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SIGNAL REPRESENTATIONS GEOMETRY AND CATASTROPHES IN THE TIME-FREQUENCY PLANE

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ABSTRACT

Generalized Wigner Distribution (GWD) is a class of time-frequency signal representations with particularly interesting properties. These properties being well known, we here take a more descriptive standpoint and discuss the "shape" of GWD. It is shown that GWD is in principle governed by the characteristic laws of interference geometry. For the important class of amplitude and frequency modulated signals, this result is confirmed and rendered more concrete by approximations using the stationary phase method and catastrophe theory. Optimality of Wigner Distribution, among the whole GWD class, is clearly revealed by its ideally concentrated and symmetric interference structure.

1. INTRODUCTION

Joint time-frequency representations of signals, like Wigner Distribution, Spectrogram, and Ambiguity Function, are valuable tools for signal analysis. This paper is concerned with Generalized Wigner Distribution (GWD), which, among the time-frequency representations of the energy density type, satisfies a great number of desirable properties and is thus most attractive from a theoretical standpoint (see Claasen and Mecklenbräuker (1980)).

Let $x(t) \leftrightarrow X(f)$, $y(t) \leftrightarrow Y(f)$ denote two signals with their spectra, where t is time and f is frequency. GWD $W_{x,y}^{(\alpha)}(t,f)$ is a family of time-frequency representations of x,y depending on a parameter $\alpha \in \mathbb{R}$,

$$W^{(\alpha)}_{x,y}(t,f) = \int_\tau x(t+(\tfrac{1}{2}+\alpha)\tau)\; y^*(t-(\tfrac{1}{2}-\alpha)\tau)\, e^{-j2\pi f\tau} d\tau$$
$$= \int_\nu X(f+(\tfrac{1}{2}-\alpha)\nu)\; Y^*(f-(\tfrac{1}{2}+\alpha)\nu)\; e^{j2\pi t\nu} d\nu. \qquad (1.1)$$

In most applications, a single signal is analyzed,

$$W^{(\alpha)}_x(t,f) \triangleq W^{(\alpha)}_{x,x}(t,f). \qquad (1.2)$$

Two special cases of GWD are noteworthy: $\alpha=0$ and $\alpha=\tfrac{1}{2}$ yield Wigner Distribution (WD) and Rihaczek Distribution (RD), respectively. GWD will be complex-valued unless $\alpha=0$.

In this paper, rather than discussing formal properties of GWD, we shall take the user's standpoint and ask what GWD will look like in typical cases. It will be demonstrated that the shape of GWD is determined by a characteristic property which we call interference geometry. The general formulation of interference geometry will then clearly show that the choice $\alpha=0$, i.e. WD, is optimum for practical applications. The interference phenomenon has been previously studied in Janssen (1982), Flandrin (1984), Hlawatsch (1984).

2. OUTER INTERFERENCE

GWD is a bilinear signal transformation and is thus governed by a bilinear superposition law: if

$$z(t) = a.x(t) + b.y(t) \qquad (2.1)$$

then

$$W^{(\alpha)}_z(t,f) = |a|^2\, W^{(\alpha)}_x(t,f) + |b|^2\, W^{(\alpha)}_y(t,f) + I^{(\alpha)}_{x,y}(t,f), \qquad (2.2)$$

with the cross term or outer interference term

$$I^{(\alpha)}_{x,y}(t,f) = ab^*\, W^{(\alpha)}_{x,y}(t,f) + ba^*\, W^{(\alpha)}_{y,x}(t,f). \qquad (2.3)$$

Note that only for $\alpha=0$ (WD) this reduces to

$$I^{(o)}_{x,y}(t,f) = 2\,\mathrm{Re}\left\{ab^*\, W^{(o)}_{x,y}(t,f)\right\}. \qquad (2.4)$$

