INTERFERENCE TERMS IN THE WIGNER DISTRIBUTION *)

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The Wigner Distribution (WD) is a joint time-frequency signal representation with very attractive properties. Application of the WD, however, is often complicated by the occurrence of interference terms. We first draw a distinction between "outer" interferences in the case of multicomponent signals and "inner" interferences that are not based on signal decomposition. It is demonstrated that both interference mechanisms show very simple geometrical properties. Inner interference geometry is then used to determine the general shape of WD supports, and finally the energy content of interference terms is investigated.

1. INTRODUCTION

The Wigner Distribution (WD) is a joint time-frequency signal representation that can be considered as a distribution of signal energy over the time-frequency plane. Unique properties and superior time-frequency resolution make it appear ideally suited for the analysis of nonstationary signals [1]. Yet interpretation of the WD is often complicated by the occurrence of interference terms which may cause serious difficulties in applications. In this paper, WD interference terms are investigated in some detail. For convenience the continuous-time WD is considered, but results carry over to the discrete-time WD, too.

The Cross Wigner Distribution (CWD) of two signals $f, g \in \mathbb{C}$ is defined by [1]

$$W_{f,g}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t+\tfrac{\tau}{2})\, g^*(t-\tfrac{\tau}{2})\, e^{-j\omega\tau}\, d\tau . \qquad (1)$$

Taking $g=f$ yields the (Auto)Wigner Distribution (WD) of f,

$$W_f(t,\omega) := W_{f,f}(t,\omega) , \qquad (2)$$

which is a real-valued, but not always nonnegative function that can be represented as a surface over the time-frequency plane ((t,ω) -plane).

2. BILINEARITY AND "OUTER" INTERFERENCE

We first consider the case of a multicomponent signal: let f be a sum of n components f_k whose WDs we know,

$$f(t) = \sum_{k=1}^{n} f_k(t) . \qquad (3)$$

What results can we expect for the WD of f? It follows from (1) that the WD is formed by a bilinear superposition principle,

$$W_f(t,\omega) = \sum_{k=1}^{n} W_{f_k}(t,\omega) + \sum_{\substack{k=1 \\ k>l}}^{n}\sum_{l=1}^{n} 2\cdot \mathrm{Re}\left\{W_{f_k,f_l}(t,\omega)\right\} : \qquad (4)$$

it consists of n "signal terms" (WDs of components) and $\binom{n}{2}$ "interference terms" (real parts of CWDs of components). Thus any signal component yields a signal term, and any pair of signal components gives rise to a corresponding interference term. This is illustrated by fig. 1.a.

For reasons of simplicity, we restrict ourselves to the case of two signal components f and g. Is there any relation between their signal terms W_f, W_g and their interference term $I_{f,g} = 2\cdot \mathrm{Re}\{W_{f,g}\}$? Some insight is gained by the "interference formula" [2]

$$|W_{f,g}(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t+\tfrac{\tau}{2}, \omega+\tfrac{\nu}{2})\, W_g(t-\tfrac{\tau}{2}, \omega-\tfrac{\nu}{2})\, d\tau\, d\nu \qquad (5)$$

that traces the modulus of the CWD of f and g back to their WDs. Suppose that W_f and W_g lie in the surroundings of points (t_f, ω_f) and (t_g, ω_g) in the (t,ω)-plane. Then (5) states that the interference term lies near the central point

$$(t_m, \omega_m) = \left(\frac{t_f+t_g}{2}, \frac{\omega_f+\omega_g}{2}\right) :$$

interference terms lie midway between corresponding signal terms. - Apart from the interferences' position, the interference formula suggests a modified interpretation of interference mechanism: not the signal components f, g, but the WD signal terms W_f, W_g themselves interfere and create an interference term between themselves. Such an interference of two WDs will be called an outer interference.

