\,dt' = \lambda_k u_k(t) . \qquad (2.3)$$

Transforming (2.2) back into the time-frequency domain by means of (1.2), we obtain a fundamental decomposition of our model $\tilde{W}(t,f)$ as a linear combination of valid WD outcomes of orthonormal signals,

$$\tilde{W}(t,f) = \sum_{k=1}^{\infty} \lambda_k W_{u_k}(t,f) . \qquad (2.4)$$

We shall call the λ_k and $u_k(t)$ the eigenvalues/eigensignals of the model $\tilde{W}(t,f)$. If indexing is done such that $\lambda_1\geq\lambda_2\geq\ldots$, and if $\lambda_1\geq 0$, then $x(t)=\sqrt{\lambda_1}u_1(t)$ is the solution of the signal synthesis problem (2.1) [5]. The derivation of *all* eigenvalues λ_k and eigensignals $u_k(t)$ from the model $\tilde{W}(t,f)$ will therefore be termed ***extended signal synthesis***.

We now study some instructive examples. Let

$$x(t) = \sum_{n=1}^{N} x_n(t) = \sum_{n=1}^{N} \sqrt{E_n}\, e_n(t)$$

be an N-component signal with time-frequency disjoint (and hence orthogonal) signal components with energies $E_n=\|x_n\|^2$; $e_n(t)=x_n(t)/\|x_n\|$ denotes the normalized signal components. Suppose, first, that the model consists only of the signal terms, i.e., the signal components' WDs,

$$\tilde{W}(t,f) = \sum_{n=1}^{N} W_{x_n}(t,f) = \sum_{n=1}^{N} E_n W_{e_n}(t,f) .$$

By comparison with (2.4), then, $\lambda_k=E_k$ and $u_k(t)=p_k e_k(t)$ for $1\leq k\leq N$ (note that the eigensignals are only defined up to phase factors $p_k=\exp(j\varphi_k)$) and $\lambda_k=0$ for $k>N$.- As a second example, let the model be the nm-th IT,

$$\tilde{W}(t,f) = I_{x_n,x_m}(t,f) = \sqrt{E_n E_m}\, I_{e_n,e_m}(t,f) .$$

To derive the eigenvalues/eigensignals for this case, we use the identity

$$I_{e_n,e_m} = \frac{1}{2}\left(W_{e_n+e_m} - W_{e_n-e_m}\right) = W_{\frac{e_n+e_m}{\sqrt{2}}} - W_{\frac{e_n-e_m}{\sqrt{2}}}$$

where it is easily shown that the signals $[e_n(t)+e_m(t)]/\sqrt{2}$ and $[e_n(t)-e_m(t)]/\sqrt{2}$ are again orthonormal. Thus, by comparison with (2.4), $\lambda_1=\sqrt{E_n E_m}$, $\lambda_2=-\sqrt{E_n E_m}$, $u_1(t)=p_1[e_n(t)+e_m(t)]/\sqrt{2}$, $u_2(t)=p_2[e_n(t)-e_m(t)]/\sqrt{2}$, and $\lambda_k=0$ for $k>2$.

We can interpret these results as follows: 1) if it were possible to mask out all ITs and retain only the signal terms, then extended signal synthesis would immediately yield the signal components $x_n(t)$ up to unknown phase factors; 2) if it were possible to isolate the nm-th IT, then extended signal synthesis would produce essentially two "rotated versions" of the (normalized) signal components $e_n(t)$ and $e_m(t)$. These results will be generalized in the next section.

3. THE WEIGHTED MODEL

Unfortunately, it is impossible to suppress (mask out) WD terms since they are generally overlaid by other terms. Let us, therefore, consider a model in which WD terms are not suppressed but *weighted*,

$$\tilde{W}(t,f) = \sum_{n=1}^{N} \alpha_{nn} W_{x_n}(t,f) + \sum_{n=1}^{N} \sum_{\substack{m=1 \\ (n>m)}}^{N} \alpha_{nm} I_{x_n,x_m}(t,f) , \qquad (3.1)$$

with arbitrary real-valued weighting coefficients α_{nm}. Any type of suppression is obviously a special case (with $\alpha_{nm}=0$ for some values of n,m). We note that, while the weighting (3.1) is impossible just as the suppression itself, it may *sometimes* be approximated fairly well (see Section 6).- Introducing