It is obvious that the occurrence of outer interference terms will strongly influence the shape of GWD in the case of multicomponent signals. To consider a simple test case, suppose that in (2.1) x and y are two signals x_+ and x_- which are derived from a given signal x_o through time-frequency shifts by $(\frac{\tau}{2},\frac{\nu}{2})$ and $(-\frac{\tau}{2},-\frac{\nu}{2})$ respectively. With the operator $s^{(\eta)}_\zeta$ denoting a shift by ζ with respect to the variable η, the components of (2.2) are then (assuming $a = b = 1$)

$$W^{(\alpha)}_{x_\pm}(t,f) = s^{(t)}_{\pm\tau/2}\, s^{(f)}_{\pm\nu/2} \left\{W^{(\alpha)}_{x_o}(t,f)\right\} \qquad (2.5)$$

$$W^{(\alpha)}_{x_\pm,x_\mp}(t,f) = s^{(t)}_{\mp\alpha\tau}\, s^{(f)}_{\pm\alpha\nu} \left\{W^{(\alpha)}_{x_o}(t,f)\; e^{-j2\pi(\nu t-\tau f\,\mp\,\alpha.\tau\nu)}\right\} \qquad (2.6)$$

While the overall position of the two "signal terms" (2.5) is thus independent from α, this is not at all true for the interference term components (2.6). In fact, (2.6) already expresses the interference geometry of GWD: the interference term consists in general of two distinct subterms which are derived from $W^{(\alpha)}_{x_o}$ by a plane-wave modulation and subsequent shifts by $(\mp\alpha\tau, \pm\alpha\nu)$. The modulation causes an oscillation in a direction orthogonal to the line connecting the two "signal terms" $W^{(\alpha)}_{x_\pm}$. Oscillation "frequency" is given by the distance between the two signal terms, it is independent of α.

A schematic illustration of interference geometry is given by Fig. 1.a. For $\alpha = \tfrac{1}{2}$ (RD) and $\alpha = -\tfrac{1}{2}$, signal and interference terms form the corners of a rectangle with lateral lengths τ,ν (Fig. 1.b). For $\alpha=0$ (WD), the two interference subterms merge into a single real-valued interference term

$$I^{(O)}_{x_+,x_-}(t,f) = 2.W^{(o)}_{x_o}(t,f)\; .\; \cos 2\pi(\nu t - \tau f) \qquad (2.7)$$

located at the centre between the two signal terms (Fig. 1.c). The uniqueness of WD is hence stressed by interference geometry through an ideal concetration and symmetry of interference terms.

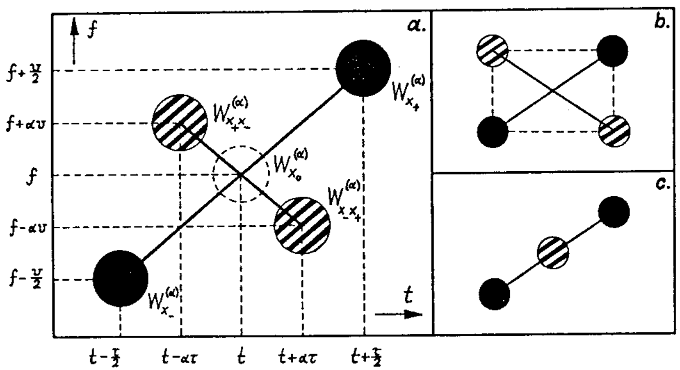

Fig. 1

The laws of interference geometry, although derived above by a simple example, are in fact a general property of GWD. A general indication of the position of outer interference terms (not, however, of their oscillations), is given by GWD's "outer interference formula" (comp. Janssen (1982))

$$|W^{(\alpha)}_{x,y}(t,f)|^2 =$$

$$\iint_{\tau\nu} W_x^{(\alpha)}(t+(\tfrac{1}{2}+\alpha)\tau,\ f+(\tfrac{1}{2}-\alpha)\nu)\ .\ W_y^{(\alpha)*}(t-(\tfrac{1}{2}-\alpha)\tau, f-(\tfrac{1}{2}+\alpha)\nu)\ d\tau d\nu: \qquad (2.8)$$

if the signal terms $W_x^{(\alpha)}$, $W_y^{(\alpha)}$ are located around points $(t\pm{}^{\tau}/_{2},\ f\pm{}^{\nu}/_{2})$ in the (t,f) -plane, then the interference term $I^{(\alpha)}_{x,y}$ is located around points $(t\mp\alpha\tau,\ f\pm\alpha\nu)$. This agrees with interference geometry.