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Further properties of interference terms can be demonstrated by studying a special case: the signal h (which, for reasons of clearness, may be thought to be concentrated around $(0,0)$) is shifted in time and frequency by (t_f,ω_f) and (t_g,ω_g) and multiplied by complex constants a,b to yield the signals f,g:

$$f(t) = a\cdot h(t-t_f)\,e^{j\omega_f t}\,, \quad g(t) = b\cdot h(t-t_g)\,e^{j\omega_g t}. \tag{6}$$

Then, the WDs of f and g are [1]

$$\begin{aligned} W_f(t,\omega) &= |a|^2\cdot W_h(t-t_f,\omega-\omega_f)\,, \\ W_g(t,\omega) &= |b|^2\cdot W_h(t-t_g,\omega-\omega_g)\,, \end{aligned} \tag{7}$$

and the interference term $I_{f,g} = 2\cdot\mathrm{Re}\{W_{f,g}\}$ can be calculated to be

$$I_{f,g}(t,\omega) = 2|ab|\cdot\cos\left[\omega_d(t-t_m) - t_d(\omega-\omega_m) + \varphi_0\right] \times$$
$$\times\, W_h(t-t_m,\omega-\omega_m) \tag{8}$$

with $(t_d,\omega_d) = (t_g-t_f,\,\omega_g-\omega_f)$

and $(t_m,\omega_m) = \left(\frac{t_f+t_g}{2},\,\frac{\omega_f+\omega_g}{2}\right)$.

These results are illustrated in fig.1.b. In particular we see from (8) that the interference term is a modulated version of the WD of the original signal h shifted to the central point (t_m,ω_m) . The modulation causes an oscillation in a direction $(-\omega_d,t_d)$ that is orthogonal to the line that connects the two "signal points" (t_f,ω_f), (t_g,ω_g) ; the "frequency" of the oscillation is simply the distance $\sqrt{t_d^2+\omega_d^2}$ between the signal points (here, t_d and ω_d must be considered as dimensionless distances). Experiments indicate that in principle these simple geometrical laws hold quite generally, even if W_f and W_g are not of the same form.

Thus in general interference terms can be recognized by their oscillations, and there exist simple relations between the positions of signal terms and the direction and "frequency" of the corresponding interference term's oscillation. It should be remarked, however, that interference terms oscillate only if the interfering signal terms are to some extent disjoint in the (t,ω)-plane. Indeed a meaningful distinction between signal terms and interference terms is possible only in this case. To demonstrate this, let us consider a trivial signal decomposition into components occupying the same region in the (t,ω)-plane:

$$h = f + g \quad \text{where} \quad f = a\cdot h\,,\; g = (1-a)\cdot h\,. \tag{9}$$

Of course, the interference term is

$$I_{f,g} = 2\cdot\mathrm{Re}\{a\cdot(1-a)^*\}\cdot W_h(t,\omega)\;: \tag{10}$$

it does not show any oscillation; even more, being the WD of h, it may be equally regarded as a signal term. This demonstrates how the WD distinguishes between a signal decomposition that is physically meaningful and one that is not.

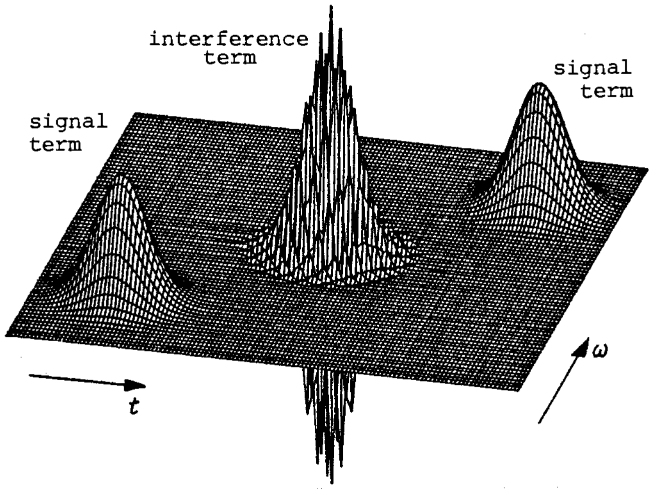

fig.1.a. WD of superposition of two gaussian signals shifted in time and frequency.