$$b_{nm} \triangleq \alpha_{nm}\sqrt{E_n E_m}\ , \qquad b_{mn} \triangleq b_{nm}\ , \qquad (3.2)$$

the model can be written as

$$\tilde{W}(t,f) = \sum_{n=1}^{N}\sum_{m=1}^{N} b_{nm}\, W_{e_n,e_m}(t,f)$$

whence, with (1.3),

$$\tilde{q}(t_1,t_2) = \sum_{n=1}^{N}\sum_{m=1}^{N} b_{nm}\, q_{e_n,e_m}(t_1,t_2) = \sum_{n=1}^{N}\sum_{m=1}^{N} b_{nm}\, e_n(t_1)\, e_m^*(t_2)$$

$$= \underline{e}^+(t_2)\,\underline{B}\,\underline{e}(t_1) \qquad (3.3)$$

with the vector $\underline{e}(t) \triangleq (e_n(t))$ and the symmetric matrix $\underline{B} \triangleq (b_{nm})$ ("+" stands for complex transposition). The bilinear form (3.3) can be diagonalized by inserting

$$\underline{B} = \underline{C}\,\underline{\Lambda}\,\underline{C}^+$$

where the diagonal elements λ_k of the diagonal matrix $\underline{\Lambda}$ are the (real-valued) eigenvalues, and the columns $\underline{c}_k$ of the orthogonal (real-valued) matrix $\underline{C}$ are the associated (orthonormal) eigenvectors of the symmetric matrix $\underline{B}$. We thus obtain

$$\tilde{q}(t_1,t_2) = \underline{u}^+(t_2)\,\underline{\Lambda}\,\underline{u}(t_1) = \sum_{k=1}^{N} \lambda_k\, q_{u_k}(t_1,t_2)$$

with

$$\underline{u}(t) = \underline{C}^+\underline{e}(t)\ , \qquad u_k(t) = \underline{c}_k^+\,\underline{e}(t) = \sum_{n=1}^{N} c_{nk}^*\, e_n(t)\ . \qquad (3.4)$$

Based on the orthonormality of the $e_n(t)$ and the orthogonality of $\underline{C}$, it is easily shown that the $u_k(t)$ are equally orthonormal. By comparison with (2.2), then, we see that λ_k and $u_k(t)$ are the eigenvalues/eigensignals of the weighted model (3.1).

To summarize, the N nonzero eigenvalues λ_k of the weighted model (3.1) are the eigenvalues of $\underline{B}$, and the eigensignals $u_k(t)$ are given by $u_k(t)=\sum_n c_{nk}^* e_n(t)$, where the coefficients c_{nk} are the elements of the k-th eigenvector $\underline{c}_k$ of $\underline{B}$. Extended signal synthesis here does not yield our original signal components $e_n(t)$ but an orthonormal basis $\{u_k(t)\}$ of the signal space spanned by the $e_n(t)$, where the new basis $\{u_k(t)\}$ is derived from the original basis $\{e_n(t)\}$ by an orthogonal transformation, i.e., a *rotation*. *Note that the original signal components* $e_n(t)$ are time-frequency disjoint by assumption while the "rotated" versions $u_k(t)$ are generally not time-frequency disjoint (but still orthonormal).

4. INVERSE BASIS ROTATION

Theoretically, we could re-derive our desired basis $\{e_n(t)\}$ from the basis $\{u_k(t)\}$ by inversion of (3.4). However, in practice the matrix $\underline{B}$, and thus also the transformation matrix $\underline{C}$, is unknown. All we know is that $\underline{e}(t)=\underline{C}\,\underline{u}(t)$ with some orthogonal but unknown matrix $\underline{C}$. To see how we may yet obtain the correct rotation matrix, let

$$\underline{v}(t) = \underline{D}^+\underline{u}(t) \qquad (4.1)$$

with $\underline{D}$ being orthogonal (the "+" in (4.1) is for notational reasons). We want to choose $\underline{D}$ such that $\underline{v}(t)=\underline{e}(t)$ results. Defining the *moment matrix* $\underline{P}_v$ of $\underline{v}(t)$ by $\underline{P}_v=(\rho_{v_n,v_m})$ and the moment matrix $\underline{P}_u$ of $\underline{u}(t)$ accordingly, it follows from (4.1) that

$$\underline{P}_v = \underline{D}^+\underline{P}_u\,\underline{D}\ .$$

If $\underline{v}(t)=\underline{e}(t)$, then $\rho_{v_n,v_m}=\rho_{e_n,e_m}=0$ for $n\neq m$ since the $e_n(t)$ were assumed *time-frequency disjoint* (note that zero time-frequency moment was shown in Chapter 1 to be necessary for time-frequency disjointness); the moment matrix $\underline{P}_v$ must then be diagonal. Hence the correct matrix $\underline{D}$ yielding $\underline{v}(t)=\underline{e}(t)$ diagonalizes $\underline{P}_v$; *its columns* $\underline{d}_n$ are thus the eigenvectors of $\underline{P}_u$.

