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# Optimum Time-Frequency Synthesis of Signals, Random Processes, Signal Spaces, and Time-Varying Filters: A Unified Framework

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**Abstract**—We review Wigner distribution-type time-frequency (TF) representations for deterministic and random signals, linear signal spaces, and linear, time-varying filters. Using these TF representations, a unified framework for optimum TF synthesis is developed. All synthesis methods are based on an "eigendecomposition" of the prescribed TF model function.

## 1 Introduction

The Wigner distribution (WD) is a time-frequency (TF) representation of deterministic signals that is of central importance in the field of quadratic TF analysis [1]-[5]. The WD has also been used for the optimum *TF synthesis* of deterministic signals [6]-[9], i.e, for the calculation of a signal whose WD is closest to a given "TF model function."

Recently, the basic WD definition has been extended to random processes [10, 11], linear signal spaces [12], and linear, time-varying filters [13]. This allows an analogous extension of the basic TF synthesis method to random processes, linear signal spaces, and linear, time-varying filters. In this paper, the new synthesis methods are formulated in a unified framework. It is shown that all methods are based on a fundamental "TF eigendecomposition."

## 2 Time-Frequency Analysis and Synthesis

This section reviews the WD definition for deterministic signals and its extension to nonstationary random processes, linear signal spaces, and linear, time-varying filters. Using the respective WD-type TF representation, the problem of optimum TF synthesis is formally stated for each case. The solution of the various TF synthesis problems will be discussed in Section 3.

### 2.1 Deterministic Signals

The WD of a deterministic signal $x(t)$ [1]-[4] is defined as

$$W_x(t,f) \triangleq \int_\tau x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau$$

where $t$ and $f$ denote time and frequency, respectively, and integrations are from $-\infty$ to $\infty$. The WD is a real-valued function of time and frequency which can be interpreted (with certain restrictions due to the uncertainty principle [14]) as a TF energy distribution.

Optimum TF signal synthesis is the computation of a signal $x_{\text{opt}}(t)$ whose WD is closest, in a least-square sense, to a given real-valued TF "model function" $M(t,f)$ [6]-[9],

$$x_{\text{opt}}(t) \triangleq \arg\min_x \epsilon_x \qquad \text{with} \quad \epsilon_x^2 = \|M - W_x\|^2 = \int_t \int_f [M(t,f) - W_x(t,f)]^2\, dt\, df\ . \qquad (1)$$

The model $M(t,f)$ expresses the desired TF energy distribution of the signal to be synthesized; it is typically not a valid WD.

## 2.2 Nonstationary Random Processes

We next consider a nonstationary random process $x(t)$ with autocorrelation function (ACF) $R_x(t_1, t_2) = \mathrm{E}\{x(t_1)\, x^*(t_2)\}$ [15]. Under certain conditions, $x(t)$ can be represented by its Karhunen-Loève expansion [15]

$$x(t) = \sum_{k=1}^{\infty} \alpha_k\, x_k(t)\ . \tag{2}$$

Here, the random variables $\alpha_k$ are statistically orthogonal with correlation $\mathrm{E}\{\alpha_k\, \alpha_l^*\} = \lambda_k\, \delta_{kl}$ where $\lambda_k \geq 0$, and the deterministic signals $x_k(t)$ are orthonormal. The $\lambda_k$ and $x_k(t)$ are the eigenvalues and eigenfunctions, respectively, of the ACF. Hence, the ACF can be written as

$$R_x(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k\, x_k(t_1)\, x_k^*(t_2)\ . \tag{3}$$

The WD of $x(t)$ is a random function of time and frequency. The *expected* WD, known as the *Wigner-Ville spectrum* (WVS) [10, 11], is a deterministic TF function which can be written as

$$\overline{W}_x(t,f) = \int_\tau R_x\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau\ , \tag{4}$$

i.e., as the *Weyl symbol* [16]-[19] of the ACF $R_x(t_1, t_2)$. The WVS describes the spectral distribution of the expected instantaneous power; for a stationary process, it reduces to the conventional power spectral density. Inserting (3) in (4), it follows that the WVS can be decomposed as

$$\overline{W}_x(t,f) = \sum_{k=1}^{\infty} \lambda_k\, W_{x_k}(t,f)$$

with nonnegative $\lambda_k$ and orthonormal $x_k(t)$.