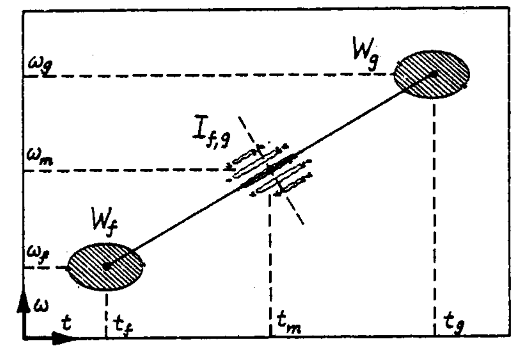

fig.1.b. Contour plot to fig.1.a.

3. "INNER" INTERFERENCE

In the above discussion, we started from a multicomponent signal: the WD's decomposition into signal terms and outer interference terms was based on a decomposition of the signal itself. Instead of starting from the signal and trying to determine its WD, let us now start from a given WD. We try to decide in what regions of the (t,ω)-plane signal energy is located, without any knowledge about signal components. It is an empirical fact that, even in those cases where a natural signal decomposition is not possible (monocomponent signals), the WD consists of "robust" (i.e. non-negative and smooth) terms and oscillatory regions. Thus a distinction between signal terms and interference terms again seems to be appropriate. Experiments indicate that the geometrical interference laws formulated above are still valid.

If in (5) we set $g=f$, we obtain [3]

$$|W_f(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t+\tfrac{\tau}{2},\omega+\tfrac{v}{2})\, W_f(t-\tfrac{\tau}{2},\omega-\tfrac{v}{2})\, d\tau\, dv\,. \qquad (11)$$

This relation describes the WD's internal interference structure: for any point (t,ω), $W_f(t,\omega)$ is made up by "interference" of all pairs of points that lie symmetrically to (t,ω); at the same time, $W_f(t,\omega)$ itself causes "interferences" at other points. In particular, if W_f shows robust terms in the vicinity of two points (t_1,ω_1), (t_2,ω_2) it may again be concluded that near the central point $(t_m,\omega_m)=\left(\frac{t_1+t_2}{2},\frac{\omega_1+\omega_2}{2}\right)$ there must be a further term. Experiments show that this term oscillates in exactly the same manner as the outer interference terms mentioned in the previous section. It can thus be considered to be an "inner" interference term. Note that this distinction between signal terms and interference terms is not based on signal decomposition but starts from the WD itself.

Fig.2 and fig.3 demonstrate the geometrical interference laws in the case of inner interferences in the WDs of monocomponent signals. In theory, signal terms interfere no matter how distant

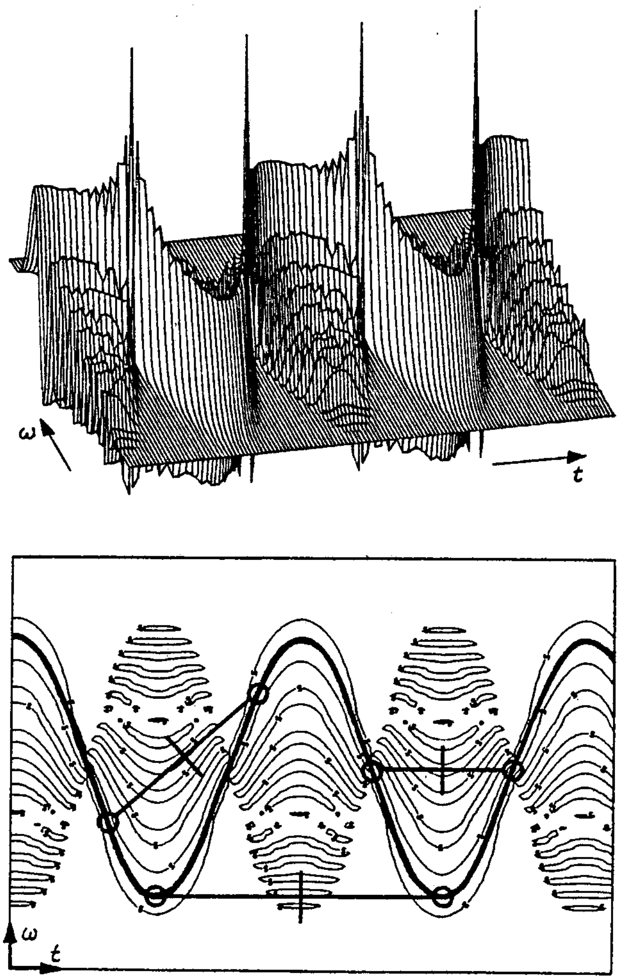

fig.2. WD (windowed version) of sinusoidal FM signal. The signal term is located along the instantaneous-frequency curve. Concave parts of this curve are filled by (inner) interference terms.

