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ROBUST TIME-VARYING WIENER FILTERS: THEORY AND TIME-FREQUENCY FORMULATION*

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ABSTRACT

We propose a minimax robust time-varying Wiener filter that is based on a novel uncertainty model for nonstationary random processes. This filter maintains a certain performance for all second-order statistics within prescribed uncertainty classes. Furthermore, it requires less detailed prior knowledge than the ordinary Wiener filter. We also present an intuitively appealing time-frequency formulation of the robust time-varying Wiener filter in which signal subspaces are replaced with time-frequency regions.

1 INTRODUCTION

We consider the estimation of a random signal $s(t)$ from a noisy observation $r(t) = s(t) + n(t)$ by means of a linear, generally time-varying system $\mathbf{H}$,

$$\hat{s}(t) = (\mathbf{H}r)(t) = \int_{t'} h(t,t')\, r(t')\, dt' ,$$

where $h(t,t')$ denotes the impulse response (kernel) of $\mathbf{H}$. Signal $s(t)$ and noise $n(t)$ are assumed to be uncorrelated, real or circular complex, zero-mean, nonstationary random processes with correlation operators[1] $\mathbf{R}_s$ and $\mathbf{R}_n$, respectively. As is well known [1, 2], minimization of the mean square error (MSE) $\varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) \triangleq \mathrm{E}\{\|\mathbf{H}r - s\|^2\}$ with respect to $\mathbf{H}$ yields the (time-varying) *Wiener filter*

$$\mathbf{H}_W \triangleq \arg\min_{\mathbf{H}} \varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) = \mathbf{R}_s(\mathbf{R}_s+\mathbf{R}_n)^{-1}. \quad (1)$$

Calculation of the Wiener filter requires complete knowledge of the correlations $\mathbf{R}_s$ and $\mathbf{R}_n$, which is rarely available in practice. If the actual correlations $\mathbf{R}_s$ and $\mathbf{R}_n$ deviate from the nominal correlations (hereafter denoted by $\mathbf{R}_s^0$ and $\mathbf{R}_n^0$) for which the Wiener filter $\mathbf{H}_W^0 = \mathbf{R}_s^0(\mathbf{R}_s^0+\mathbf{R}_n^0)^{-1}$ was designed, the filter's performance may degrade significantly.

This paper proposes a *robust* time-varying Wiener filter that maintains a certain performance for all correlations within prescribed uncertainty classes and is thus insensitive to limited deviations from the nominal operating conditions. Our results extend a previously proposed minimax robust time-*in*variant Wiener filter based on the so-called p-point uncertainty model for *stationary* processes [3, 4].

The paper is organized as follows. Section 2 introduces a p-point uncertainty model for nonstationary processes, and Section 3 derives the corresponding minimax robust time-varying Wiener filter. Intuitively appealing time-frequency formulations are presented in Section 4. Finally, numerical simulations are provided in Section 5.

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[1] The correlation operator $\mathbf{R}_x$ of a (generally nonstationary) random process $x(t)$ is the linear, positive (semi-)definite operator whose kernel equals the correlation function $r_x(t,t') = \mathrm{E}\{x(t)\, x^*(t')\}$. In a discrete-time setting, $\mathbf{R}_x$ would be a matrix.

2 NONSTATIONARY p-POINT UNCERTAINTY MODEL

Generalizing the p-point uncertainty model for stationary random processes [3, 4], we propose an uncertainty model for *nonstationary* random processes which describes the designer's uncertainty about the actual correlations. Let the orthogonal subspaces $\mathcal{X}_i$, $i = 1, 2, \ldots, N$ be a partition of the space $L_2(\mathbb{R})$ of square-integrable functions, i.e., $\bigoplus_{i=1}^N \mathcal{X}_i = L_2(\mathbb{R})$ and $\mathcal{X}_i \perp \mathcal{X}_j$ for $i \neq j$. The associated orthogonal projection operators $\mathbf{P}_i$ then satisfy $\sum_{i=1}^N \mathbf{P}_i = \mathbf{I}$. The mean energy of a nonstationary process $x(t)$ in a subspace $\mathcal{X}_i$ is given by $\mathrm{E}\{\|\mathbf{P}_i x\|^2\} = \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_x\}$, where $\mathrm{tr}\{\cdot\}$ denotes the trace. Note that $\bar{E}_x \triangleq \mathrm{E}\{\|x\|^2\} = \sum_{i=1}^N \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_x\}$. By definition, the *$p$-point uncertainty class* $\mathcal{U}$ comprises all nonstationary processes $x(t)$ (i.e., correlations $\mathbf{R}_x$) having given mean subspace energies $x_i \geq 0$,

$$\mathcal{U} \triangleq \left\{\mathbf{R}_x\colon\ \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_x\} = x_i\,,\ \ i = 1,2,\ldots,N\right\}.$$

Note that this implies equal mean energies $\bar{E}_x = \sum_{i=1}^N x_i$.