Based on the WVS, the optimum TF synthesis of a nonstationary random process has been proposed in [20] as the solution to the minimization problem

$$x_{\mathrm{opt}}(t) \triangleq \arg\min_x\, \epsilon_x \qquad \text{with} \quad \epsilon_x^2 = \|M - \overline{W}_x\|^2 = \int_t\int_f [M(t,f) - \overline{W}_x(t,f)]^2\, dt\, df\ , \tag{5}$$

i.e., the optimum process $x_{\mathrm{opt}}(t)$ is defined as the process whose WVS is closest to a given real-valued TF model function $M(t,f)$. Since the WVS contains strictly the same information about a process as the ACF, $x_{\mathrm{opt}}(t)$ is only specified with respect to its ACF (see Section 3).

## 2.3 Linear Signal Spaces

Linear signal spaces and the associated concepts of orthogonal projections and orthonormal expansions are fundamental to signal and system theory as well as to modern signal processing methods [21]-[23]. A linear signal space $\mathcal{X}$ is a set of signals $x(t)$ such that $x_1(t) \in \mathcal{X}$ and $x_2(t) \in \mathcal{X}$ implies $c_1 x_1(t) + c_2 x_2(t) \in \mathcal{X}$ [21, 23]. Here, we shall consider subspaces of the space $\mathcal{L}_2(\mathbb{R})$ of finite-energy signals, so that $\mathcal{X}$ is equipped with inner product $\langle x_1, x_2\rangle = \int_t x_1(t)\, x_2^*(t)\, dt$ and norm $\|x\| = \langle x, x\rangle^{1/2} = [\int_t |x(t)|^2\, dt]^{1/2}$. A space $\mathcal{X}$ can be represented mathematically by its orthogonal projection operator $\mathbf{P}_{\mathcal{X}}$ with kernel $P_{\mathcal{X}}(t_1, t_2)$ or by an orthonormal basis $\{x_k(t)\}_{k=1}^{N_{\mathcal{X}}}$ spanning $\mathcal{X}$ ($N_{\mathcal{X}}$ is the dimension of $\mathcal{X}$). Any signal $x(t) \in \mathcal{X}$ can be expanded into the basis as

$$x(t) = \sum_{k=1}^{N_{\mathcal{X}}} \alpha_k\, x_k(t) \qquad \text{with} \quad \alpha_k = \langle x, x_k\rangle\ ,$$

and the kernel of the orthogonal projection operator $\mathbf{P}_{\mathcal{X}}$ can be written in terms of the basis as

$$P_{\mathcal{X}}(t_1,t_2) = \sum_{k=1}^{N_{\mathcal{X}}} x_k(t_1)\, x_k^*(t_2) \; . \tag{6}$$

A joint TF energy distribution of a linear signal space can be obtained by suitably averaging the WD of a signal $x(t)$ over all elements $x(t) \in \mathcal{X}$ of the space. This idea leads to the *WD of a linear signal space* $\mathcal{X}$ [12]

$$W_{\mathcal{X}}(t,f) = \int_\tau P_{\mathcal{X}}\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\; e^{-j2\pi f\tau}\, d\tau \; , \tag{7}$$

which is the Weyl symbol of the projection operator $\mathbf{P}_{\mathcal{X}}$. Inserting (6) in (7), we see that the WD of a space is the sum of the WDs of all orthonormal basis signals $x_k(t)$,

$$W_{\mathcal{X}}(t,f) = \sum_{k=1}^{N_{\mathcal{X}}} W_{x_k}(t,f) \; .$$

The optimum TF synthesis of a linear signal space $\mathcal{X}$ is formulated as [24, 25]

$$\mathcal{X}_{\rm opt} \triangleq \arg\min_{\mathcal{X}} \epsilon_{\mathcal{X}} \qquad \text{with} \quad \epsilon_{\mathcal{X}}^2 = \|M - W_{\mathcal{X}}\|^2 = \int_t\int_f [M(t,f) - W_{\mathcal{X}}(t,f)]^2\, dt\, df \; , \tag{8}$$

i.e., the optimum space is defined as the space whose WD is closest to a given real-valued TF model function $M(t,f)$. Typically, $M(t,f)$ will be the indicator function of a TF region $R$,

$$M(t,f) = \begin{cases} 1, & (t,f) \in R \\ 0, & (t,f) \notin R \; . \end{cases}$$

An application of the TF synthesis of signal spaces will be considered in Section 4.

