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TIME–FREQUENCY REPRESENTATION OF LINEAR TIME-VARYING SYSTEMS USING THE WEYL SYMBOL

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Abstract. The Weyl Symbol (WS) representation of linear operators is introduced for the time–frequency analysis of linear time–varying systems. We show that, from a joint time–frequency point of view, the WS and the usual time–varying transfer function as introduced by Zadeh (ZF) have similar theoretical justification. Interpretation, estimation, properties, and relations of both the WS and the usual ZF are elaborated.

1 INTRODUCTION

A general linear time–varying (LTV) system $\mathbf{H}$ is uniquely determined by its impulse response (kernel) $h(t,t')$. It establishes a mapping of an input signal $x(t)$ onto an output signal $(\mathbf{H}x)(t)$ according to

$$(\mathbf{H}x)(t) = \int_{t'} h(t,t')x(t')dt' . \qquad (1)$$

In a classical work [1], Zadeh introduced the time–varying transfer function (ZF) of a linear system as

$$Z_H(t,f) = \int_{\tau} h(t,t-\tau)e^{-j2\pi f\tau}d\tau . \qquad (2)$$

The ZF can be considered as a generalization of the transfer function of linear time–invariant (LTI) systems since

$$Z_H(t,f) = \left.\frac{(\mathbf{H}x)(t)}{x(t)}\right|_{x(t) = e^{j2\pi ft}} .$$

This relation, however, is only of moderate importance since the complex sinusoids $e^{j2\pi ft}$ are no longer eigensignals of LTV systems. In the special case of an LTI system, the ZF reduces to the conventional transfer function $H(f)$ (Fourier transform of the impulse response).

2 WEYL SYMBOL

The *Weyl Symbol* (WS) $L_H(t,f)$ of a linear system $\mathbf{H}$ has originally been defined in a quantum mechanical context for hermitian operators (other designations are Weyl quantization, Weyl correspondence) [2,3]. It is based on an invertible one–to–one mapping of the impulse response $h(t,t')$ of $\mathbf{H}$

$$L_H(t,f) \stackrel{\text{def}}{=} \int_{\tau} h\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}d\tau$$

with the inversion formula

$$h(t,t') = \int_{f} L_H\left(\frac{t+t'}{2}, f\right) e^{j2\pi f(t-t')}df .$$

Normal system. A normal system commutes with its adjoint, i. e., $\mathbf{HH}^+ = \mathbf{H}^+\mathbf{H}$. If its impulse response is square–integrable, the discrete spectral decomposition

$$h(t,t') = \sum_{k=0}^{\infty} \lambda_k u_k(t) u_k^*(t') \qquad (3)$$

holds, where λ_k are the eigenvalues and $u_k(t)$ are the respective (unit–energy) eigensignals of $\mathbf{H}$ [4]. Then, the WS is given by

$$L_H(t,f) = \sum_{k=0}^{\infty} \lambda_k W_{u_k}(t,f) , \qquad (4)$$

where

$$W_{u_k}(t,f) = \int_{\tau} u_k\left(t+\frac{\tau}{2}\right) u_k^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}d\tau$$

is the Wigner distribution (WD) [5] of the eigensignal $u_k(t)$. The WD of a signal can be interpreted as the signal's time–frequency energy distribution [5].

3 INTERPRETATION OF THE WEYL SYMBOL

One of the major goals of linear system theory is to replace the general input–output relation (1) by a more transparent description of the system's effect on input signals. For LTI systems, the transfer function $H(f)$ allows to describe the system's behaviour as a frequency–selective (complex) weighting. However, for a general LTV system such an efficient tool does not exist. Nevertheless, one can split up the output signal according to

$$(\mathbf{H}x)(t) = cx(t) + \epsilon(t) ,$$

where c is a *time–independent* weighting factor and $\epsilon(t)$ is the remaining 'error signal'. Minimization of $\epsilon(t)$ in a least square–sense

$$\int_{t} |\epsilon(t)|^2 dt \rightarrow \min$$

yields the unique solution

$$c = \rho_H(x) \stackrel{\text{def}}{=} \frac{(\mathbf{H}x, x)}{(x, x)}$$

$$= \frac{\int\limits_{t'}\int\limits_{t} h(t,t')x(t')x^*(t)dt'dt}{\int\limits_{t} |x(t)|^2 dt},$$