3. INNER INTERFERENCE

So far, the interference character of GWD has been considered as a direct consequence of GWD's bilinear superposition law. Interference terms, however, are not always explicit bilinear cross terms of distinct signal components (outer interference). Rather, GWD also shows an inherent interference structure which is equally present in the GWD of monocomponent signals (inner interference).

Consider the GWD of a signal x which may be monocomponent. With y=x, (2.8) yields the "inner interference formula"

$$|W_x^{(\alpha)}(t,f)|^2 =$$

$$\iint_{\tau\nu} W_x^{(\alpha)}(t+(\tfrac{1}{2}+\alpha)\tau, f+(\tfrac{1}{2}-\alpha)\nu)\ .\ W_x^{(\alpha)*}(t-(\tfrac{1}{2}-\alpha)\tau, f-(\tfrac{1}{2}+\alpha)\nu)\,d\nu. \qquad (3.1)$$

If $W_x^{(\alpha)}$ is large and non-oscillatory around points $(t\pm{}^{\tau}/_{2}, f\pm{}^{\nu}/_{2})$ (we could then say that some part of signal energy is located there), then $W_x^{(\alpha)}$ is also large around the points $(t\mp\alpha\tau, f\pm\alpha\nu)$. These, then, are points of inner interference, in perfect agreement with interference geometry.

Inner interference terms also show the characteristic oscillations described by interference geometry. This will be demonstrated for an important class of signals in the next section.

4. GWD OF AM-FM SIGNALS

We consider a complex signal x(t) with amplitude and frequency modulation (AM-FM)

$$x(t) = a(t)\ e^{j\phi(t)},\ a \geq 0,\ \phi \in \mathbb{R}, \qquad (4.1)$$

where the AM term a(t) is slowly varying and the instantaneous phase $\phi(t)$ is determined by instantaneous frequency $f_i(t)$,

$$f_i(t) = \frac{1}{2\pi}\ \phi'(t). \qquad (4.2)$$

For such a signal, GWD can be written as

$$W_x^{(\alpha)}(t,f) = \int_{\tau} L(t,\tau)\ e^{j\Phi(t,f;\tau)}d\tau \qquad (4.3)$$

with

$$L(t,\tau) = a(t+(\tfrac{1}{2}+\alpha)\tau).a(t-(\tfrac{1}{2}-\alpha)\tau) \qquad (4.4)$$

$$\Phi(t,f;\tau) = \phi(t+(\tfrac{1}{2}+\alpha)\tau)-\phi(t-(\tfrac{1}{2}-\alpha)\tau) - 2\pi f.\tau. \qquad (4.5)$$

GWD (4.3) is an oscillatory integral for which an approximate solution is obtained by the method of stationary phase (see e.g. Papoulis (1977)): for a specific point (t_o,f_o), let $\{\tau_i\}$,

$i=1,\dots,n$ be the values of τ for which

$$\Phi_o'(\tau_i) = 0 \quad \text{and} \quad \Phi_o''(\tau_i) \neq 0, \tag{4.6}$$

where $\Phi_o(\tau)$ is short for $\Phi(t_o,f_o;\tau)$. Then each of the τ_i will yield a distinct contribution to $W_x^{(\alpha)}(t_o,f_o)$:

$$W_x^{(\alpha)}(t_o,f_o) \simeq \sum_{i=1}^{n} \frac{L(t_o;\tau_i)}{\sqrt{|\Phi_o''(\tau_i)|}}\, e^{j[\Phi_o(\tau_i) + \frac{\pi}{4}\,\mathrm{sign}\,\Phi_o''(\tau_i)]}\,. \tag{4.7}$$