### 2.4 Linear Time-Varying Filters

The effect of a linear, time-varying (LTV) system (linear operator) $\mathbf{H}$ on an input signal $x(t)$ is described by

$$(\mathbf{H}\, x)(t) = \int_{t'} H(t,t')\, x(t')\, dt' \; ,$$

where $H(t,t')$ denotes the impulse response (or kernel) of the linear operator $\mathbf{H}$. In the following, we restrict our attention to *normal* operators for which $\mathbf{H}^+\mathbf{H} = \mathbf{H}\mathbf{H}^+$, where $\mathbf{H}^+$ is the adjoint[1] of $\mathbf{H}$ [21, 23]. For a normal operator with square-integrable kernel $H(t,t')$, there exists the eigendecomposition

$$H(t_1,t_2) = \sum_{k=1}^{\infty} \eta_k\, h_k(t_1)\, h_k^*(t_2) \tag{9}$$

with complex-valued eigenvalues $\eta_k$ and orthonormal eigenfunctions $h_k(t)$. We also introduce the composite operator

$$\mathbf{R}_H = \mathbf{H}^+\mathbf{H} = \mathbf{H}\mathbf{H}^+ \tag{10}$$

with kernel

$$R_H(t_1,t_2) = \int_t H^*(t,t_1)\, H(t,t_2)\, dt = \int_t H(t_1,t)\, H^*(t_2,t)\, dt \tag{11}$$

$$= \sum_{k=1}^{\infty} |\eta_k|^2\, h_k(t_1)\, h_k^*(t_2) \; . \tag{12}$$

An LTV system $\mathbf{H}$ causes a *TF weighting* (i.e., components of the input signal located in different regions of the TF plane are amplified or attenuated), as well as a *TF displacement* of signal components [26]. The TF weighting can be described by the *WD of an LTV system* [13].

[1] The kernel $H^+(t,t')$ of the adjoint operator $\mathbf{H}^+$ is $H^+(t,t') = H^*(t',t)$.

The WD of a normal LTV system $\mathbf{H}$ is defined as the WVS of the output $(\mathbf{H}w)(t)$ when the input is white noise $w(t)$ with power spectral density $S_w(f) \equiv 1$. The WD of $\mathbf{H}$ can be expressed as

$$W_{\mathbf{H}}(t,f) = \int_\tau R_H\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\; e^{-j2\pi f\tau}\, d\tau \;, \tag{13}$$

i.e., as the Weyl symbol of the "squared" operator $\mathbf{R}_H$ in (10)-(12). The WD reduces to the squared magnitude of the frequency response for a time-invariant system, and to the WD of a linear space for an orthogonal projection operator. Inserting (12) in (13) yields

$$W_{\mathbf{H}}(t,f) = \sum_{k=1}^{\infty} |\eta_k|^2\, W_{h_k}(t,f) \;,$$

which does not depend on the phases of the eigenvalues $\eta_k$. In fact, nonzero eigenvalue phases can be associated with TF displacement effects which are not described by the WD [26].

We now consider the problem of designing a linear filter with specified TF weighting characteristic and minimum TF displacement effects. Expressing the desired TF weighting characteristic via a real-valued TF model function $M(t,f)$, this problem can be formulated as [13]

$$\mathbf{H}_{\mathrm{opt}} \triangleq \arg\min_{\mathbf{H}\geq\mathbf{0}} \epsilon_{\mathbf{H}} \qquad \text{with} \quad \epsilon^2_{\mathbf{H}} = \|M - W_{\mathbf{H}}\|^2 = \int_t\int_f [M(t,f) - W_{\mathbf{H}}(t,f)]^2\, dt\, df \;. \tag{14}$$

Hence, $\mathbf{H}_{\mathrm{opt}}$ is defined as the LTV system whose WD is closest to the given TF model function $M(t,f)$, under the side constraint that the system be positive semidefinite ($\mathbf{H} \geq \mathbf{0}$). This side constraint is motivated by the requirement of minimum TF displacement: it is shown in [26] that, among all LTV systems with identical WD, the positive semidefinite system introduces minimum TF displacement. Note that positive semidefiniteness implies $\eta_k \geq 0$, i.e., zero eigenvalue phases.

## 3 A Unified Framework for Time-Frequency Synthesis

In the previous section, the TF synthesis of deterministic signals, nonstationary random processes, linear signal spaces, and LTV filters has been formulated as a minimization of the "distance" between a prescribed TF model function $M(t,f)$ and a Wigner-type TF representation. In this section, we present the solutions to all synthesis problems without proof.