namely, the *Rayleigh quotient* (RQ) of $\mathbf{H}$ for $x(t)$. If the input signal $x(t)$ is an eigensignal of the system $\mathbf{H}$, the error signal $\epsilon(t)$ vanishes. We will henceforth presuppose $(x, x) = 1$, i. e., a unit-energy input signal. Then,

$$\rho_H(x) = (\mathbf{H}x, x).$$

The fundamental property of the WS is to represent a *time-frequency weighting characteristic* of an LTV system $\mathbf{H}$ by virtue of the relation [3]

$$\rho_H(x) = (L_H, W_x) = \iint\limits_{t\,f} L_H(t,f)W_x(t,f)dtdf. \quad (5)$$

That is, the RQ $\rho_H(x)$ is expressed as a weighted integral of the input signal's WD $W_x(t,f)$; the time-frequency weighting function is the WS. If there existed a signal $x(t)$ with perfect time–frequency concentration in a time–frequency point $t = t_0$, $f = f_0$, such that $W_x(t,f) \stackrel{?}{=} \delta(t - t_0)\delta(f - f_0)$, then the Rayleigh quotient $\rho_H(x)$ would take on the value of the WS at $t = t_0$ and $f = f_0$, $\rho_H(x) = L_H(t_0, f_0)$. We emphasize, however, that a signal $x(t)$ with perfect time–frequency concentration does not exist.

System analysis by means of the RQ is consistent with LTI system theory. Indeed, the RQ of an LTI system with transfer function $H(f)$ is

$$\rho_H(x) = \left(H, |X|^2\right) = \int\limits_f H(f)|X(f)|^2 df, \quad (6)$$

where $|X(f)|^2$ is the energy density spectrum of the input signal $x(t)$. Eq. (6) shows that if the input signal is well concentrated in frequency with center ν, i.e., $|X(f)|^2 \simeq \delta(f-\nu)$, then the RQ approximately yields the value of the LTI system's transfer function $H(f)$ for $f = \nu$. Thus, we can interpret the transfer function of an LTI system as a *frequency*-parametrized RQ, whereas the WS can be interpreted as a *time-frequency-parametrized* RQ for an LTV system.

For a time-varying transfer function, the consistency with the LTI system transfer function is a natural requirement. In view of the consistency of the Eqs. (5) and (6) for arbitrary input signals $x(t)$, we can formulate a 'rule of generalization' as follows:

If the WD $W_x(t,f)$ is regarded as the time-varying generalization of the energy density spectrum $|X(f)|^2$ of the input signal $x(t)$, then the WS $L_H(t,f)$ represents the time-varying generalization of the LTI system transfer function $H(f)$ in the sense of the RQ,

$$\begin{array}{llll} \text{if} & |X(f)|^2 & \longrightarrow & W_x(t,f), \\ \text{then} & H(f) & \longrightarrow & L_H(t,f). \end{array} \quad (7)$$

We emphasize that the WS is uniquely determined by (5), i.e., $L_H(t,f)$ cannot be replaced by another function of t and f such that (5) holds for any input signal $x(t)$. Note, also, that the WS of an LTI system reduces to the transfer function $H(f)$.

4 NEW INTERPRETATION OF THE ZF

There is a certain amount of freedom in the definition of a time–varying spectrum [6]. An alternative to the WD is the *Rihaczek distribution* (RD) [6]

$$R_x(t,f) = x(t)X^*(f)e^{-j2\pi ft}$$
$$= \int\limits_\tau x(t)x^*(t-\tau)e^{-j2\pi f\tau}d\tau. \quad (8)$$

