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TWO SIGNAL SYNTHESIS ALGORITHMS FOR PSEUDO WIGNER DISTRIBUTION

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Abstract - The *pseudo Wigner distribution* (PWD) is a time-frequency signal representation particularly suited for analyzing and processing "long" signals. Signal processing by means of PWD involves a *signal synthesis* step. This paper presents two signal synthesis algorithms for PWD, the *pseudo power method* which allows optimal signal synthesis but is computationally expensive for longer signals, and the *partial sum method* which is suboptimal but suited for the synthesis of signals with arbitrary length. The performance of the two algorithms is demonstrated by simple synthesis experiments.

1. INTRODUCTION

Wigner distribution (WD) is a time-frequency signal representation which has been successfully used for signal analysis, detection and estimation, and signal processing. In the case of "long" signals (e.g. speech), however, a short-time modification of WD known as *pseudo Wigner distribution* (PWD) is better suited [1]. To perform *signal processing* by means of PWD, the PWD of the *input signal* is modified in some meaningful manner, yielding a new time-frequency function that is generally not a valid PWD of any signal. The *output signal* is then *synthesized* from this "non-valid" function (called *model* henceforth). Ideally, the synthesized signal is such that its PWD is closest (in a least-squares sense) to the model; this will be referred to as *optimal synthesis* [2]. Unfortunately, optimal signal synthesis requires the entire model to be known; all signal samples must be synthesized simultaneously, rather than sample-by-sample. This sets practical limits to the optimal synthesis of long signals and, at the same time, establishes the importance of suboptimal synthesis schemes with sequential mode of operation.

While methods for optimal signal synthesis are known for WD [2],[3], no optimal PWD synthesis algorithm has been reported so far. Two *suboptimal* algorithms have been presented in [4]: 1) The "outer product approximation" synthesizes longer signals on a segment-by-segment basis. A difficulty with this scheme (not discussed in [4]) seems to be the adequate combination of the synthesized segments. 2) The "overlapping method" has the advantage of synthesizing signals recursively and sample-by-sample; as demonstrated in [5], however, synthesis results may be very poor.

This paper presents two new PWD signal synthesis algorithms. The *pseudo power method* (PPM; briefly presented in [6]) is an *optimal* synthesis method where all signal samples are synthesized simultaneously. The *partial sum method* (PSM) is a suboptimal scheme for recursive sample-by-sample synthesis. While practical applications of the PPM will be limited to "short" signals, the PSM is suited for synthesizing signals with arbitrary length.

2. THE PWD SIGNAL SYNTHESIS PROBLEM

WD and PWD. The (discrete-time) WD of a signal x(n) is defined as [1]

$$WD_x(n,\Theta) = 2\sum_k x(n+k)\,x^*(n-k)\,e^{-j2k\Theta} , \qquad (2.1)$$

where n is discrete time and Θ is normalized angular frequency; summation is infinite unless otherwise indicated. WD has unique mathematical properties, but its practical application is restricted to "short" signals for three reasons: 1) When calculating WD, all signal samples have to be known simultaneously, i.e. the total signal has to be stored; 2) The summation length in (2.1) equals the signal length; 3) Signal components give rise to WD *interference terms* [7] irrespective of their time distance; these interference terms may make WD results unreadable. All these problems are avoided or alleviated by a short-time WD version known as PWD and defined as

$$PWD_x^{(h)}(n,\Theta) = 2\sum_{k=-K}^{K} x(n+k)\,x^*(n-k)\,h^2(k)\,e^{-j2k\Theta} ,$$

where h(k) is a real-valued, even window of length 2K+1; we assume h(0)=1. WD and PWD are real-valued and π-periodic w.r.t. Θ. Formally, WD is a PWD with $K=\infty$ and $h(n)\equiv1$.