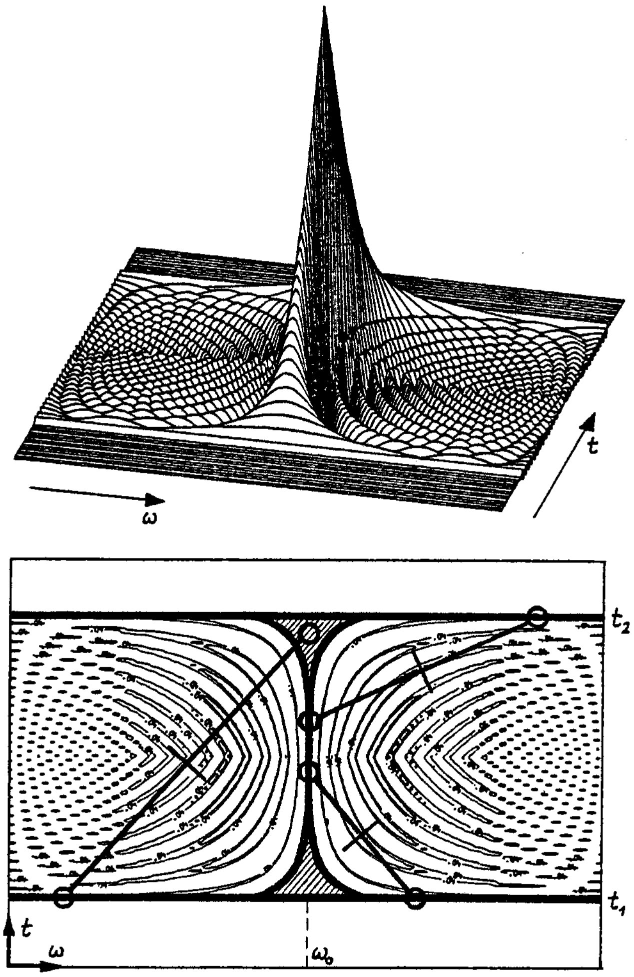

fig.3. WD of tone burst (frequency ω_0; switched on (off) at t_1 (t_2)). The signal term consists of a "spectral line" at ω_0 and two broad-band "clicks" at t_1, t_2.

they are from each other, which renders WD results quite complicated. Yet in practice a windowed version of the WD [1] is employed, and then the time distance of interfering signal terms is limited by the window length.

4. SHAPE OF WD SUPPORTS

Let us denote by the support of a given WD the smallest closed region (where part of its boundary, or all of it, may lie at infinity) outside of which the WD vanishes. Inner interference geometry gives some insight into the possible shapes of WD supports. Consider, for example, the region shown in fig.4. Such a finite WD support is not possible, for it would postulate a signal with finite support both in time and frequency. This impossibility of a finite WD support can also be made plausible by the WD's internal interference structure described by (11): a marginal point (P) in a convex part of the boundary does not possess any pair of points that would make it up by interference. On the other hand, a concave boundary as in fig.5 is not possible either, for there would be interference points (e.g. Q).

It is thus made plausible that, if the WD vanishes at all outside a region, this region must be an (infinite) strip or a half plane. What does the boundary look like? Let us for the moment assume a W_f that is a strip with arbitrary boundaries. This strip can again be bounded by two parallel lines l_1, l_2 as is shown in fig.6. Now the WD's "rotation property" [3] is used: rotating W_f until l_1, l_2 are parallel to the ω-axis again yields the WD of some other signal $\tilde{f}$ (fig.7). Obviously, $\tilde{f}$ is constrained to a finite time interval $[t_1,t_2]$:

$$\tilde{f}(t) = 0, \quad t \notin [t_1,t_2]. \tag{12}$$

Let us investigate $W_{\tilde f}$ in the vicinity of, say, t_2. We want to evaluate $W_{\tilde f}(t_2-\varepsilon,\omega)$ for a small positive ε. From (1) we have

$$W_{\tilde f}(t_2-\varepsilon,\omega) = \frac{1}{2\pi}\cdot\int_{-2\varepsilon}^{2\varepsilon}\tilde f\left((t_2-\varepsilon)+\frac{\tau}{2}\right)\tilde f^*\left((t_2-\varepsilon)-\frac{\tau}{2}\right)e^{-j\omega\tau}\,d\tau$$

$$\approx \frac{1}{2\pi}\cdot\tilde f(t_2-\varepsilon)\tilde f^*(t_2-\varepsilon)\cdot 4\varepsilon\,. \tag{13}$$