Replacing the WD by the RD in the discussion of Section 3 leads to the ZF $Z_H(t,f)$ as defined by (2). More explicitly, one can show the fundamental property of the ZF

$$\rho_H(x) = (Z_H, R_x) = \iint\limits_{t\,f} Z_H(t,f)R_x^*(t,f)dtdf,$$

which is analogous to (5). For a normal system with square–integrable impulse response the ZF can be written as (note the analogy to (4))

$$Z_H(t,f) = \sum_{k=0}^{\infty} \lambda_k R_{u_k}(t,f), \quad (9)$$

where $R_{u_k}(t,f)$ is the RD (8) of the eigensignal $u_k(t)$ of the system $\mathbf{H}$. Consequently, the ZF can be regarded as a time–frequency weighting characteristic of an LTV system, with the prerequisite that the input signal's time–frequency energy distribution is given by the RD instead of the WD. Hence, one can formulate an alternative to (7):

If the RD $R_x(t,f)$ is regarded as the time–varying generalization of the energy density spectrum $|X(f)|^2$ of the input signal $x(t)$, then the ZF $Z_H(t,f)$ represents the time–varying generalization of the LTI system's transfer function $H(f)$ in the sense of the RQ,

$$\begin{array}{llll} \text{if} & |X(f)|^2 & \longrightarrow & R_x(t,f), \\ \text{then} & H(f) & \longrightarrow & Z_H(t,f). \end{array} \quad (10)$$

From this 'joint time–frequency' point of view, the ZF and the WS have equal theoretical justification.

Clearly, for the same system the ZF and the WS yield different time–frequency characteristics. The relation between the ZF $Z_H(t,f)$ and the WS $L_H(t,f)$ can be compactly expressed by ($**$ denotes double convolution)

$$Z_H(t,f) = 2L_H(t,f) ** e^{-j4\pi tf},$$

which is equivalent to the relation between RD $R_x(t,f)$ and WD $W_x(t,f)$

$$R_x(t,f) = 2W_x(t,f) ** e^{-j4\pi tf}.$$

It must be emphasized that the freedom in the definition of a time–varying spectrum gives rise to an infinite number of possible generalizations analogous to (7) and (10). However, in the present paper we concentrate on the WS and the ZF since they are distinguished both by their popularity and mathematical properties.

5 ESTIMATION

Deterministic approach. As in the foregoing sections, we recall LTI system theory. Eq. (6) shows that if a test signal $g(t)$ is a narrowband lowpass signal, $|G(f)|^2 \simeq \delta(f)$, then the RQ $\rho_H(g)$ yields an estimate of the transfer function $H(f)$ at $f=0$. An estimate $\hat{H}^{(g)}(f)$ of the LTI system's transfer function $H(f)$ for any value of f can be achieved by constructing a manifold of *frequency*–shifted test signals $g^{(f)}(t) = g(t)e^{j2\pi ft}$ and evaluating the RQ,

$$\hat{H}^{(g)}(f) = \rho_H(g^{(f)}) = \\ = \int_{f'} H(f')\,|G(f'-f)|^2 df' = H(f) * |G(-f)|^2. \quad (11)$$

The generalization to LTV systems is straightforward: we obtain an estimate $\hat{L}_H^{(g)}(t,f)$ of the WS $L_H(t,f)$ by constructing a manifold of *time-frequency* shifted test signals $g^{(t,f)}(t') = g(t'-t)e^{j2\pi ft}$ and evaluating the RQ,

$$\hat{L}_H^{(g)}(t,f) = \rho_H(g^{(t,f)}). \quad (12)$$

Indeed, $\hat{L}_H^{(g)}(t,f)$ can be shown to be the double convolution of the WS of $\mathbf{H}$ and the WD of the elementary test signal $g(t)$,

$$\hat{L}_H^{(g)}(t,f) = L_H(t,f) ** W_g(-t,-f). \quad (13)$$

Alternatively, $\hat{L}_H^{(g)}(t,f)$ is also the convolution of the ZF and the RD of the elementary test signal $g(t)$,

$$\hat{L}_H^{(g)}(t,f) = Z_H(t,f) ** R_g^*(-t,-f). \quad (14)$$

The LTI transfer function estimation (11) can be enhanced arbitrarily by increasing the frequency concentration of the test signal $g(t)$. In contrast, the accuracy of the time–frequency estimate (12) is limited by Heisenberg's uncertainty principle, i. e., the choice of the elementary test signal $g(t)$ is governed by a *time-frequency resolution tradeoff* as follows: a sharp time concentration of $g(t)$ yields good time resolution but deteriorates the frequency resolution; conversely, a sharp frequency concentration of $g(t)$ yields good frequency resolution but deteriorates the time resolution.