### 3.1 Time-Frequency Eigendecomposition

The solutions to all synthesis problems considered so far involve an "eigendecomposition" of the TF model $M(t,f)$ [27, 28, 18, 25]. It can be shown that any square-integrable, real-valued function $M(t,f)$ can be expressed as a linear combination of the WDs of orthonormal signals $u_k(t)$,

$$M(t,f) = \sum_{k=1}^{\infty} \lambda_k\, W_{u_k}(t,f) \qquad \text{with } \lambda_k \in \mathbb{R} \text{ and } \langle u_k, u_l\rangle = \delta_{kl} \;.$$

The real-valued $\lambda_k$ and the orthonormal $u_k(t)$ are the solutions to the eigenequation

$$\int_{t_2} H_M(t_1,t_2)\, u_k(t_2)\, dt_2 = \lambda_k\, u_k(t_1) \;, \tag{15}$$

where the Hermitian[2] kernel $H_M(t_1,t_2)$ is related to $M(t,f)$ as

$$M(t,f) = \int_\tau H_M\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau \;, \qquad H_M(t_1,t_2) = \int_f M\Big(\frac{t_1+t_2}{2}, f\Big)\, e^{j2\pi(t_1-t_2)f}\, df \;. \tag{16}$$

[2] *Hermitian* means $H^*_M(t_2,t_1) = H_M(t_1,t_2)$.

Note that, according to (16), $M(t,f)$ is the Weyl symbol of $H_M(t_1,t_2)$, and that

$$H_M(t_1,t_2) = \sum_{k=1}^{\infty} \lambda_k \, u_k(t_1) \, u_k^*(t_2) \, .$$

The real-valued $\lambda_k$ and the orthonormal $u_k(t)$ will be called the *eigenvalues* and *eigensignals*, respectively, of the TF function $M(t,f)$. We assume that the eigenvalues are ordered such that $\lambda_{k+1} \le \lambda_k$. Thus, for any $N$, the first $N$ eigenvalues will be the $N$ largest eigenvalues.

The eigenvalues $\lambda_k$ and eigensignals $u_k(t)$ can be calculated by first transforming $M(t,f)$ into the kernel $H_M(t_1,t_2)$ as in (16) and then solving the Hermitian eigenproblem (15). An alternative method using an orthonormal basis $\{b_k(t)\}_{k=1}^{\infty}$ of $\mathcal{L}_2(\mathbb{R})$ consists of the following steps [7]:

1. The Hermitian matrix $\mathbf{M}$ with elements

$$(\mathbf{M})_{kl} = \langle M, W_{b_k,b_l} \rangle = \int_t \int_f M(t,f) \, W_{b_k,b_l}^*(t,f) \, dt \, df$$

   is formed, where $W_{b_k,b_l}(t,f) = \int_\tau b_k(t+\frac{\tau}{2}) \, b_l^*(t-\frac{\tau}{2}) \, e^{-j2\pi f\tau} \, d\tau$ is the cross-WD of the basis signals $b_k(t)$ and $b_l(t)$.

2. The eigenvalues $\lambda_k$ and eigenvectors $\mathbf{e}_k$ of $\mathbf{M}$ are calculated.

3. The looked-for eigenvalues $\lambda_k$ are those of the matrix $\mathbf{M}$; the eigensignals $u_k(t)$ are derived from the eigenvectors $\mathbf{e}_k$ of $\mathbf{M}$ as (here, $(\mathbf{e}_k)_l$ denotes the $l$th element of $\mathbf{e}_k$)

$$u_k(t) = \sum_{l=1}^{\infty} (\mathbf{e}_k)_l \, b_l(t) \, .$$

Discrete-time versions of these methods are discussed in [8, 20].

### 3.2 Solving the Synthesis Problems

With the eigenvalues $\lambda_k$ and eigensignals $u_k(t)$ of the TF model function $M(t,f)$, we can now state the solutions to the TF synthesis problems formulated in Section 2.

- *Deterministic signals.* The optimum signal $x_{\mathrm{opt}}(t) = \arg\min_x \|M - W_x\|$ (cf. (1)) is [6, 7]

$$x_{\mathrm{opt}}(t) = \sqrt{\lambda_1} \, u_1(t) \, ,$$

  where the first (= largest) eigenvalue $\lambda_1$ is assumed positive (otherwise $x_{\mathrm{opt}}(t) \equiv 0$). Note that $x_{\mathrm{opt}}(t)$ is defined only up to a constant phase factor [29, 6]. The residual (minimum) synthesis error is

$$\epsilon_{x_{\mathrm{opt}}}^2 = \|M - W_{x_{\mathrm{opt}}}\|^2 = \sum_{k=2}^{\infty} \lambda_k^2 \, .$$

