2. THE PWD SIGNAL SYNTHESIS PROBLEM

WD and PWD. The (discrete-time) WD of a signal \( x(n) \) is defined as [1]

\[
WD_x(n,\theta) = \sum_k x(n+k) x^*(n-k) e^{-2\pi i k \theta},
\]

where \( n \) is discrete time and \( \Theta \) is normalized angular frequency; summation is infinite unless otherwise indicated. WD has unique mathematical properties, but its practical application is restricted to "short" signals for three reasons: 1) When calculating WD, all signal samples have to be known simultaneously, i.e., the total signal has to be stored; 2) The summation length in (2.1) equals the signal length; 3) Signal components give rise to WD interference terms [7] irrespective of their time distance; these interference terms may make WD results unreadable. All these problems are avoided or alleviated by a short-time WD version known as PWD and defined as

\[
PWD_x^{(h)}(n,\theta) = \sum_k x(n+k) x^*(n-k) h^2(k) e^{-2\pi i k \theta},
\]

where \( h(k) \) is a real-valued, even window of length \( 2K+1 \); we assume \( \text{h}(0) = 1 \). WD and PWD are real-valued and \( \pi \)-periodic w.r.t. \( \Theta \). Formally, WD is a PWD with \( K = \infty \) and \( h(n) = 1 \).

PWD signal synthesis. The problem of optimal PWD signal synthesis is formulated as follows: given a real-valued model \( Y(n,\theta) \) defined for \( \pi/2 \leq \Theta \leq \pi/2 \), find the signal \( x(n) \) whose PWD (with given window \( h \)) is closest to this model, i.e., which minimizes the error norm

\[
N^2 = \int_{-\pi/2}^{\pi/2} \left| Y(n,\theta) - PWD_x^{(h)}(n,\theta) \right|^2 d\theta.
\]

Using Parseval's theorem and separating the even-indexed signal samples \( x_e(n) = x(2n) \) and the odd-indexed signal samples \( x_o(n) = x(2n+1) \), this error norm can be rewritten as

\[
N^2 = N_e^2 + N_o^2,
\]

where

\[
N_e^2 = \sum_{l=1}^{K} \sum_{m=1}^{K} \left| y_e(l,m) - \bar{y}_e(l) \bar{y}_o(m) h^2(l-m) \right|^2
\]

with

\[
y_e(l,m) = y(l+m,n-m), \quad y(n,k) = 1/2 \int_{-\pi/2}^{\pi/2} Y(n,\theta) e^{2\pi i k \theta} d\theta;
\]

a similar expression involving \( x_o(n) \) holds for \( N_o^2 \). According to (2.2), the minimization problem \( N^2 \rightarrow \text{min} \) splits up into two separate and independent minimization problems \( N_e^2 \rightarrow \text{min} \) and \( N_o^2 \rightarrow \text{min} \) yielding the even- and odd-indexed signal samples,
respectively (note that this decoupling of even and odd indices causes troublesome phase ambiguities in the synthesis result [2][3]). As the two minimization problems have identical structures, we shall further consider minimization for \( y \) only. Letting the gradient of (2.3) be zero [2] leads to the following necessary condition for the synthesis solution:

\[
\sum_{m \in \mathbb{Z}} \left[ \gamma_x(i, m) - y_x(i) x_n^*(m) h^2(i-m) \right] x_n^*(m) h^2(i-m) = 0 \, . \tag{2.4}
\]

This equation must be satisfied for all \( i \).

3. The Pseudo Power Method

Since a closed-form solution of the third-order equation (2.4) does not seem to be available, we try to solve (2.4) iteratively. Our iteration scheme is motivated by an iterative synthesis method for WD.

**Optimal WD synthesis – the power method.** For WD (PVD with \( K = 1 \) and \( h(n)=1 \), eq. (2.4) reduces to

\[
\sum_{m \in \mathbb{Z}} \gamma_x(i, m) x(m) = \left( \sum_{m \in \mathbb{Z}} |x(m)|^2 \right) x(i)
\]

(the index \( e \) has been suppressed for the sake of simplicity) or, with obvious vector-matrix notation, to the eigenvector-eigenvalue equation

\[
\mathbf{y} \mathbf{x} = |\mathbf{x}|^2 \mathbf{x} \quad . \tag{3.1}
\]

It is thus necessary that the optimal \( \mathbf{x} \) is an eigenvector of the (constant) matrix \( \mathbf{y} \), with \( |\mathbf{x}|^2 \) equal to the corresponding eigenvalue, and it can be shown that the error norm is minimized if the maximal eigenvalue is taken [2]. The following iterative scheme yields the locked-for vector: starting with some initial vector \( \mathbf{x}_0 \), calculate

\[
1) \quad \mathbf{z}_{n+1} = \mathbf{y} \mathbf{x}_n / |\mathbf{x}_n| \quad ; \tag{3.2}
\]

\[
2) \quad \mathbf{x}_{n+1} = \sqrt{\mathbf{z}_{n+1}} \quad . \tag{3.3}
\]

This, in fact, is essentially the well-known power method for calculating the eigenvector corresponding to the maximal eigenvalue; the normalization has been chosen such that, after convergence, \( |\mathbf{x}|^2 \) equals that eigenvalue.