Eqs. (4.6) amd (4.5) yield a relation between t_o, f_o and τ,

$$f_o = \frac{1}{2}[f_i(t_1) + f_i(t_2)] + \alpha\,[f_i(t_1) - f_i(t_2)] \tag{4.8}$$

and the condition

$$(\tfrac{1}{2}+\alpha)^2 . f_i'(t_1) \neq (\tfrac{1}{2}-\alpha)^2 . f_i'(t_2) \tag{4.9}$$

where $t_1 = t_o + (\frac{1}{2}+\alpha)\tau$, $t_2 = t_o - (\frac{1}{2}-\alpha)\tau$. Defining t, f and ν by

$$t = \tfrac{1}{2}[t_1+t_2]\,,\ \tau = t_1 - t_2\ ;\ f = \tfrac{1}{2}[f_i(t_1)+f_i(t_2)]\,,\ \nu = f_i(t_1) - f_i(t_2), \tag{4.10}$$

this can be formulated as follows: two points $f_i(t_1)$, $f_i(t_2)$ of the instantaneous frequency curve $f_i(t)$ interfere and create two subterms of inner interference at points $(t_o,f_o) = (t\mp\alpha\tau, t\pm\alpha\nu)$, according to whether $+\tau$ or $-\tau$ is inserted in (4.7). This is shown by Fig. 2. The connection with interference geometry is easily found by regarding the instantaneous frequency curve $f_i(t)$ as signal term; different points of this one signal term then interfere according to the laws of interference geometry (comp. Fig. 1).

The interference terms' oscillation, too, can be derived from the stationary phase approximation (4.7). Consider, for a specific point (t_o,f_o), a single contribution to (4.7) given by some τ. In local surroundings of (t_o, f_o), i.e. for small $|\Delta t|$, $|\Delta f|$, (4.7) then yields

$$W_x^{(\alpha)}(t_o+\Delta t, f_o+\Delta f) \simeq C(t_o,f_o).a(t_1)a(t_2).e^{j2\pi[\nu.(t_o+\Delta t) - \tau(f_o+\Delta f)]}\,, \tag{4.11}$$

where $f_i(t_1)$ and $f_i(t_2)$ are the interfering points of the instantaneous frequency curve (see above) and $\tau = t_1 - t_2$, $\nu = f_i(t_1) - f_i(t_2)$ are the time and frequency differences between these points. This is exactly the plane-wave oscillation described by interference geometry.

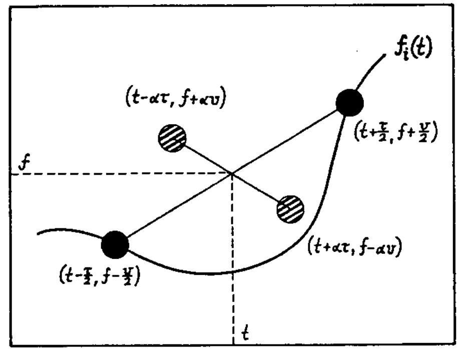

Fig. 2

The strong dependence of the interference terms' position on α will sometimes lead to dramatic differences between the shapes of different GWDs. Fig. 3 compares WD (α=0) and real part of RD ($\alpha=\frac{1}{2}$) of a chirp signal (FM linear, AM according to raised-cosine window). Note that these results can be predicted from interference geometry (comp. Fig. 1.c, Fig. 1.b).

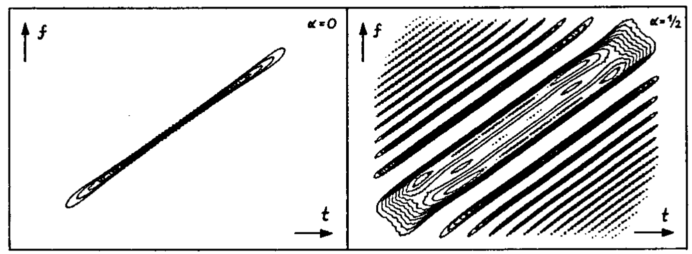

Fig. 3

5. CATASTROPHES OF WD

We consider again the AM-FM signal (4.1) but restrict to the WD case (α=0) for simplicity. For a specific (t_o,f_o), the stationary points are given by (4.8)

$$f_o = \frac{1}{2}[f_i(t_o + {}^{\tau}/_2) + f_i(t_o - {}^{\tau}/_2)], \qquad (5.1)$$