PWD signal synthesis. The problem of optimal PWD signal synthesis is formulated as follows: given a real-valued model $Y(n,\Theta)$ defined for $-\pi/2\le\Theta<\pi/2$, find the signal x(n) whose PWD (with given window h) is closest to this model, i.e. which minimizes the error norm

$$N_x^2 = \sum_n \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \left| Y(n,\Theta) - PWD_x^{(h)}(n,\Theta)\right|^2 d\Theta .$$

Using Parseval's theorem and separating the even-indexed signal samples $x_e(n)=x(2n)$ and the odd-indexed signal samples $x_o(n)=x(2n-1)$, this error norm can be rewritten as

$$N_x^2 = N_{x_e}^2 + N_{x_o}^2 , \qquad (2.2)$$

where

$$N_{x_e}^2 = 2\sum_i \sum_{m=i-K}^{i+K} \left| y_e(i,m) - x_e(i)\,x_e^*(m)\,h^2(i-m)\right|^2 \qquad (2.3)$$

with

$$y_e(i,m) = y(i+m,i-m) , \qquad y(n,k) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} Y(n,\Theta)\,e^{j2k\Theta}\,d\Theta ;$$

a similar expression involving $x_o(n)$ holds for $N_{x_o}^2$. According to (2.2), the minimization problem $N_x\to$min splits up into two separate and independent minimization problems $N_{x_e}\to$min and $N_{x_o}\to$min yielding the even- and odd-indexed signal samples,

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respectively (note that this decoupling of even and odd indices causes troublesome phase ambiguities in the synthesis result [2],[3]). As the two minimization problems have identical structures, we shall further consider minimization for x_e only. Letting the gradient of (2.3) be zero [2] leads to the following necessary condition for the synthesis solution:

$$\sum_{m=i-K}^{i+K} \left[y_e(i,m) - x_e(i)\, x_e^*(m)\, h^2(i-m) \right] x_e(m)\, h^2(i-m) = 0 \quad . \tag{2.4}$$

This equation must be satisfied for all i.

3. THE PSEUDO POWER METHOD

Since a closed-form solution of the third-order equation (2.4) does not seem to be available, we try to solve (2.4) iteratively. Our iteration scheme is motivated by an iterative synthesis method for WD.

Optimal WD synthesis - the power method. For WD (PWD with $K=\infty$ and $h(n)\equiv 1$), eq. (2.4) reduces to

$$\sum_m y(i,m)\,x(m) = \Big(\sum_m |x(m)|^2\Big)\,x(i)$$

(the index e has been suppressed for the sake of simplicity) or, with obvious vector-matrix notation, to the eigenvector-eigenvalue equation

$$\mathbf{Y}\mathbf{x} = \|\mathbf{x}\|^2\,\mathbf{x} \quad . \tag{3.1}$$

It is thus necessary that the optimal $\mathbf{x}$ is an eigenvector of the (hermitian) matrix $\mathbf{Y}$, with $\|\mathbf{x}\|^2$ equal to the corresponding eigenvalue, and it can be shown that the error norm is minimized if the *maximal* eigenvalue is taken [2]. The following iterative scheme yields the looked-for vector: starting with some initial vector $\mathbf{x}_o$, calculate

$$1)\quad \mathbf{z}_{n+1} = \mathbf{Y}\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \quad ; \tag{3.2}$$

$$2)\quad \mathbf{x}_{n+1} = \frac{\mathbf{z}_{n+1}}{\sqrt{\|\mathbf{z}_{n+1}\|}} \quad . \tag{3.3}$$

This, in fact, is essentially the well-known *power method* for calculating the eigenvector corresponding to the maximal eigenvalue; the normalization has been chosen such that, after convergence, $\|\mathbf{x}\|^2$ equals that eigenvalue.

Optimal PWD synthesis - the pseudo power method. In the general PWD case, eq. (2.4) can be written similar to (3.1),

$$\mathbf{Y}_x^{(h)}\mathbf{x} = \|\mathbf{x}\|^2\,\mathbf{x} \quad , \tag{3.4}$$

where, unlike the WD case, the matrix $\mathbf{Y}_x^{(h)}$ now depends on $\mathbf{x}$ according to

$$\mathbf{Y}_x^{(h)} = \mathbf{Y}^{(h)} + \mathbf{D}_x\mathbf{H}\mathbf{D}_x^* \tag{3.5}$$

with $(\mathbf{Y}^{(h)})_{im} = y(i,m)\,h^2(i-m)$,

$(\mathbf{H})_{im} = 1 - h^4(i-m)$,

$(\mathbf{D}_x)_{im} = x(m)\,\delta_{im}$.