5. THE IDEALIZED METHOD

The derivation given so far can be summarized to yield the following "idealized" (since not realizable) signal separation method. As above, we suppose that we have given an N-component signal $x(t)$ whose components $x_n(t)$ are (nearly) time-frequency disjoint. ***Step 1:*** Calculate the WD of $x(t)$ and apply real-valued weights α_{nm} to signal terms and ITs as in (3.1). This generally yields a time-frequency model $\tilde{W}(t,f)$ which is not a valid WD outcome. ***Step 2:*** Calculate the N nonzero eigenvalues λ_k and the associated eigensignals $u_k(t)$ of $\tilde{W}(t,f)$. ***Step 3:*** Form the moment matrix $\underline{P}_u$ of the eigensignals $u_k(t)$ and calculate the eigenvectors $\underline{d}_k$ of $\underline{P}_u$. ***Step 4:*** Apart from an unknown phase factor p_n, the normalized n-th signal component $e_n(t)$ is $\tilde{e}_n(t)=p_n e_n(t)=\sum_k d_{kn}^* u_k(t)$. ***Step 5:*** The signal component $x_n(t)$ is finally given by $x_n(t)=a_n\tilde{e}_n(t)$ with a complex coefficient a_n which accounts for both the normalization and unknown phase of $\tilde{e}_n(t)$. By virtue of $x(t)=\sum_n x_n(t)=\sum_n a_n\tilde{e}_n(t)$ and the orthonormality of the $\tilde{e}_n(t)$, this coefficient is derived from $\tilde{e}_n(t)$ and the overall signal $x(t)$ according to $a_n=(x,\tilde{e}_n)$.

We note that the weighting coefficients α_{nm} can be chosen rather arbitrarily; we only have to require that the matrix $\underline{B}$ defined by (3.2) be nonsingular. A complication occurs if two eigenvalues of the moment matrix $\underline{P}_u$ happen to be equal since the associated eigenvectors are then ambiguous. This can be easily detected and resolved by repeating Step 3 using a different time parameter T in the moment definition (1.4).

6. THE PRACTICAL METHOD

In practice, weighting of WD terms as required by Step 1 is impossible in general. However, due to the oscillatory shape of ITs [3], a *time-frequency smoothing* will generally result in IT attenuation without severely affecting signal terms. On certain conditions, this then approximates a weighting of ITs. To arrive at a realizable signal separation method, we thus propose to replace Step 1 of the idealized method by the simple smoothing operation

$$\tilde{W}(t,f) = \int_{t'}\int_{f'} H(t-t',f-f')\,W_x(t',f')\,dt'df' \qquad (6.1)$$

where $H(t,f)$ is some smoothing kernel. Note that the smoothing need not be massive since *suppression* of ITs is not required.

Performance and applicability of the practical method are limited by the parasitic effects of smoothing. 1. If an IT's oscillation is not homogeneous, i.e., oscillation frequency and/or direction change over the IT's time-frequency support, then smoothing will *deform* the IT instead of simply weighting it. Besides, smoothing always deforms (broadens) the non-oscillatory signal terms and the IT's envelope. Any a priori knowledge regarding the IT's oscillation should be used to optimize the smoothing such that the deformation of ITs or signal terms is minimal. 2. For the *ideal* model (3.1), the number of nonzero eigenvalues equals the number N of signal components. With the practical model (6.1), however, the parasitic effects of smoothing create an infinite number of eigenvalues. Only the N most significant eigensignals (corresponding to the N largest eigenvalues) can be used for further processing. This is a problem when the number N of signal components is not known a priori. 3. A problem that applies equally well to the idealized method is that the signal components encountered in practice will seldom be exactly orthogonal or time-frequency disjoint. Both idealized and practical method then introduce errors since they yield signals which are strictly orthogonal and whose time-frequency cross moments are exactly zero.