This is constant and positive and therefore does not vanish for any ω. Hence it follows that the lines l_1, l_2 are themselves the boundaries of $W_{\tilde f}$ (fig.8), and after rotating back we obtain the original W_f according to fig.9. We can thus formulate the following basic WD property: WD supports are strips that are bounded by two parallel lines (where one line or both may be located at infinity).

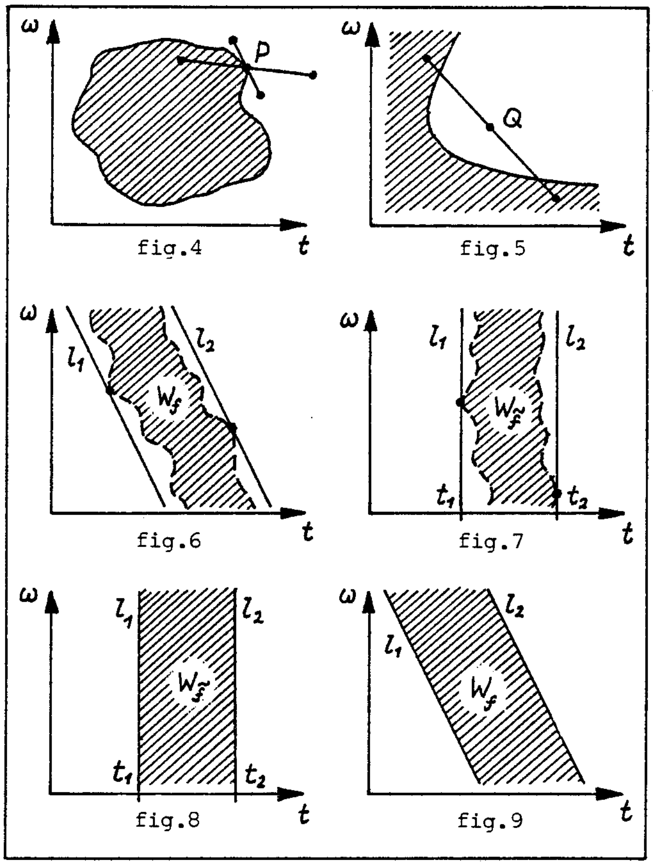

fig.4 fig.5 fig.6 fig.7 fig.8 fig.9

5. ENERGY CONTENT OF INTERFERENCE TERMS

In this section we will be concerned with outer interferences. For a superposition $h=f+g$, signal energy is

$$E_h = E_f + E_g + C_{f,g} \tag{14}$$

with $C_{f,g}$ being an energy cross-term,

$$C_{f,g} = 2\cdot Re\{E_{f,g}\}\ , \quad E_{f,g} = \int_{-\infty}^{\infty} f(t)g^*(t)\,dt\,. \tag{15}$$

For the WD we find

$$W_h(t,\omega) = W_f(t,\omega) + W_g(t,\omega) + I_{f,g}(t,\omega) \tag{16}$$

where $I_{f,g}$ is the interference term of f and g:

$$I_{f,g}(t,\omega) = 2\cdot Re\{W_{f,g}(t,\omega)\}\,. \tag{17}$$

A connection is established by the WD's energy property [1]:

$$E_f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\,dt\,d\omega \quad \text{(analogously for } g \text{ and } h\text{)}, \tag{18}$$

$$E_{f,g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{f,g}(t,\omega)\,dt\,d\omega\,, \tag{19}$$

$$C_{f,g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_{f,g}(t,\omega)\,dt\,d\omega\,. \tag{20}$$

Thus the integral of a WD signal- or interference term may be called its energy content. According to (14), signal energy is made up by the energy contents of the various WD terms. While in principle both signal- and interference terms contribute to signal energy, the interference terms' oscillations suggest that their energy content is small compared to the energy content of the robust signal terms. In particular we may ask whether there are cases in which the energy content of an interference term is exactly zero.