The p-point uncertainty model captures the prior knowledge in a simple and flexible manner via the mean subspace energies x_i. If a nominal correlation $\mathbf{R}_x^0$ is given, then $x_i \triangleq \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_x^0\}$. In the extreme case $N = 1$, i.e., $\mathcal{X}_1 = L_2(\mathbb{R})$, only the total mean energy $\bar{E}_x$ is determined, corresponding to a minimal amount of prior knowledge and a maximally wide uncertainty class $\mathcal{U}$.

It is easily shown that the convex combination $(1-\alpha)\mathbf{R}_1 + \alpha\mathbf{R}_2$ of two correlations $\mathbf{R}_1 \in \mathcal{U}$ and $\mathbf{R}_2 \in \mathcal{U}$ is again in $\mathcal{U}$, which means that the p-point uncertainty class $\mathcal{U}$ is a *convex* set. This will be important in what follows.

3 ROBUST TIME-VARYING WIENER FILTER

Returning to our estimation problem, we assume that signal $s(t)$ and noise $n(t)$ belong to p-point uncertainty classes

$$\begin{aligned}\mathcal{S} &= \left\{\mathbf{R}_s\colon\ \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_s\} = s_i\,,\ \ i = 1,2,\ldots,N\right\},\\ \mathcal{N} &= \left\{\mathbf{R}_n\colon\ \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_n\} = n_i\,,\ \ i = 1,2,\ldots,N\right\},\end{aligned} \quad (2)$$

respectively, with given signal subspaces $\mathcal{X}_i$ (identical for $\mathcal{S}$ and $\mathcal{N}$) and given mean subspace energies $s_i \geq 0$ and $n_i \geq 0$. Adopting a minimax approach, we define the *robust time-varying Wiener filter* $\mathbf{H}_R$ as the system that optimizes the worst-case performance (i.e., maximal MSE) obtained within the uncertainty classes $\mathcal{S}$, $\mathcal{N}$:

$$\mathbf{H}_R \triangleq \arg\min_{\mathbf{H}} \left\{ \max_{\substack{\mathbf{R}_s\in\mathcal{S}\\ \mathbf{R}_n\in\mathcal{N}}} \varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)\right\}. \quad (3)$$

This optimization problem is difficult to solve in general. However, an important simplification occurs if

$$\min_{\mathbf{H}} \max_{\substack{\mathbf{R}_s \in \mathcal{S} \\ \mathbf{R}_n \in \mathcal{N}}} \varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) = \max_{\substack{\mathbf{R}_s \in \mathcal{S} \\ \mathbf{R}_n \in \mathcal{N}}} \min_{\mathbf{H}} \varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n) \equiv \max_{\substack{\mathbf{R}_s \in \mathcal{S} \\ \mathbf{R}_n \in \mathcal{N}}} \varepsilon^2_{\min}(\mathbf{R}_s,\mathbf{R}_n), \tag{4}$$

where $\varepsilon^2_{\min}(\mathbf{R}_s,\mathbf{R}_n) \triangleq \min_{\mathbf{H}} \varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$. Let us assume that (4) is valid. Since $\varepsilon^2_{\min}(\mathbf{R}_s,\mathbf{R}_n)$ is achieved by the ordinary Wiener filter $\mathbf{H}_W = \mathbf{R}_s(\mathbf{R}_s+\mathbf{R}_n)^{-1}$ in (1), equation (4) implies that $\mathbf{H}_R$ is equal to the *ordinary* Wiener filter

$$\mathbf{H}_W^L \triangleq \mathbf{R}_s^L(\mathbf{R}_s^L+\mathbf{R}_n^L)^{-1}$$

obtained for those correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ that are *least favorable* in the sense that they lead to the maximal $\varepsilon^2_{\min}(\mathbf{R}_s,\mathbf{R}_n)$ among all $\mathbf{R}_s \in \mathcal{S}$ and $\mathbf{R}_n \in \mathcal{N}$, i.e.,

$$(\mathbf{R}_s^L,\mathbf{R}_n^L) = \arg \max_{\substack{\mathbf{R}_s \in \mathcal{S} \\ \mathbf{R}_n \in \mathcal{N}}} \varepsilon^2_{\min}(\mathbf{R}_s,\mathbf{R}_n). \tag{5}$$