- *Nonstationary random processes.* The optimum process $x_{\mathrm{opt}}(t) = \arg\min_x \|M - \overline{W}_x\|$ (cf. (5)) is given via its Karhunen-Loève expansion (2) as

$$x_{\mathrm{opt}}(t) = \sum_{k=1}^{N_+} \alpha_k \, u_k(t) \, ,$$

  where $N_+$ denotes the number of positive eigenvalues $\lambda_k > 0$ [20]. The random coefficients $\alpha_k$ are statistically orthogonal with mean powers

$$\mathrm{E}\{|\alpha_k|^2\} = \lambda_k \, , \qquad k = 1, \ldots, N_+ \, .$$

  The means $\mathrm{E}\{\alpha_k\}$ of the coefficients can be chosen arbitrarily apart from the condition [20]

$$\sum_{k=1}^{N_+} \frac{|\mathrm{E}\{\alpha_k\}|^2}{\lambda_k} \le 1 \, .$$

In particular, this condition will always be satisfied if $\mathrm{E}\{\alpha_k\} \equiv 0$, which yields a zero-mean process $x_{\mathrm{opt}}(t)$. The residual (minimum) synthesis error is

$$\epsilon^2_{x_{\mathrm{opt}}} = \|M - \overline{W}_{x_{\mathrm{opt}}}\|^2 = \sum_{k=N_+ +1}^{\infty} \lambda_k^2 \, .$$

- *Linear signal spaces.* The optimum signal space $\mathcal{X}_{\mathrm{opt}} = \arg\min_{\mathcal{X}} \|M - W_{\mathcal{X}}\|$ (cf. (8)) is spanned by the orthonormal basis $\{u_k(t)\}_{k=1}^{N_{1/2}}$, where $N_{1/2}$ denotes the number of eigenvalues $\lambda_k > 1/2$ [24, 25],

$$\mathcal{X}_{\mathrm{opt}} = \mathrm{span}\,\{u_k(t)\}_{k=1}^{N_{1/2}} \, .$$

  The residual (minimum) synthesis error is

$$\epsilon^2_{\mathcal{X}_{\mathrm{opt}}} = \|M - W_{\mathcal{X}_{\mathrm{opt}}}\|^2 = \sum_{k=1}^{N_{1/2}} (1-\lambda_k)^2 + \sum_{k=N_{1/2}+1}^{\infty} \lambda_k^2 \, .$$

- *Linear time-varying filters.* The optimum LTV filter $\mathbf{H}_{\mathrm{opt}} = \arg\min_{\mathbf{H}\ge\mathbf{0}} \|M - W_{\mathbf{H}}\|$ (cf. (14)) is given via its eigendecomposition (cf. (9)) as

$$H_{\mathrm{opt}}(t_1,t_2) = \sum_{k=1}^{N_+} \sqrt{\lambda_k}\, u_k(t_1)\, u_k^*(t_2) \, ,$$

  where $N_+$ denotes the number of positive eigenvalues $\lambda_k > 0$ [13]. The residual (minimum) synthesis error is

$$\epsilon^2_{\mathbf{H}_{\mathrm{opt}}} = \|M - W_{\mathbf{H}_{\mathrm{opt}}}\|^2 = \sum_{k=N_+ +1}^{\infty} \lambda_k^2 \, .$$

# 4 Simulation Results

The TF synthesis of a random process is shown in *Fig. 1* [20]. The TF model $M(t,f)$ consists of two plateaus, one of which models a narrowband noise with sinusoidal frequency modulation, while the other models a process component located in an elliptical TF region. Also shown are the WVS of the optimum process $x_{\mathrm{opt}}(t)$ and two realizations of $x_{\mathrm{opt}}(t)$ (assuming zero mean).

*Fig. 2* shows the TF synthesis of a linear signal space and its application to TF filtering [24, 25]. The TF model $M(t,f)$ is the indicator function of a TF region $R$. The number of eigenvalues above 1/2, and thus the dimension of $\mathcal{X}_{\mathrm{opt}}$, is $N_{1/2} = 6$. Figs. 2(d)-(f) show the application to a TF filtering problem. The orthogonal projection operator $\mathbf{P}_{\mathcal{X}_{\mathrm{opt}}}$ on $\mathcal{X}_{\mathrm{opt}}$ is used as a "TF filter" that passes all signals inside the TF region $R$ and suppresses all signals outside $R$ [24, 25].