Fig. 3 shows a simulation result where the practical method has worked fairly well. The signal here consists of two chirps with gaussian envelope which are sufficiently distant in the time-frequency plane so that the condition of time-frequency disjointness is approximately satisfied. Note that the applied smoothing does not totally suppress the IT; also, it does not significantly deform the IT since the ITs oscillation is homogeneous. For $k > N=2$, some additional eigenvalues exist as a result of the broadening effect of smoothing; however, these extra eigenvalues are small enough so that there cannot be any doubt concerning the number of signal components. This simulation example, however, is not meant to obscure the fact that the practical method will *not* work well if the ITs' oscillations are not sufficiently homogeneous (in these cases, the time-frequency invariant smoothing (6.1) is obviously insufficient and more sophisticated smoothing schemes would be necessary).

On practical implementation of the signal separation method, the WD of x(t) need not be calculated since the smoothing (6.1) can be directly done using the signal product of x(t): it is easily shown that the function $\mathfrak{q}(t_1,t_2)$ (from which the eigenvalues/eigensignals are derived according to (2.3)) is given by

$$\mathfrak{q}(t_1,t_2) = \int_{t'} h\left(\frac{t_1+t_2}{2}-t', t_1-t_2\right) x\left(t'+\frac{t_1-t_2}{2}\right) x^*\left(t'-\frac{t_1-t_2}{2}\right) dt'$$

with $h(t,\tau) = \int_f H(t,f)\, e^{j2\pi\tau f}\, df$.

7. CONCLUSION

We have presented a method for the approximate separation of time-frequency disjoint signal components which consists of time-frequency smoothing, extended signal synthesis and basis rotation involving diagonalization of a time-frequency moment matrix. In contrast to an existing method involving the Wigner distribution, our method is automatic since it does not require a time-frequency masking procedure. A simulation result has been provided, and some problems and performance limitations of our method have been pointed out.

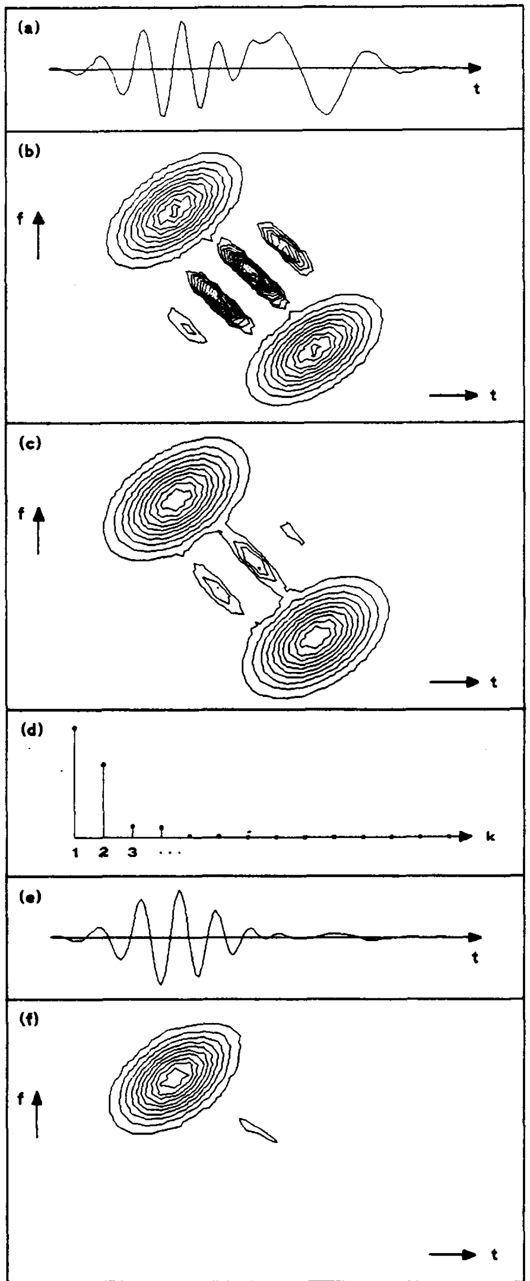

Fig. 3: Simulation result (practical method): (a) two-component signal $x(t)=x_1(t)+x_2(t)$; (b) WD of (a); (c) time-frequency model $\tilde{W}(t,f)$ (smoothed version of (b)); (d) eigenvalues λ_k of $\tilde{W}(t,f)$; (e) separation result for $x_1(t)$; (f) WD of (e)

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