It can be shown [5] that the pivotal relation (4) is valid if and only if there exist a filter $\mathbf{H}_L$ and correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ forming a *saddle point* of $\varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$ in the sense that

$$\varepsilon^2(\mathbf{H}_L;\mathbf{R}_s,\mathbf{R}_n) \le \varepsilon^2(\mathbf{H}_L;\mathbf{R}_s^L,\mathbf{R}_n^L) \le \varepsilon^2(\mathbf{H};\mathbf{R}_s^L,\mathbf{R}_n^L) \tag{6}$$

for all $\mathbf{H}$ and $\mathbf{R}_s \in \mathcal{S}$, $\mathbf{R}_n \in \mathcal{N}$. If $\mathbf{H}_L$ is chosen as $\mathbf{H}_L = \mathbf{H}_W^L = \mathbf{R}_s^L(\mathbf{R}_s^L+\mathbf{R}_n^L)^{-1}$, i.e., as the ordinary Wiener filter for the correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$, the right-hand inequality in (6) is trivially true (since $\mathbf{H}_W^L$ minimizes $\varepsilon^2(\mathbf{H};\mathbf{R}_s^L,\mathbf{R}_n^L)$), and it only remains to find correlation operators $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ satisfying the left-hand inequality in (6):

$$\varepsilon^2(\mathbf{H}_W^L;\mathbf{R}_s,\mathbf{R}_n) \le \varepsilon^2(\mathbf{H}_W^L;\mathbf{R}_s^L,\mathbf{R}_n^L) \equiv \varepsilon^2_{\min}(\mathbf{R}_s^L,\mathbf{R}_n^L). \tag{7}$$

Using the convexity of the sets $\mathcal{S}$ and $\mathcal{N}$ (see Section 2), it can be shown [6] that (7) is satisfied if and only if $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ are chosen as the least favorable correlations in (5).

Hence, the robust Wiener filter $\mathbf{H}_R$ equals the ordinary Wiener filter designed for the least favorable correlations, and thus its construction essentially reduces to the easier task of finding least favorable correlations. Using this simplification, the following result is shown in the Appendix.

Theorem 3.1. *For the p-point uncertainty classes $\mathcal{S}$ and $\mathcal{N}$ in* (2)*, the robust time-varying Wiener filter as defined in* (3) *is given by*

$$\mathbf{H}_R = \sum_{i=1}^{N} \frac{s_i}{s_i+n_i}\,\mathbf{P}_i\,, \tag{8}$$

and the MSE achieved by $\mathbf{H}_R$ for any $\mathbf{R}_s \in \mathcal{S}$, $\mathbf{R}_n \in \mathcal{N}$ is

$$\varepsilon^2(\mathbf{H}_R;\mathbf{R}_s,\mathbf{R}_n) = \sum_{i=1}^{N} \frac{s_i n_i}{s_i+n_i}\,.$$

We see that $\mathbf{H}_R$ simply forms a weighted sum of the orthogonal projections of the input signal onto the subspaces $\mathcal{X}_i$. Thus, it treats all signal components lying in a given subspace $\mathcal{X}_i$ alike and it does not exploit cross-correlations between process components in different subspaces $\mathcal{X}_i$. Whereas in general the ordinary Wiener filter $\mathbf{H}_W$ is not even a normal operator, the robust Wiener filter $\mathbf{H}_R$ is self-adjoint and nonnegative definite. In the extreme case $N=1$ where only $\bar{E}_s$ and $\bar{E}_n$ are known, $\mathbf{H}_R$ reduces to a simple gain factor, i.e., $\mathbf{H}_R = \frac{\bar{E}_s}{\bar{E}_s+\bar{E}_n}\mathbf{I}$. Furthermore, the MSE achieved by $\mathbf{H}_R$ does not depend on the actual correlation operators $\mathbf{R}_s$, $\mathbf{R}_n$ as long as these lie in the respective uncertainty classes $\mathcal{S}$, $\mathcal{N}$.