*Fig. 3* illustrates the TF design of an LTV filter and its application [13]. The TF model $M(t,f)$ defines a quadrangular TF pass region with a TF-dependent amplification factor. A filtering experiment using the synthesized filter $\mathbf{H}_{\mathrm{opt}}$ is shown in Figs. 3(c)-(d). One of the two input signal components is located outside the TF pass region and is duly suppressed; the other is inside the pass region and is duly passed, with a partial amplification as desired. Note that such an arbitrary TF-dependent amplification (or TF weighting) cannot be achieved by an orthogonal projection operator, which can only implement a binary pass/suppress TF weighting.

## 5 Extensions

The TF synthesis methods described in this paper are all based on the WD and the Weyl symbol. All methods can however be extended to other quadratic TF representations provided that these are *unitary* [7, 30]. Important examples of unitary quadratic TF representations other than the

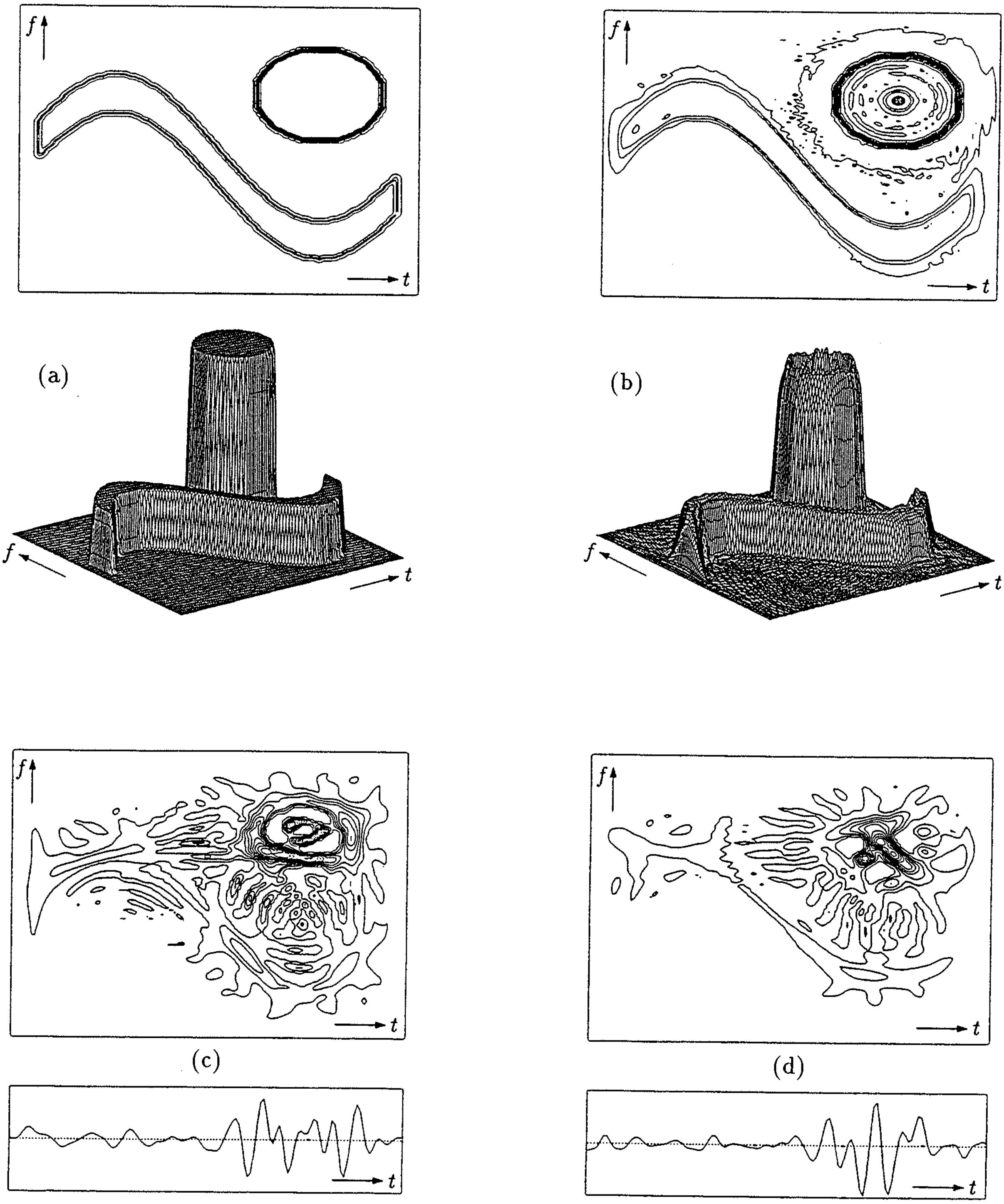


*Fig. 1: Optimum TF synthesis of a nonstationary random process from a two-component plateau model. (a) TF model $M(t,f)$, (b) WVS of the synthesized process $x_{\mathrm{opt}}(t)$, (c) and (d) real parts and (slightly smoothed) WDs of two realizations of $x_{\mathrm{opt}}(t)$.*