This resolution tradeoff is analogous to the resolution tradeoff encountered in *signal analysis* via the spectrogram [6]. The spectrogram $S_x^{(g)}(t,f)$ of a signal $x(t)$ (using an analysis window $g(t)$) is defined as the squared magnitude of the short–time Fourier transform (STFT),

$$S_x^{(g)}(t,f) = \left| \int_{t'} x(t')g^*(t'-t)e^{-j2\pi ft'}dt' \right|^2.$$

It is a smoothed version of either the WD $W_x(t,f)$ or the RD $R_x(t,f)$ [6]

$$\begin{aligned} S_x^{(g)}(t,f) &= W_x(t,f) ** W_g(-t,-f) \\ &= R_x(t,f) ** R_g^*(-t,-f), \end{aligned} \quad (15)$$

where the smoothing kernels are essentially the respective distributions of the window function $g(t)$. Using (4) and (15) in (13), we can write the WS estimate $\hat{L}_H^{(g)}(t,f)$ of a normal system (with square–integrable kernel, cf. (3)) as

$$\hat{L}_H^{(g)}(t,f) = \sum_{k=0}^{\infty} \lambda_k S_{u_k}^{(g)}(t,f),$$

i.e., an eigenvalue–weighted sum of the spectrograms of the eigensignals (analogous to (4) and (9)).

Stochastic approach. Let $n(t)$ be white noise with autocorrelation function $R_n(t,t') = E\{n(t)n^*(t')\} = \delta(t-t')$. Then, the WS can be written as

$$L_H(t,f) = E\left\{W_{Hn,n}(t,f)\right\}, \quad (16)$$

i. e., as the expectation of the cross WD

$$W_{Hn,n}(t,f) = \int_{\tau} (\mathbf{H}n)\left(t+\frac{\tau}{2}\right) n^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$$

of the output process $(\mathbf{H}n)(t)$ and the input (white-noise) process $n(t)$. For LTI systems, (16) reduces to the well-known relation

$$H(f) = S_{Hn,n}(f),$$

where $S_{Hn,n}(f)$ is the cross–power spectral density of the output process $(\mathbf{H}n)(t)$ and the input process $n(t)$.

In analogy to (16), the ZF satisfies

$$Z_H(t,f) = E\{R_{Hn,n}(t,f)\} ,$$

with $R_{Hn,n}(t,f)$ denoting the cross-RD of the processes $(\mathbf{H}n)(t)$ and $n(t)$. In practice, the expectation value of (16) must be *estimated* based on a single or several realizations of the processes $n(t)$ and $(\mathbf{H}n)(t)$.

6 PROPERTIES AND EXAMPLES

The foregoing discussion has mathematically corroborated the interpretation of the WS as a time–frequency weighting characteristic that is strongly related to the ZF. We now turn to the question of how the WS and the ZF actually behave for specific linear systems.

LTI system. The WS and the ZF of an LTI system $\mathbf{H}$ are equal, independent of time, and identical with the LTI system's transfer function $H(f)$ (Fourier transform of the impulse response $h(t)$),

$$L_H(t,f) = Z_H(t,f) = H(f) .$$

Multiplication–type system. The WS and the ZF of a multiplication–type system $\mathbf{H}$ (modulator, windowing system) with impulse response $h(t,t') = m(t)\delta(t-t')$ are equal, independent of frequency, and identical with the multiplier signal $m(t)$,

$$L_H(t,f) = Z_H(t,f) = m(t) .$$

Chirped LTI system. A less trivial, but interesting system is the 'chirped LTI system' (see Fig. 1) consisting of a down-chirp modulator with chirp rate $-\alpha$, an inner LTI system with transfer function $\tilde{H}(f)$, and an up-chirp modulator with chirp rate α.