4 TIME-FREQUENCY FORMULATION

We shall now establish an approximate, intuitively appealing time-frequency (TF) formulation of the p-point uncertainty classes $\mathcal{S}$, $\mathcal{N}$ and the robust time-varying Wiener filter $\mathbf{H}_R$. Let $L_{\mathbf{H}}(t,f)$ denote the Weyl symbol of a linear time-varying system $\mathbf{H}$ [7]–[9] and $\overline{W}_x(t,f)$ the Wigner-Ville spectrum (WVS) of a nonstationary random process $x(t)$ [10, 11]. Using [7, 8] $\mathrm{tr}\{\mathbf{P}_i\mathbf{R}_x\} = \langle L_{\mathbf{P}_i}, \overline{W}_x\rangle = \int_t\int_f L_{\mathbf{P}_i}(t,f)\,\overline{W}_x(t,f)\,dt\,df$, a TF formulation of the p-point uncertainty class $\mathcal{S}$ in (2) is obtained as $\{\overline{W}_s(t,f) : \langle L_{\mathbf{P}_i}, \overline{W}_s\rangle = s_i\,,\ i=1,2,\ldots,N\}$, and similarly for $\mathcal{N}$.

For non-sophisticated [12] subspaces $\mathcal{X}_i$, each $\mathcal{X}_i$ corresponds to a TF region $\mathcal{R}_i$ such that $L_{\mathbf{P}_i}(t,f) \approx I_{\mathcal{R}_i}(t,f)$ and hence $\langle L_{\mathbf{P}_i}, \overline{W}_x\rangle \approx \langle I_{\mathcal{R}_i}, \overline{W}_x\rangle \equiv \iint_{\mathcal{R}_i} \overline{W}_x(t,f)\,dt\,df$, where $I_{\mathcal{R}_i}(t,f)$ is the indicator function of $\mathcal{R}_i$ [12, 13]. The TF regions $\mathcal{R}_i$ form a partition of the TF plane, i.e., $\bigcup_{i=1}^N \mathcal{R}_i = \mathbb{R}^2$ and $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for $i \neq j$ [12, 14]. Hence, approximate TF formulations of the p-point uncertainty classes $\mathcal{S}$, $\mathcal{N}$ are obtained as

$$\widetilde{\mathcal{S}} = \Big\{\overline{W}_s(t,f) : \iint_{\mathcal{R}_i} \overline{W}_s(t,f)\,dt\,df = \tilde{s}_i\,,\ i=1,2,\ldots,N\Big\},$$

$$\widetilde{\mathcal{N}} = \Big\{\overline{W}_n(t,f) : \iint_{\mathcal{R}_i} \overline{W}_n(t,f)\,dt\,df = \tilde{n}_i\,,\ i=1,2,\ldots,N\Big\}.$$

Note that $\sum_{i=1}^N \tilde{s}_i = \bar{E}_s$ and $\sum_{i=1}^N \tilde{n}_i = \bar{E}_n$. The "TF uncertainty classes" $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ comprise all nonstationary processes $s(t)$, $n(t)$ having prescribed amounts $\tilde{s}_i$, $\tilde{n}_i$ of mean energy in given TF regions $\mathcal{R}_i$, which is quite intuitive. If nominal WVS $\overline{W}_s^0(t,f)$, $\overline{W}_n^0(t,f)$ are given, then $\tilde{s}_i \triangleq \iint_{\mathcal{R}_i} \overline{W}_s^0(t,f)\,dt\,df$ and $\tilde{n}_i \triangleq \iint_{\mathcal{R}_i} \overline{W}_n^0(t,f)\,dt\,df$. If $s_i \triangleq \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_s^0\}$ and $n_i \triangleq \mathrm{tr}\{\mathbf{P}_i\mathbf{R}_n^0\}$ (cf. Section 2), then we can expect $\tilde{s}_i \approx s_i$ and $\tilde{n}_i \approx n_i$ (cf. Table 1 in Section 5). In practice, the "mean regional energies" (WVS integrals) $\tilde{s}_i$, $\tilde{n}_i$ can be estimated from realizations of $s(t)$, $n(t)$ much more accurately and efficiently than the WVS themselves or the correlations.