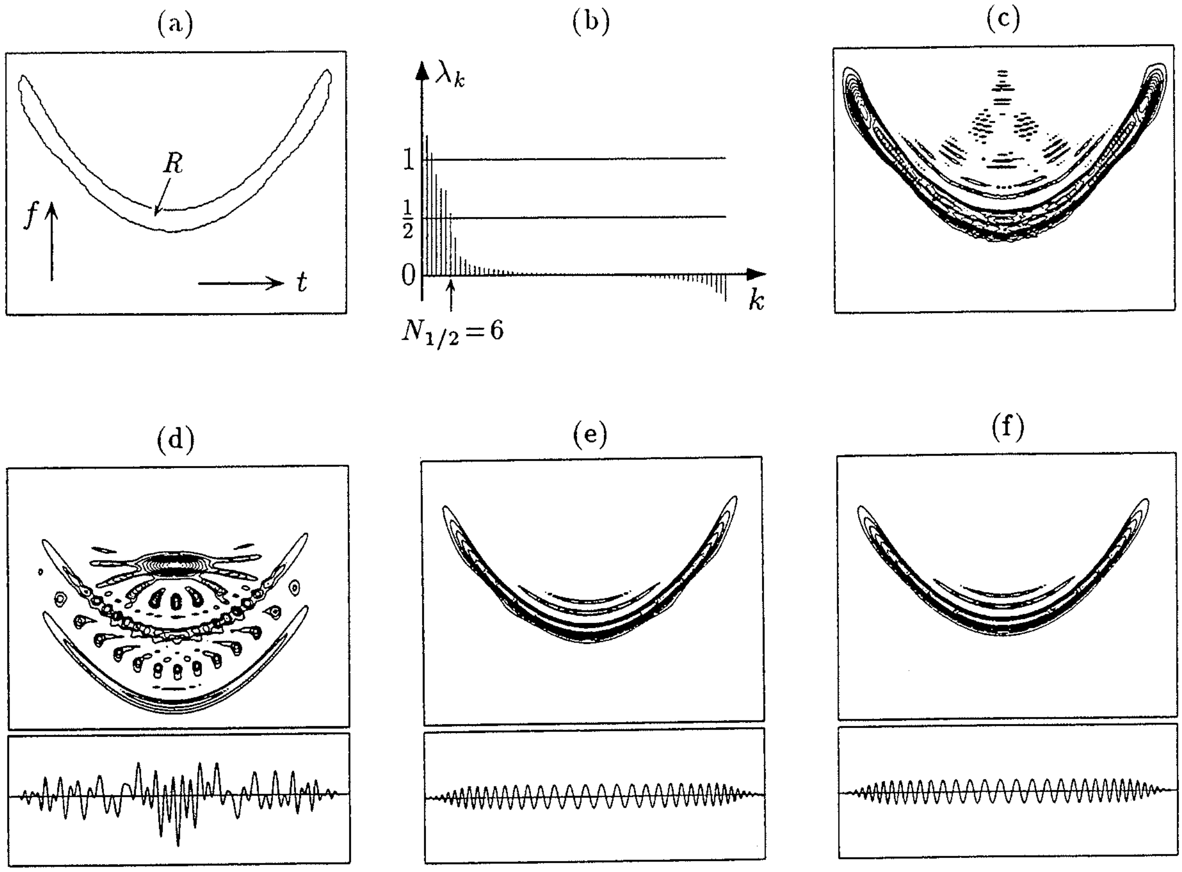


*Fig. 2: Optimum TF synthesis of a linear signal space and application to TF filtering. (a) TF model $M(t,f)$, (b) eigenvalues $\lambda_k$, (c) WD of the synthesized space $\mathcal{X}_{\rm opt}$, (d) real part and (slightly smoothed) WD of the three-component input signal applied to the projection operator (TF filter) $\mathbf{P}_{\mathcal{X}_{\rm opt}}$, (e) real part and WD of the output signal, (f) real part and WD of the desired (true) signal component.*

WD are the ambiguity function [31], the generalized WD and the Rihaczek distribution [32, 2, 3], the Bertrand $P_0$ distribution [33]-[37], and the Altes-Marinovic $Q$ distribution [35]-[38].