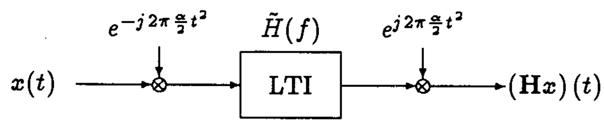

Figure 1 Chirped LTI system

The impulse response of the overall system is given by

$$h(t,t') = e^{j2\pi\frac{\alpha}{2}(t^2-t'^2)}\tilde{h}(t-t') .$$

The WS is

$$L_H(t,f) = \tilde{H}(f-\alpha t),$$

i. e., the result of 'shearing' $\tilde{H}(f)$ along the chirp's instantaneous-frequency line $f_i(t) = \alpha t$. The ZF yields the more complicated result

$$Z_H(t,f) = \left[\tilde{H}(f') * \sqrt{\frac{1}{j\alpha}}e^{-\frac{\pi}{j\alpha}f'^2}\right]\Bigg|_{f'=f-\alpha t}$$

Periodically time–varying system. The impulse response $h(t,t')$ of a periodically time–varying system is periodic with respect to t and t', $h(t,t') = h(t+T_0, t'+T_0)$. The WS and the ZF are both periodic with respect to time,

$$\begin{aligned} L_H(t,f) &= L_H(t+T_0,f), \\ Z_H(t,f) &= Z_H(t+T_0,f). \end{aligned}$$

Hermitian system. A hermitian system $\mathbf{H}$ can be considered as a *real weighting system* since its RQ $\rho_H(x)$ is real for any input signal $x(t)$. As may be expected, the WS of a hermitian system is real (this is one of the reasons for its popularity in quantum mechanics) and the WS of an antihermitian system is purely imaginary. As a further consequence, one can easily show that the real part of the WS of any system $\mathbf{H}$ corresponds to the hermitian part of $\mathbf{H}$, and the imaginary part corresponds to the antihermitian part of $\mathbf{H}$,

$$\begin{aligned} \mathrm{Re}\{L_H(t,f)\} &= L_{\frac{H+H^+}{2}}(t,f), \\ \mathrm{Im}\{L_H(t,f)\} &= L_{\frac{H-H^+}{2j}}(t,f) . \end{aligned}$$

This property of the WS does not hold for the ZF. Indeed, *the ZF of a hermitian system is generally complex.* This is an important disadvantage of the ZF that can be traced back to a respective disadvantage of the RD, the fact that it is generally complex–valued even for real signals.

Projection filter. A projection filter $\mathbf{P}_\mathcal{S}$ (orthogonal projection operator of a linear signal space $\mathcal{S}$ [4]) is a linear system that yields the orthogonal projection $x_\mathcal{S}(t)$ of the input signal $x(t)$ onto a linear signal space $\mathcal{S}$,

$$(\mathbf{P}_\mathcal{S}x)(t) = x_\mathcal{S}(t) \in \mathcal{S}.$$

Given an orthonormal basis $\{u_k(t)\}$ of the signal space $\mathcal{S}$, the WS $L_H(t,f)$ and the ZF $Z_H(t,f)$ of the projection filter $\mathbf{P}_\mathcal{S}$ can be written as (cf. (4),(9))

$$\begin{aligned} L_{P_\mathcal{S}}(t,f) &= \sum_{k=0}^{N} W_{u_k}(t,f), \\ Z_{P_\mathcal{S}}(t,f) &= \sum_{k=0}^{N} R_{u_k}(t,f), \end{aligned}$$

where N is the dimension of $\mathcal{S}$. The WS and the ZF of the projection filter can be regarded as time–frequency representations of the corresponding signal space $\mathcal{S}$ and, in fact, equal the recently defined *WD and RD (respectively) of a linear signal space* [7].