Next, we develop a TF designed approximation $\widetilde{\mathbf{H}}_R$ to the robust time-varying Wiener filter $\mathbf{H}_R$ by taking the Weyl symbol of $\mathbf{H}_R$ in (8) and using $L_{\mathbf{P}_i}(t,f) \approx I_{\mathcal{R}_i}(t,f)$ and $s_i \approx \tilde{s}_i$, $n_i \approx \tilde{n}_i$:

$$L_{\mathbf{H}_R}(t,f) = \sum_{i=1}^N \frac{s_i}{s_i+n_i} L_{\mathbf{P}_i}(t,f) \approx \sum_{i=1}^N \frac{\tilde{s}_i}{\tilde{s}_i+\tilde{n}_i} I_{\mathcal{R}_i}(t,f) \triangleq L_{\widetilde{\mathbf{H}}_R}(t,f)\,. \tag{9}$$

Note that $L_{\widetilde{\mathbf{H}}_R}(t,f)$ is piecewise constant, expressing an equal TF weighting of all process components lying in a given TF region $\mathcal{R}_i$ [9]. With (9), $\widetilde{\mathbf{H}}_R$ is obtained as

$$\widetilde{\mathbf{H}}_R = \sum_{i=1}^N \frac{\tilde{s}_i}{\tilde{s}_i+\tilde{n}_i}\,\tilde{\mathbf{P}}_i\,.$$

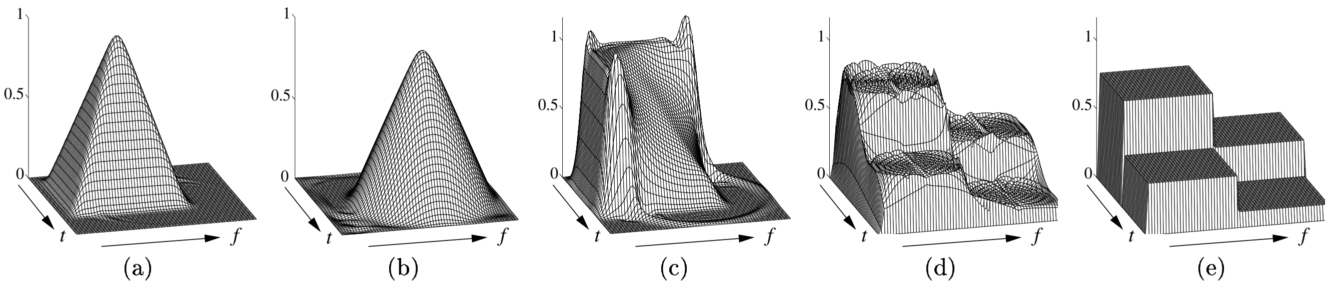

Figure 1. *TF representations of signal and noise statistics and various Wiener filters: (a) WVS of $s(t)$, (b) WVS of $n(t)$, (c) Weyl symbol of $\mathbf{H}_W^0$, (d) Weyl symbol of $\mathbf{H}_R$, and (e) Weyl symbol of $\widetilde{\mathbf{H}}_R$.*

Here, the impulse response $\tilde{p}_i(t,t')$ of $\tilde{\mathbf{P}}_i$ is the inverse Weyl transform [7, 8] of the indicator function $I_{\mathcal{R}_i}(t,f)$, which can be computed efficiently using FFT methods. Note that $\tilde{\mathbf{P}}_i$ approximates $\mathbf{P}_i$ but is not exactly an orthogonal projection operator [12, 13].

The prior knowledge necessary for designing this "robust TF Wiener filter" $\widetilde{\mathbf{H}}_R$ is given by the mean regional energies $\tilde{s}_i$, $\tilde{n}_i$ of $s(t)$, $n(t)$ in the prescribed TF regions $\mathcal{R}_i$, which is more intuitive and physically relevant than the mean subspace energies s_i, n_i required for the design of $\mathbf{H}_R$. Note also that the task of choosing a partition of $L_2(\mathbb{R})$ into orthogonal subspaces $\mathcal{X}_i$ has been replaced by the much simpler and more intuitive task of choosing a partition of the TF plane into disjoint TF regions $\mathcal{R}_i$. This partition may have a regular structure (e.g., a rectangular or wavelet-type tiling of the TF plane) or the $\mathcal{R}_i$ may correspond to individual components of the processes $s(t)$, $n(t)$ (if prior knowledge about the TF localization of such components is available). In the first case, efficient multi-window Gabor or multi-wavelet implementations of the resulting robust Wiener filter can be derived [6].