A second extension is the inclusion of a *subspace constraint* in the TF synthesis problems [7, 8, 20]. This subspace constraint can be used to enforce cerain properties of the synthesized signal etc. (e.g., bandlimitation in a given frequency band); it is also necessary for avoiding aliasing effects in a discrete-time implementation of TF synthesis [8, 20].

The extension of TF projection filters to perfect-reconstruction TF filter banks is discussed in [24]. We note that, besides the optimum methods discussed here, several other methods for a TF design and/or TF implementation of LTV filters have been proposed in the literature (see [39] for a review and comparison). Some of these methods (e.g., Weyl filters and STFT filters [39]) can be viewed as reduced-complexity approximations to the methods discussed here.

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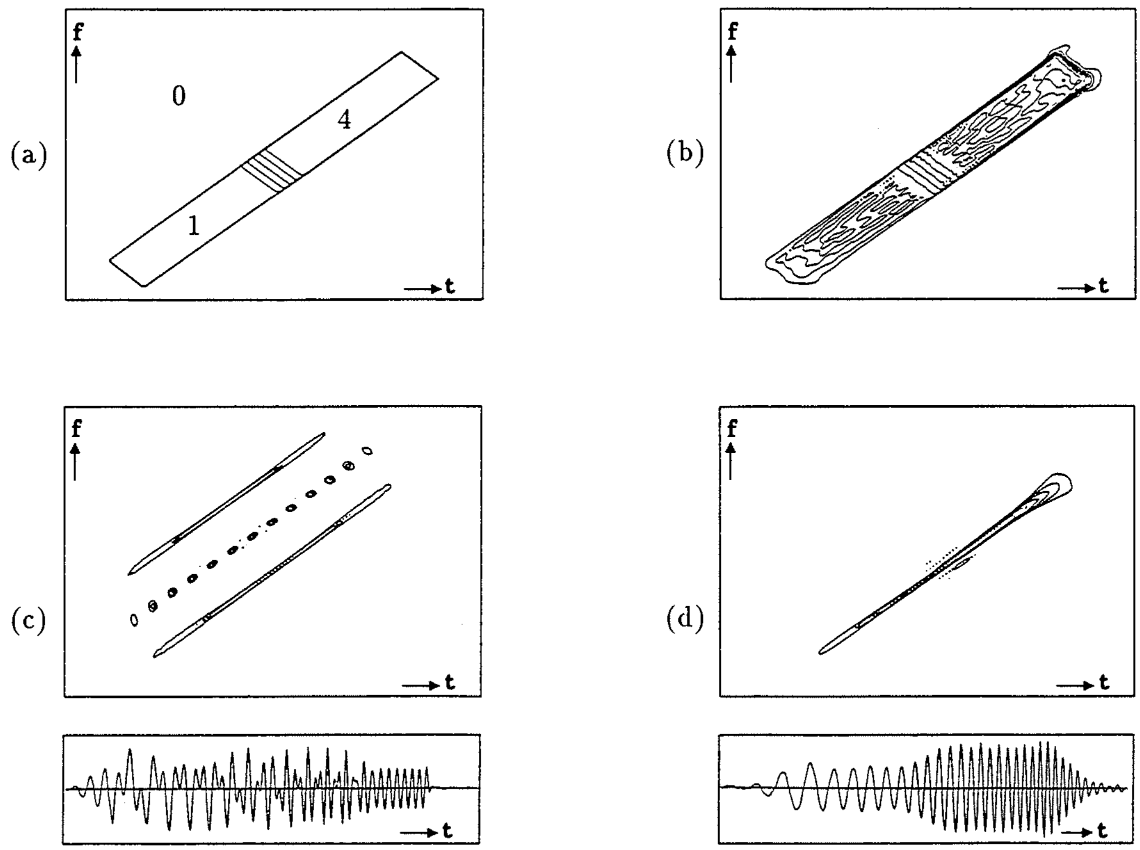


*Fig. 3: Optimum TF synthesis of an LTV filter and application to TF filtering. (a) TF model $M(t,f)$, (b) WD of the synthesized filter $\mathbf{H}_{\rm opt}$, (c) real part and WD of the input signal, (d) real part and WD of the resulting output signal.*