**Optimal PVD synthesis – the pseudo power method.** In the general PVD case, eq. (2.4) can be written similar to (3.1),

\[
\mathbf{y}_n \mathbf{x} = |\mathbf{x}|^2 \mathbf{x} \quad , \tag{3.4}
\]

where, unlike the WD case, the matrix \( \mathbf{y}_n \) now depends on \( \mathbf{x} \) according to

\[
\mathbf{y}_n = \mathbf{y} + \mathbf{D}_x \mathbf{H} \mathbf{D}^*_x \quad , \tag{3.5}
\]

with

\[
(\mathbf{y}^n)_{lm} = y(l, m) h^2(i-m) \quad ,
\]

\[
(\mathbf{H})_{lm} = 1 - h^2(i-m) \quad ,
\]

\[
(\mathbf{D}_x)_{lm} = x(m) h^2(i-m) \quad .
\]

To solve (3.4) iteratively, we use the power-method recursion (3.2), (3.3) but update the matrix \( \mathbf{y}_n \) in each step according to (3.5):

1) \( \mathbf{y}_n \mathbf{x} = \mathbf{y}_n \mathbf{x} + \mathbf{D}_x \mathbf{H} \mathbf{D}^*_x \quad ; \tag{3.6}
\]

2) \( \mathbf{z}_{n+1} = \mathbf{y} \mathbf{x}_n / |\mathbf{x}_n| \quad ; \tag{3.7}
\]

3) \( \mathbf{x}_{n+1} = \sqrt{\mathbf{z}_{n+1}} \quad . \tag{3.8}
\]

We call this the PPM (Pseudo Power Method). When the iteration (3.6)–(3.8) converges, \( \mathbf{x}_{n+1} \rightarrow \mathbf{x} \), the resulting vector \( \mathbf{x} \) is guaranteed to solve (3.4) since

\[
\mathbf{x}_{n+1} \mathbf{x} = \mathbf{y} \mathbf{x} = |\mathbf{x}|^2 \mathbf{x} \quad .
\]

In experiments, convergence has invariably been observed with practically arbitrary initial vector \( \mathbf{x}_0 \). Note, also, that the power method for WD synthesis (where convergence is guaranteed) is a special case of the PPM with \( h(n)=1 \). The convergence speed of the PPM strongly depends on the WD window length: larger windows (i.e., closer similarity to WD) yield faster convergence. Storage requirements and computation per iteration of the PPM are \( O(L) \), where \( L \) is the length of the synthesized signal and thus also the dimension of matrices and vectors. This sets practical limits to the signal length. In principle, PPM can be adapted to longer signals by a segment-by-segment mode of operation but then a method for segment combination has to be found and, anyway, optimality is lost.

A **PPM synthesis experiment.** Fig. 1 shows the synthesis of a signal of length \( L=100 \); the PWD window is a Hamming window of length \( 2K+1=63 \). A noise signal is used for the initial vector \( \mathbf{x}_0 \). From the iteration signals \( \mathbf{x}_n \) and the sequence of error norms, it is seen that convergence is essentially complete after some fifteen iterations. The non-zero residual error \( N_{res} \) is due to the fact that the model is not a valid PWD.

4. The Partial Sum Method

The following suboptimal algorithm, termed partial sum method (PSM), is particularly suited for the synthesis of long signals since it operates sample-by-sample. Computational expense per synthesized sample is independent of the total signal length, and only a local model interval must be known (stored) at any time.