Since $\mathbf{H}_R$ is minimax robust for the uncertainty classes $\mathcal{S}$, $\mathcal{N}$, and since furthermore $\widetilde{\mathbf{H}}_R \approx \mathbf{H}_R$ and $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ are approximately equivalent to $\mathcal{S}$, $\mathcal{N}$, the filter $\widetilde{\mathbf{H}}_R$ is approximately minimax robust[2] for the TF uncertainty classes $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$.

5 SIMULATION RESULTS

Figs. 1 (a), (b) depict the WVS of (TF designed [17]) nominal correlations $\mathbf{R}_s^0$ and $\mathbf{R}_n^0$. The Weyl symbol of the ordinary Wiener filter $\mathbf{H}_W^0 = \mathbf{R}_s^0(\mathbf{R}_s^0 + \mathbf{R}_n^0)^{-1}$ is shown in Fig. 1 (c). Figs. 1 (d), (e) depict the Weyl symbols of the robust Wiener filter $\mathbf{H}_R$ and the robust TF Wiener filter $\widetilde{\mathbf{H}}_R$ obtained for p-point uncertainty classes $\mathcal{S}$, $\mathcal{N}$ and $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$, respectively, both with $N = 4$. The rectangular TF regions $\mathcal{R}_i$ underlying the TF uncertainty classes $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{N}}$ are clearly visible in the Weyl symbol of $\widetilde{\mathbf{H}}_R$. Since the subspaces $\mathcal{X}_i$ underlying $\mathcal{S}$, $\mathcal{N}$ were derived from the TF regions $\mathcal{R}_i$ by means of TF space synthesis [12]–[14], this rectangular TF tiling is also visible in the Weyl symbol of $\mathbf{H}_R$.

Table 1 compares the values of the mean subspace energies s_i, n_i and the mean regional energies $\tilde{s}_i$, $\tilde{n}_i$ as well as the resulting coefficients of $\mathbf{H}_R$ and $\widetilde{\mathbf{H}}_R$. It can be verified that the TF approximations are quite good.

Finally, Fig. 2 compares the performance (output SNR $\bar{E}_s/\varepsilon^2(\mathbf{H};\mathbf{R}_s,\mathbf{R}_n)$ vs. input SNR[3] $\bar{E}_s/\bar{E}_n$) of the ordinary Wiener filter $\mathbf{H}_W^0 = \mathbf{R}_s^0(\mathbf{R}_s^0+\mathbf{R}_n^0)^{-1}$, the robust Wiener filter $\mathbf{H}_R$, the robust TF Wiener filter $\widetilde{\mathbf{H}}_R$, and a trivial filter $\mathbf{H}_T$ that suppresses (passes) all signals in the case of negative (positive) SNR. It was verified that the performance of $\mathbf{H}_R$ is indeed independent of the operating conditions within the uncertainty classes $\mathcal{S}$, $\mathcal{N}$. For all possible operating conditions, the output SNR obtained with $\widetilde{\mathbf{H}}_R$ was observed to be within 0,07dB of that obtained with $\mathbf{H}_R$, thereby confirming the quality of the TF approximation (9). Furthermore, it is seen that at nominal operating conditions $\mathbf{H}_W^0$ performs only slightly better than $\mathbf{H}_R$ or $\widetilde{\mathbf{H}}_R$ but at its worst-case operating conditions $\mathbf{H}_W^0$ performs much worse than $\mathbf{H}_R$ or $\widetilde{\mathbf{H}}_R$ or even $\mathbf{H}_T$. Hence, in this example, the robust Wiener filters $\mathbf{H}_R$ and $\widetilde{\mathbf{H}}_R$ achieve a drastic perfor-

i	s_i	$\tilde{s}_i$	n_i	$\tilde{n}_i$	$\frac{s_i}{s_i+n_i}$	$\frac{\tilde{s}_i}{\tilde{s}_i+\tilde{n}_i}$
1	5.272	5.283	1.602	1.582	0.766	0.769
2	1.579	1.608	2.690	2.731	0.369	0.370
3	0.602	0.589	4.848	4.836	0.110	0.108
4	1.634	1.608	2.747	2.715	0.373	0.371

Table 1. *Comparison of the mean subspace energies s_i, n_i and the mean regional energies $\tilde{s}_i$, $\tilde{n}_i$ as well as the resulting coefficients $\frac{s_i}{s_i+n_i}$ of $\mathbf{H}_R$ and $\frac{\tilde{s}_i}{\tilde{s}_i+\tilde{n}_i}$ of $\widetilde{\mathbf{H}}_R$.*