Multiplicative modification of the STFT. Multiplicative modification of the short–time Fourier transform (STFT) is a well–known concept for the design of linear systems [8]. The overall system consists of three parts:

1. STFT analysis with analysis window $\gamma(t)$,
$$X^{(\gamma)}(t,f) = \int_{t'} x(t')\gamma^*(t'-t)e^{-j2\pi f t'}dt',$$
2. multiplicative modification of the STFT outcome $X^{(\gamma)}(t,f)$ by a time–frequency weighting function $M(t,f)$,
$$\tilde{X}^{(\gamma)}(t,f) = M(t,f)\, X^{(\gamma)}(t,f),$$
3. STFT synthesis applied to the modified STFT outcome $\tilde{X}^{(\gamma)}(t,f)$, using a "synthesis window" $g(t)$; this yields the output signal $(\mathbf{M}x)(t)$
$$(\mathbf{M}x)(t) = \iint_{t'f'} \tilde{X}^{(\gamma)}(t',f')g(t-t')e^{j2\pi f't}dt'df'.$$

This concept allows a direct definition of a time–frequency transmission characteristic $M(t,f)$. However, the actual overall system M also depends on the choice of the window functions $\gamma(t)$ and $g(t)$. This becomes apparent in the impulse response of M

$$m^{(\gamma,g)}(t,t'') = \iint_{t'f'} M(t',f')\gamma^*(t''-t')g(t-t')e^{j2\pi f'(t-t'')}dt'df'.$$

In this expression for the impulse response, the influence of the window functions $\gamma(t)$ and $g(t)$ on the actual behaviour of the overall system M is quite non-transparent. The WS $L_M(t,f)$ or the ZF $Z_M(t,f)$ yield a more lucid characterization of the overall system M :

$$\begin{aligned} L_M(t,f) &= M(t,f) ** W_{\gamma,g}(t,f), \\ Z_M(t,f) &= M(t,f) ** R_{\gamma,g}(t,f). \end{aligned} \tag{17}$$

The WS and the ZF of the overall system are equal to the two-dimensional convolution of the modification function $M(t,f)$ with the cross WD and the cross RD (respectively) of the window functions $\gamma(t)$ and $g(t)$. In the special case $g(t)=\gamma(t)$ and $M(t,f)\in\mathbf{R}$, the overall system M is hermitian and (17) shows how the choice of the window function $g(t)$ affects the weighting behaviour of M. Specifically, the STFT-based system *design* discussed above entails a time-frequency resolution tradeoff analogous to the *analysis* resolution tradeoff discussed in Section 5, Eqs. (12)—(14).

7 SPREADING FUNCTIONS

An ideal time–frequency shifting system $\mathrm{S}^{(\tau,\nu)}$ consists of a delay element with lag τ and a complex modulator with frequency ν. The input–output relation of $\mathrm{S}^{(\tau,\nu)}$ is then given by

$$\left(\mathrm{S}^{(\tau,\nu)}x\right)(t) = x(t-\tau)e^{j2\pi\nu t}. \tag{18}$$

A *time–frequency shifting characteristic* of a linear system **H** can be specified by the *asymmetrical spreading function* (ASF) [9]

$$S_H^{(a)}(\tau,\nu) = \int_t h(t,t-\tau)e^{-j2\pi\nu t}dt.$$

The ASF establishes an infinitesimal decomposition of **H** into ideal time–frequency shifting systems $\mathrm{S}^{(\tau,\nu)}$,

$$\mathbf{H} = \iint_{\tau\,\nu} S_H^a(\tau,\nu)\mathrm{S}^{(\tau,\nu)}d\tau d\nu. \tag{19}$$

The ASF is essentially the double Fourier transform of the ZF,

$$\begin{aligned} S_H^{(a)}(\tau,\nu) &= \iint_{t\,f} Z_H(t,f)e^{-j2\pi(\nu t-\tau f)}dtdf \\ &= \mathcal{F}^2_{(t,f)\to(\nu,-\tau)}\{Z_H(t,f)\}. \end{aligned}$$

However, in the definition of a time–frequency shift of $x(t)$ by time τ and frequency ν, there is a considerable amount of freedom that leads to a manifold of signals which are equal up to constant phase factors. This comes from the fact that time and frequency shift operators are not commutable.