Let us call two signals disjoint if their WD supports do not overlap. (Time disjointness and frequency disjointness are just two special cases of disjointness). We want to show that the interference term of disjoint signals has zero energy content.

We start by noting that, according to the previous section, the WDs of two disjoint signals f,g can be bounded only by two parallel lines l_1 , l_2 as is shown in fig.10. Then again a suitable rotation can be employed so that l_1 , l_2 are parallel to the ω-axis (fig. 11.) This rotation transforms f,g into

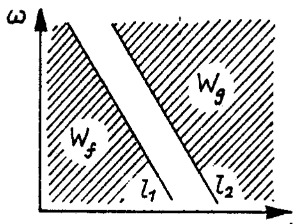

fig.10

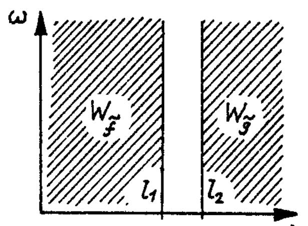

fig.11

signals $\tilde{f},\tilde{g}$ without altering the energy contents of signal- and interference terms:

$$E_{\tilde{f}}=E_f\,,\ E_{\tilde{g}}=E_g\,;\ E_{\tilde{f},\tilde{g}}=E_{f,g}\,;\ C_{\tilde{f},\tilde{g}}=C_{f,g}\,. \tag{21}$$

But now $\tilde{f}$ and $\tilde{g}$ are disjoint in time, and thus it follows from (15) that $E_{\tilde{f},\tilde{g}}=0$ and, because of (21), also

$$E_{f,g} = \int_{-\infty}^{\infty} f(t)g^*(t)\,dt = 0 \tag{22}$$

(which shows that two disjoint signals are orthogonal), and finally

$$C_{f,g} = 0\,, \qquad E_h = E_f + E_g\,. \tag{23}$$

We remark that another, more general and formal proof may be given by using the relation [1]

$$|E_{f,g}|^2 = 2\pi\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(t,\omega)\,W_g(t,\omega)\,dt\,d\omega\,, \tag{24}$$

which states that two signals are orthogonal if and only if their WDs are orthogonal. Thus orthogonality of W_f and W_g is a sufficient condition for an energyless interference term, and disjointness is just a special case of orthogonality.

6. CONCLUSION

The WD interference principle was shown to be a basic WD property and important for the prediction and interpretation of WD results. Two interference mechanisms have been discussed, "outer" interferences in the case of multicomponent signals, and "inner" interferences that can be found in any WD. In both cases interference terms are oscillatory if corresponding signal terms are to some extent disjoint, and in principle oscillation obeys simple geometrical laws. These geometrical laws were found to be useful for determining possible shapes of WD supports and for investigating the energy content of interference terms.

While interference terms tend to obscure the WD itself, they help to form a very clear signal representation if a windowed WD version (Pseudo Wigner Distribution [1]) is used. Yet there are applications in which an isolation or suppression of interference terms is desirable. An approach to this problem that is based on the interference terms' oscillations is time-frequency smoothing [4].

7. REFERENCES

[1] T.A.C.M. Claasen, W.F.G. Mecklenbräuker: The Wigner Distribution - a Tool for Time-Frequency Signal Analysis. Part I: Continuous-Time Signals. Philips J. Res. 35, 217-250, 1980.

[2] A.J.E.M. Janssen: Application of the Wigner Distribution to Harmonic Analysis of Generalized Stochastic Processes. Mathematical Centre Tracts 114, Mathematisch Centrum, Amsterdam 1979.

[3] A.J.E.M. Janssen: On the Locus and Spread of Pseudo-Density Functions in the Time-Frequency Plane. Philips J. Res. 37, 79-110, 1982.

[4] P. Flandrin: Some Features of Time-Frequency Representations of Multicomponent Signals, ICASSP '84, San Diego.