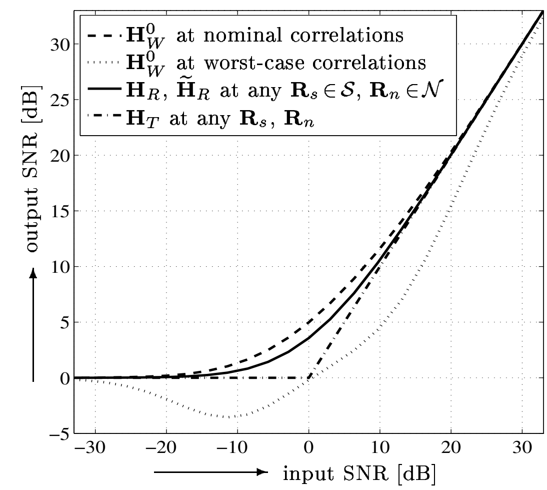

Figure 2. *Performance of the ordinary Wiener filter $\mathbf{H}_W^0$, the robust Wiener filters $\mathbf{H}_R$ and $\widetilde{\mathbf{H}}_R$, and the trivial filter $\mathbf{H}_T$. (The SNR curves of $\widetilde{\mathbf{H}}_R$ at different operating conditions all coincide with the SNR curve of $\mathbf{H}_R$.)*

[2] For underspread processes [15], based on approximate TF formulations of the ordinary Wiener filter and the MSE [16], this can be verified via a derivation in the TF domain that is analogous to the derivation given in the Appendix.

[3] The input SNR was varied by scaling $\mathbf{R}_s$.

mance improvement over $\mathbf{H}_W^0$ at its worst-case operating conditions with only a slight performance loss at nominal operating conditions.

6 CONCLUSION

We have derived a minimax robust time-varying Wiener filter that is based on a novel p-point uncertainty model for nonstationary random processes. This filter is insensitive to limited deviations from the nominal operating conditions, and it requires less detailed prior knowledge than the ordinary Wiener filter. Furthermore, we presented time-frequency formulations of the uncertainty model and robust Wiener filter which are particularly intuitive since they use simple time-frequency regions instead of signal subspaces. We note that a generalized theory of minimax robust time-frequency Wiener filters is given in [6].

APPENDIX: PROOF OF THEOREM 3.1

According to Section 3, $\mathbf{H}_R$ equals the ordinary Wiener filter $\mathbf{H}_W^L = \mathbf{R}_s^L \big(\mathbf{R}_s^L + \mathbf{R}_n^L\big)^{-1}$ obtained for least favorable correlations $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ that satisfy the inequality (7). Let

$$\mathbf{R}_s^L \triangleq \sum_{i=1}^N \mathbf{R}_{s,i}\,, \qquad \mathbf{R}_n^L \triangleq \sum_{i=1}^N \mathbf{R}_{n,i}\,,$$

where $\mathbf{R}_{s,i}$, $\mathbf{R}_{n,i}$ $(i = 1, 2, \ldots, N)$ are positive (semi-)definite operators with domain and range in $\mathcal{X}_i$, chosen such that $n_i \mathbf{R}_{s,i} = s_i \mathbf{R}_{n,i}$. (For finite-dimensional $\mathcal{X}_i$, we could choose $\mathbf{R}_{s,i}$, $\mathbf{R}_{n,i}$ to be proportional to $\mathbf{P}_i$.) $\mathbf{R}_s^L$ and $\mathbf{R}_n^L$ are positive (semi-)definite and thus valid correlation operators, and the $\mathbf{R}_{s,i}$, $\mathbf{R}_{n,i}$ can be normalized such that $\mathbf{R}_s^L \in \mathcal{S}$, $\mathbf{R}_n^L \in \mathcal{N}$. We now show that $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ satisfy (7). The ordinary Wiener filter for $\mathbf{R}_s^L$, $\mathbf{R}_n^L$ can be written as