To solve (3.4) iteratively, we use the power-method recursion (3.2), (3.3) but update the matrix $\mathbf{Y}_x^{(h)}$ in each step according to (3.5):

$$1)\quad \mathbf{Y}_{x_n}^{(h)} = \mathbf{Y}^{(h)} + \mathbf{D}_{x_n}\mathbf{H}\mathbf{D}_{x_n}^* \quad ; \tag{3.6}$$

$$2)\quad \mathbf{z}_{n+1} = \mathbf{Y}_{x_n}^{(h)}\frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \quad ; \tag{3.7}$$

$$3)\quad \mathbf{x}_{n+1} = \frac{\mathbf{z}_{n+1}}{\sqrt{\|\mathbf{z}_{n+1}\|}} \quad . \tag{3.8}$$

We call this the *pseudo power method* (PPM). When the iteration (3.6)-(3.8) converges, $\mathbf{x}_{n+1}=\mathbf{x}_n=\mathbf{x}$, the resulting vector $\mathbf{x}$ is guaranteed to solve (3.4) since

$$\mathbf{x}_{n+1}=\mathbf{x}_n=\mathbf{x} \quad \Rightarrow \quad \mathbf{Y}_x^{(h)}\mathbf{x} = \|\mathbf{x}\|^2\,\mathbf{x} \ .$$

In experiments, convergence has invariably been observed with practically arbitrary initial vector $\mathbf{x}_o$. Note, also, that the *power method* for WD synthesis (where convergence is guaranteed) is a special case of the PPM with $h(n)\equiv 1$. The convergence speed of the PPM strongly depends on the PWD window length: *longer* windows (i.e., closer similarity to WD) yield faster convergence. Storage requirements and computation per iteration of the PPM are $O(L^2)$, where L is the length of the synthesized signal and thus also the dimension of matrices and vectors. This sets practical limits to the signal length. In principle, PPM can be adapted to longer signals by a segment-by-segment mode of operation but then a method for segment combination has to be found and, anyway, optimality is lost.

A PPM synthesis experiment. Fig. 1 shows the synthesis of a signal of length L=100; the PWD window is a Hamming window of length 2K+1=63. A noise signal is used for the initial vector $\mathbf{x}_o$. From the iteration signals $\mathbf{x}_n$ and the sequence of error norms, it is seen that convergence is essentially complete after some fifteen iterations. The non-zero residual error $N_{x\infty}$ is due to the fact that the model is not a valid PWD.

4. THE PARTIAL SUM METHOD

The following suboptimal algorithm, termed *partial sum method* (PSM), is particularly suited for the synthesis of long signals since it operates sample-by-sample. Computational expense per synthesized sample is independent of the total signal length, and only a local model interval must be known (stored) at any time.

The error norm of even-indexed signal samples can be rewritten as

$$N_{x_e}^2 = \sum_i \Delta N_{x_e}^2(i) \ ,$$

where
$$\Delta N_{x_e}^2(i) = 4\sum_{m=i-K}^{i-1} \left| y_e(i,m) - x_e(i)\, x_e^*(m)\, h^2(i-m) \right|^2 + 2\left| y_e(i,i) - x_e(i)\, x_e^*(i) \right|^2 \tag{4.1}$$

will be termed *causal error component* since it only depends on the signal samples $x_e(n)$ for $n\le i$. Suppose, now, that the synthesized signal $x_e(n)$ is already known for $n\le i-1$. Based on this knowledge, we calculate the sample $x_e(i)$ such that the i-th causal error component (4.1) is minimized, $\Delta N_{xe}(i)\rightarrow\min$. We obtain as a necessary condition

$$\sum_{m=i-K}^{i} \left[y_e(i,m) - x_e(i)\, x_e^*(m)\, h^2(i-m) \right] x_e(m)\, h^2(i-m) = 0 \quad . \tag{4.2}$$