provided that (4.9)

$$f_i'(t_o + {}^{\tau}/_2) \neq f_i'(t_o - {}^{\tau}/_2). \qquad (5.2)$$

For a geometrical interpretation, consider the simple example of a sinusoidal FM shown in Fig. 4. Evidently, there are three cases:

i) For (t_o, f_o) inside the shaded area (point P_1), there are two stationary points $\pm\tau$ whose combination (see (4.7)) yields an oscillatory structure (the local cosine terms of inner WD interference). According to (5.1), the two stationary points $\pm\tau$ are defined by the chord whose endpoints lie on $f_i(t)$ and whose midpoint is P_1 (interference geometry, comp. Fig. 2 with α=0).

ii) On $f_i(t)$ (point P_2), the two stationary points coalesce ($\tau \to 0$, zero length chord). Since (5.2) is no longer valid, the stationary phase approximation (4.7) diverges.

iii) Outside the shaded area (point P_3), there is no τ satisfying (5.1), and the stationary phase approximation vanishes.

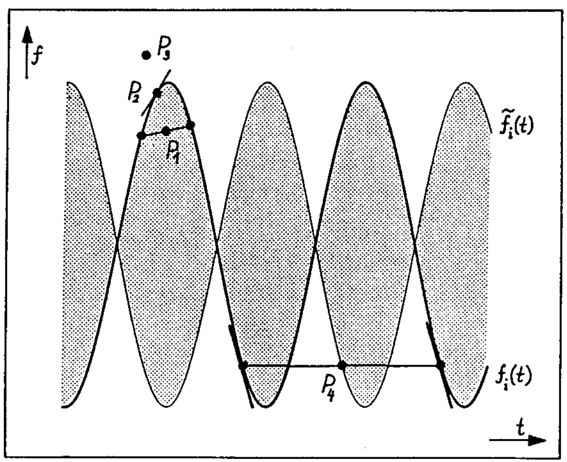

Fig. 4

The sudden change in behaviour which occurs when passing through the boundaries of the shaded area in Fig. 4 can be viewed as a "catastrophe" of WD. In fact catastrophe theory (CT) can be applied to yield a qualitative description of these changes, see Poston and Stewart (1978), Berry (1977), Flandrin and Escudié with contributions of Bouachache (1981). According to CT, the occurrence of "catastrophic" changes of oscillatory integrals like (4.3) depends only on the singularities (vanishing of derivatives) of the phase function $\Phi_o(\tau)$. This, in a sense, generalizes the stationary phase method. The specific interest of CT lies in the fact that it yields a classification of the basic geometrical structures (morphologies) which typically occur. In our case, only two such typical structures ("catastrophes") exist, namely, the "fold" and the "cusp" catastrophe (see Poston and Stewart (1978)).

It follows from this that, in the qualitative behaviour of WD, four cases can be distinguished, according to the derivatives of the phase function $\Phi_o(\tau)$ (see (4.5) with α=0):

i) *Stationary point areas* (e.g. the shaded area in Fig. 4) consist of those points (t_o, f_o) for which there exist two τ's such that $\Phi_o'(\tau) = 0$, $\Phi_o''(\tau) \neq 0$. An approximation of WD is here given by the stationary phase method (cosine function).

ii) *Fold lines* consist of those points (t_o, f_o) for which $\Phi_o'(\tau) = \Phi_o''(\tau) = 0$, $\Phi_o'''(\tau) \neq 0$. This, in particular, implies

$$f_i'(t_o + {}^{\tau}/_2) = f_i'(t_o - {}^{\tau}/_2), \qquad (5.3)$$

so that here the slopes of $f_i(t)$ at the chord endpoints (interfering points) are equal. As shown by Fig. 4, there are two types of fold lines: a) the instantaneous frequency curve $f_i(t)$ itself (e.g. point P_2) for which τ=0; an approximation of WD is here given by the Airy function; b) a "ghost" curve $\tilde{f}_i(t)$ (e.g. point P_4) for which $\tau \neq 0$.