The shifting operator $\mathrm{S}^{(\tau,\nu)}$ of (18) corresponds to first time–shifting and then frequency–shifting. Alternatively, we may split up time shifts and frequency shifts into fractional shifts and interchange these fractional shifts arbitrarily. Each version of the shift operator leads to a corresponding version of the spreading function defined by the decomposition relation (19). In particular, if the splitting–up of time and frequency shifts is done alternatingly and infinitesimally, then we obtain the shift operator $\bar{\mathrm{S}}^{(\tau,\nu)}$ as

$$\left(\bar{\mathrm{S}}^{(\tau,\nu)}x\right)(t) = x(t-\tau)e^{j2\pi\nu t}e^{-j2\pi\frac{\nu\tau}{2}}.$$

The corresponding spreading function is

$$S_H^{(s)}(\tau,\nu) = \int_t h\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)e^{-j2\pi\nu t}dt$$

and will be called *symmetrical spreading function* (SSF). Analogously to (19), we have the decomposition

$$\mathbf{H} = \iint_{\tau\,\nu} S_H^s(\tau,\nu)\bar{\mathrm{S}}^{(\tau,\nu)}d\tau d\nu.$$

The SSF is essentially the double Fourier transform of the WS,

$$\begin{aligned} S_H^{(s)}(\tau,\nu) &= \iint_{t\,f} L_H(t,f)e^{-j2\pi(\nu t-\tau f)}dtdf \\ &= \mathcal{F}^2_{(t,f)\to(\nu,-\tau)}\{L_H(t,f)\}. \end{aligned}$$

The spreading functions are closely related with two correlative time–frequency signal representations known as (symmetrical and asymmetrical) *ambiguity functions* [6]. For a normal system **H** with square–integrable impulse response (3), the symmetrical spreading function can be written as

$$S_H^{(s)}(\tau,\nu) = \sum_{k=0}^{\infty} \lambda_k A_{u_k}^{(s)}(\tau,\nu),$$

where

$$A_{u_k}^{(s)}(\tau,\nu) = \int_t u_k\left(t+\frac{\tau}{2}\right) u_k^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f t} dt$$

is the symmetrical ambiguity function of the signal $u_k(t)$. Similarly, the asymmetrical spreading function is

$$S_H^{(a)}(\tau,\nu) = \sum_{k=0}^{\infty} \lambda_k A_{u_k}^{(a)}(\tau,\nu),$$

where

$$A_{u_k}^{(a)}(\tau,\nu) = \int_t u_k(t) u_k^*(t-\tau) e^{-j2\pi f t} dt$$

is the asymmetrical ambiguity function of the signal $u_k(t)$. The relation between the symmetrical spreading function $S_H^{(s)}(\tau,\nu)$ and its asymmetrical counterpart $S_H^{(a)}(\tau,\nu)$ is given by

$$S_H^{(a)}(\tau,\nu) = S_H^{(s)}(\tau,\nu) e^{-j2\pi\frac{\nu\tau}{2}}. \tag{20}$$

In particular, it follows from (20) that $|S_H^{(a)}(\tau,\nu)| = |S_H^{(s)}(\tau,\nu)|$. If **H** is a stochastic system corresponding to wide–sense stationary uncorrelated scattering [10], then it can be shown that

$$E\left\{\left|S_H^{(a)}(\tau,\nu)\right|^2\right\} = E\left\{\left|S_H^{(s)}(\tau,\nu)\right|^2\right\}$$

equals the well–known *scattering function* of **H**.

8 CONCLUSION

The Weyl symbol (WS) represents a time–frequency weighting characteristic of an LTV system that is consistent with the conventional transfer function of an LTI system. Formally, the WS is largely analogous to Zadeh's time–varying transfer function (ZF). In fact, it has been shown that both WS and ZF are closely related to a corresponding time–frequency *signal* representation, namely, the Wigner distribution (WD) in the case of WS and the Rihaczek distribution (RD) in the case of ZF. The difference between WS and ZF can thus be attributed to the differences between WD and RD.

The Fourier transforms of WS and ZF can both be interpreted as the system's time–frequency spreading functions (based on different definitions of the underlying time–frequency shift operators). These spreading functions are closely related to two correlative time–frequency signal representations known as symmetrical and asymmetrical ambiguity functions, respectively.

The application of the WS to a joint time–frequency *design* of LTV systems has been considered in [11].

After completion of this manuscript we became aware of the paper [12], where the Weyl correspondence is introduced as a time–frequency signal analysis tool.

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