This third-order equation has to be solved for $x_e(i)$. Letting

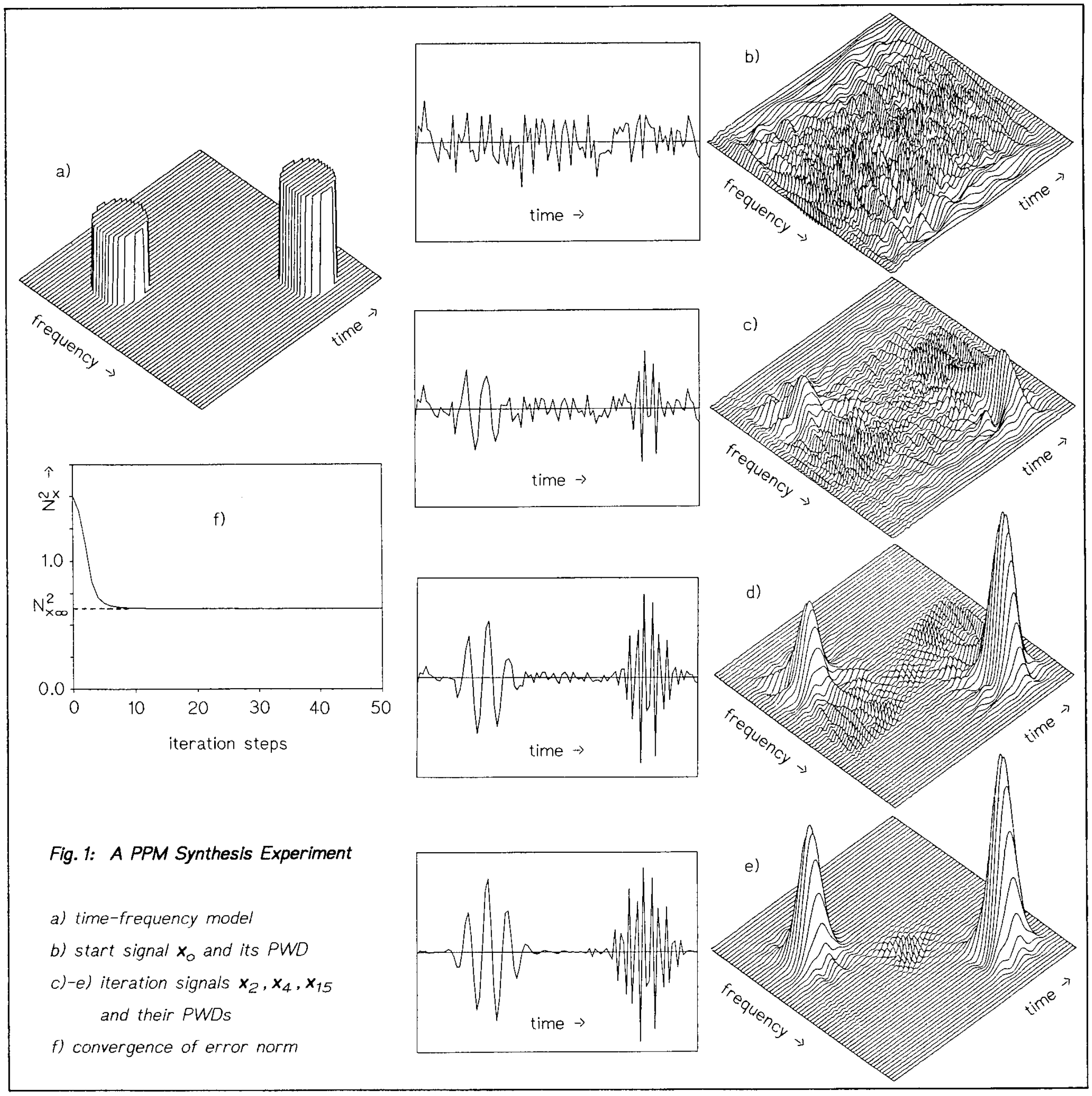

Fig. 1: A PPM Synthesis Experiment

a) time-frequency model

b) start signal $\boldsymbol{x}_o$ *and its PWD*

c)-e) iteration signals $\boldsymbol{x}_2, \boldsymbol{x}_4, \boldsymbol{x}_{15}$ *and their PWDs*

f) convergence of error norm

$$p = \sum_{m=i-K}^{i-1} |x_e(m)\, h^2(i-m)|^2 - y_e(i,i) \quad \epsilon\ \mathbb{R}$$

$$q = \sum_{m=i-K}^{i-1} y_e(i,m)\, x_e(m)\, h^2(i-m) ,$$

magnitude and phase of $x_e(i)$ are given by

$$|x_e(i)|^3 + p\,|x_e(i)| - |q| = 0 , \qquad (4.3)$$

$$\mathrm{Arg}\{x_e(i)\} = \mathrm{Arg}\{q\} - \mathrm{Arg}\{p + |x_e(i)|^2\} .$$

It follows from $p \epsilon \mathbb{R}$ and $|q| \geq 0$ that (4.3) has a unique real-valued and non-negative solution

$$|x_e(i)| = \sqrt[3]{a+b} + \sqrt[3]{a-b} \qquad \text{with} \quad a = \frac{|q|}{2} , \quad b = \sqrt{\frac{|q|^2}{4} + \frac{p^3}{27}} .$$