iii) *Cusp points* (t_o, f_o) are characterized by $\Phi_o'(\tau) = \Phi_o''(\tau) = \Phi_o'''(\tau) = 0$, $\Phi_o''''(\tau) \neq 0$. They are points of fold lines (ghost type) for which, in addition to equal slopes (see (5.3)), the curvatures of $f_i(t)$ at the chord endpoints are equal in magnitude but opposite in sign (see Fig. 5, point P),

$$f_i''(t_o+\tau/2) = - f_i''(t_o-\tau/2). \qquad (5.4)$$

At cusp points, the WD is peaked and can be approximated by a Pearcey function.

iv) *Points of higher-order singularities* are midpoints of an infinity of chords. They occur only in the case of a perfectly (anti-) symmetrical $f_i(t)$. At these points, the WD is highly peaked. Fig. 6 gives three examples (point P).

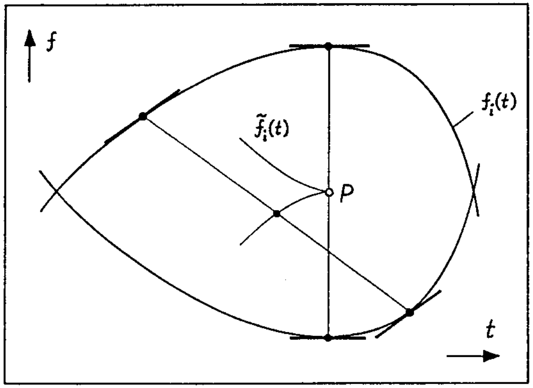

Fig. 5

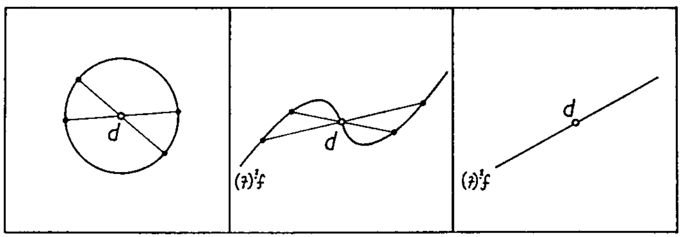

Fig. 6

6. CONCLUSION

GWD is in principle governed by the simple laws of interference geometry. Variation of GWD's parameter α will essentially lead to a variation of the interference terms'

position which may have a dramatic influence on GWD's shape. Thus different GWDs of the same signal may look totally different even though most of the mathematical properties are the same. It is thus from the standpoint of interference geometry that "optimality" of WD ($\alpha=0$) is most evident.

The description of the GWD and WD of AM-FM signals which was based on the stationary phase method and catastrophe theory is consistent with interference geometry but at the same time yields more concrete results. We stress that the approach taken here could be extended to bilinear signal representations other than GWD. In particular, the class of Generalized Ambiguity Functions (which is dual to GWD by Fourier transform, see Hlawatsch (1985)) shows an interference geometry which is analogous to that of GWD (see Flandrin and Escudié (1981), Flandrin (1984)).

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A UNIFIED THEORY OF MODEL REDUCTION VIA GLEASON MEASURES

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ABSTRACT

Earlier work has cast the stochastic realization and approximation problem in the framework of the RV-coefficient. This allowed the introduction of a common measure for the "goodness of fit" for the different realization algorithms. This paper explores the deeper geometrical basis for this common measure in a unified theory for the data driven and exact covariance approaches.

1. INTRODUCTION

1.1 Scope of the Paper

In the theory of identification, signal processing, and digital filtering, a problem of fundamental importance is that of finding a finite dimensional Markovian representation of a stochastic process from the covariance information. This problem is known as the Stochastic Realization Problem, and has received a great deal of attention. Whenever a finite set of real data is gathered, all processing is done over finite sets, and an underlying probabilistic description is absent in most cases. As a result, covariances must be estimated by sample covariances, and a "degradation" of the theoretical realization solutions results. A more direct, data driven approach is needed. Moreover, for many applications, the Markovian representation or state space model may be too complex, due to its high dimensionality, thus barring efficient computational management. This motivates the quest for approximate lower order models, and the need for common measures to evaluate and compare different approaches.