The error norm of even-indexed signal samples can be rewritten as

\[
N^2_{re} = \sum |\Delta N_{re}^2(i)_e| \quad ,
\]

where

\[
\Delta N_{re}^2(i)_e = 4 \sum_{m \in \mathbb{Z}} \left[ \sum_{l=0}^{i-1} \left| y_x(l, m) - y_x(l) x_n^*(m) h^2(i-m) \right|^2 + 2 \left| y_x(l, i) - y_x(l) x_n^*(i) \right|^2 \right] \quad , \tag{4.1}
\]

will be termed causal error component since it only depends on the signal samples \( y_x(n) \) for \( n \leq i \). Suppose, now, that the synthesized signal \( y_n(m) \) is already known for \( m \leq i-1 \). Based on this knowledge, we calculate the sample \( y_x(i) \) such that the \( i \)-th causal error component (4.1) is minimized, \( \Delta N_{res}^2(i)-m \) has as a necessary condition

\[
\sum_{m \in \mathbb{Z}} \left[ \sum_{l=0}^{i-1} \left| y_x(l, m) - y_x(l) x_n^*(m) h^2(i-m) \right| x_n^*(m) h^2(i-m) \right] = 0 \quad . \tag{4.2}
\]

This third-order equation has to be solved for \( y_x(i) \). Letting

\[
\left[ y_x(i, m) - y_x(i) x_n^*(m) h^2(i-m) \right] x_n^*(m) h^2(i-m) = 0 \quad .
\]
Fig. 1: A PPM Synthesis Experiment

a) time-frequency model
b) start signal $x_0$ and its PWD
c)–e) iteration signals $x_2, x_4, x_{15}$ and their PWDs
f) convergence of error norm

$$p = \sum_{m=-K}^{K} |x_0(m)| \tilde{h}^2(|m|)^2 \cdot y_w(i,i) \in \mathbb{R}$$
$$q = \sum_{m=-K}^{K} y_w(i,m) \cdot x_0(m) \cdot \tilde{h}^2(|m|),$$
magnitude and phase of $x_w(i)$ are given by
$$|x_w(i)|^2 + p|x_w(i)| - |q| = 0,$$
$$\text{Arg}(x_w(i)) = \text{Arg}(q) - \text{Arg}(p + |x_w(i)|^2).$$

It follows from $p \in \mathbb{R}$ and $|q| > 0$ that (4.3) has a unique real-valued and non-negative solution
$$|x_w(i)| = \sqrt{\frac{q^2}{4} + \frac{p^2}{q^2}}.$$

With the PSM, the samples of the synthesized signal minimize individual error norms $\Delta N_w(i)$ which are local and causal; the total error $N_{\text{re}}$, on the other hand, is generally not minimized. PSM results are thus suboptimal. Note that the necessary condition (4.2) of PSM equals the necessary condition (2.4) of optimal synthesis apart from the fact that the summation range $-K \leq m \leq K$ of (2.4) has been replaced by the partial (causal) range $-K \leq m$ ("partial sum"). In general, the optimal synthesis solution will not be a solution of (4.2); however, in
the special case of signal reconstruction where the model \( Y(n,\theta) \) is a valid PWD, the optimal solution satisfies

\[
y_n(i,m) - x_n(i) x^*_n(i) h^2(i-m) = 0
\]

and is thus also a solution of (4.2). The PSM is hence an exact method for signal reconstruction from valid PWDs and an approximate method in the signal synthesis case where the model is non-valid.

A simplified version of the PSM is obtained if the summation range \( i = \text{K} \text{min} \) of (4.2) is replaced by \( i = \text{K} \text{min} - 1 \); (4.2) is then linear in \( x_n(i) \) and can directly be solved by

\[
x_n(i) = \sum_{m=K}^{K+1} x_n(m) x^*_n(m) h^2(i-m) / \sum_{m=K}^{K+1} |x_n(m) h(i-m)|^2
\]

The synthesis performance of this simplified method is slightly inferior to the original PSM. It has been compared with the “overlapping method” of [4] in [5].

A PSM synthesis experiment. Fig. 2 demonstrates the synthesis of 500 signal samples from a model which simulates the PWD of a sinusoidal FM signal; the model is clearly not a valid PWD since inner interference terms [7] are lacking. The PWD window was defined to be a Hamming window of length 2K+1=63. To start the PSM recursion, the first K=31 signal samples have to be initialized; these were chosen as \( x_n(n)=0 \) for \( 1 \leq n \leq 30 \) and \( x_n(31)=1 \). It is seen that the PSM duly adapts to the model in spite of these extremely faulty initial values.

5. CONCLUSION

Two quite different signal synthesis algorithms for PWD have been presented: the pseudo power method (PPM) yields optimal synthesis results but synthesizes all signal samples simultaneously, using the entire model. The PPM is thus best suited for offline processing of short signal records. The partial sum method, on the other hand, is suboptimal but operates recursively and sample-by-sample, using a local model segment only. In this sense, the PSM is similar to the way PWD is calculated. With the PSM, signal processing schemes can be devised which allow sequential time-frequency processing of signals with unrestricted length; the PSM is thus suited for real-time applications.

Acknowledgment

The authors are indebted to Prof. W. Mecklenbräuker whose suggestions have substantially contributed to the development of the PPM.

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