With the PSM, the samples of the synthesized signal minimize individual error norms $\Delta N_{xe}(i)$ which are local and causal; the total error N_{xe} , on the other hand, is generally not minimized. PSM results are thus suboptimal. Note that the necessary condition (4.2) of PSM equals the necessary condition (2.4) of *optimal* synthesis apart from the fact that the summation range $i-K \leq m \leq i+K$ of (2.4) has been replaced by the partial (causal) range $i-K \leq m \leq i$ ("partial sum"). In general, the optimal synthesis solution will not be a solution of (4.2); however, in the special case of signal *reconstruction* where the model $Y(n,\Theta)$ is a valid PWD, the optimal solution satisfies

$$y_e(i,m) - x_e(i)\, x_e^*(m)\, h^2(i-m) \equiv 0$$

and is thus also a solution of (4.2). The PSM is hence an exact method for signal *reconstruction* from valid PWDs and an approximate method in the signal *synthesis* case where the model is non-valid.

A simplified version of the PSM is obtained if the summation range $i-K \le m \le i$ of (4.2) is replaced by $i-K \le m \le i-1$; (4.2) is then linear in $x_e(i)$ and can directly be solved by

$$x_e(i) = \frac{\sum\limits_{m=i-K}^{i-1} y_e(i,m)\, x_e(m)\, h^2(i-m)}{\sum\limits_{m=i-K}^{i-1} |\, x_e(m)\, h^2(i-m)\,|^2} .$$

The synthesis performance of this simplified method is slightly inferior to the original PSM. It has been compared with the "overlapping method" of [4] in [5].

A PSM synthesis experiment. Fig. 2 demonstrates the synthesis of 500 signal samples from a model which simulates the PWD of a sinusoidal FM signal; the model is clearly not a valid PWD since inner interference terms [7] are lacking. The PWD window was defined to be a Hamming window of length $2K+1=63$. To start the PSM recursion, the first $K=31$ signal samples have to be initialized; these were chosen as $x_e(n)=0$ for $1 \le n \le 30$ and $x_e(31)=1$. It is seen that the PSM duly adapts to the model in spite of these extremely faulty initial values.

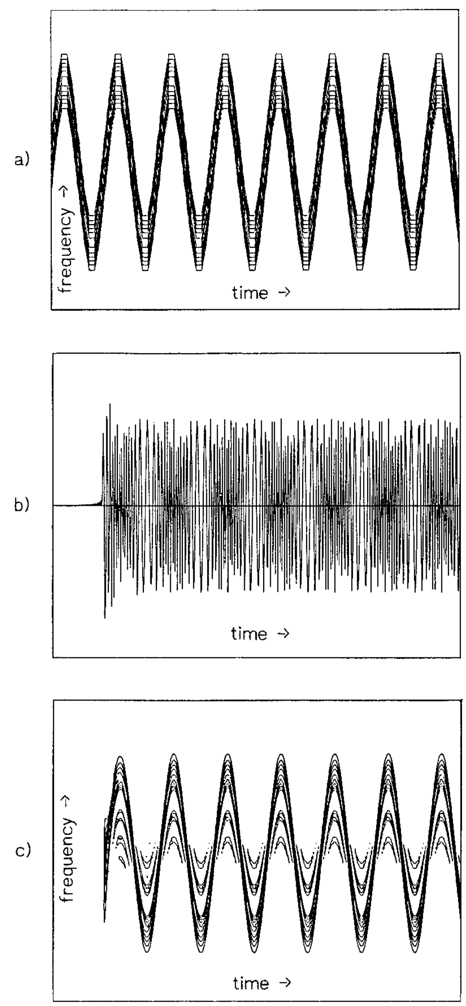

Fig. 2. A PSM Synthesis Experiment
a) time-frequency model
b) synthesized signal
c) PWD of synthesized signal

5. CONCLUSION

Two quite different signal synthesis algorithms for PWD have been presented: the *pseudo power method* (PPM) yields optimal synthesis results but synthesizes all signal samples simultaneously, using the entire model. The PPM is thus best suited for off-line processing of short signal records. The *partial sum method*, on the other hand, is suboptimal but operates recursively and sample-by-sample, using a local model segment only. In this sense, the PSM is similar to the way PWD is calculated. With the PSM, signal processing schemes can be devised which allow sequential time-frequency processing of signals with unrestricted length; the PSM is thus suited for real-time applications.

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