$$\mathbf{H}_W^L = \mathbf{R}_s^L \big(\mathbf{R}_s^L + \mathbf{R}_n^L\big)^{-1} = \sum_{i=1}^N \mathbf{R}_{s,i} \Big[\sum_{j=1}^N \big(\mathbf{R}_{s,j} + \mathbf{R}_{n,j}\big) \Big]^{-1}$$
$$= \sum_{i=1}^N \mathbf{R}_{s,i} \sum_{j=1}^N \big(\mathbf{R}_{s,j} + \mathbf{R}_{n,j}\big)^{\#} = \sum_{i=1}^N \mathbf{R}_{s,i} \big(\mathbf{R}_{s,i} + \mathbf{R}_{n,i}\big)^{\#}$$

where $(\mathbf{R}_{s,i} + \mathbf{R}_{n,i})^{\#}$ denotes the pseudo-inverse [2] on $\mathcal{X}_i$, and the last identity holds since $\mathbf{R}_{s,i}\big(\mathbf{R}_{s,j} + \mathbf{R}_{n,j}\big)^{\#} = \mathbf{0}$ for $i \neq j$. With $n_i \mathbf{R}_{s,i} = s_i \mathbf{R}_{n,i}$, we obtain further

$$\mathbf{H}_W^L = \sum_{i=1}^N \mathbf{R}_{s,i} \Big[\Big(1 + \frac{n_i}{s_i}\Big) \mathbf{R}_{s,i} \Big]^{\#} = \sum_{i=1}^N \frac{s_i}{s_i + n_i} \mathbf{R}_{s,i} \mathbf{R}_{s,i}^{\#}$$
$$= \sum_{i=1}^N \frac{s_i}{s_i + n_i} \mathbf{P}_i\,. \tag{10}$$

The MSE obtained when applying $\mathbf{H}_W^L$ to processes with correlations $\mathbf{R}_s \in \mathcal{S}$, $\mathbf{R}_n \in \mathcal{N}$ is given by

$$\varepsilon^2(\mathbf{H}_W^L; \mathbf{R}_s, \mathbf{R}_n) = \mathrm{E}\big\{ \big\| \mathbf{H}_W^L s + \mathbf{H}_W^L n - s \big\|^2 \big\}$$
$$= \mathrm{E}\big\{ \big\| \big(\mathbf{I} - \mathbf{H}_W^L\big) s \big\|^2 \big\} + \mathrm{E}\big\{ \big\| \mathbf{H}_W^L n \big\|^2 \big\}$$
$$= \mathrm{tr}\Big\{ \big(\mathbf{I} - \mathbf{H}_W^L\big)^2 \mathbf{R}_s \Big\} + \mathrm{tr}\Big\{ \big(\mathbf{H}_W^L\big)^2 \mathbf{R}_n \Big\}$$
$$= \mathrm{tr}\Big\{ \Big[\sum_{i=1}^N \frac{n_i}{s_i + n_i} \mathbf{P}_i \Big]^2 \mathbf{R}_s \Big\} + \mathrm{tr}\Big\{ \Big[\sum_{i=1}^N \frac{s_i}{s_i + n_i} \mathbf{P}_i \Big]^2 \mathbf{R}_n \Big\}$$
$$= \sum_{i=1}^N \Big[\frac{n_i}{s_i + n_i} \Big]^2 \mathrm{tr}\{\mathbf{P}_i \mathbf{R}_s\} + \sum_{i=1}^N \Big[\frac{s_i}{s_i + n_i} \Big]^2 \mathrm{tr}\{\mathbf{P}_i \mathbf{R}_n\}$$
$$= \sum_{i=1}^N \frac{n_i^2 s_i + s_i^2 n_i}{(s_i + n_i)^2} = \sum_{i=1}^N \frac{s_i n_i}{s_i + n_i}\,,$$

which is constant for all $\mathbf{R}_s \in \mathcal{S}$, $\mathbf{R}_n \in \mathcal{N}$. Hence, inequality (7) is satisfied (with equality), and thus the Wiener filter $\mathbf{H}_W^L$ in (10) equals the robust Wiener filter $\mathbf{H}_R$.

REFERENCES

[1] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I: Detection, Estimation, and Linear Modulation Theory.* New York: Wiley, 1968.

[2] L. L. Scharf, *Statistical Signal Processing.* Reading (MA): Addison Wesley, 1991.

[3] L. J. Cimini and S. A. Kassam, "Robust and quantized Wiener filters for p-point spectral classes," in *Proc. Conf. Information Sciences and Systems*, (Princeton, NJ), pp. 314–318, March 1980.

[4] S. A. Kassam and H. V. Poor, "Robust techniques for signal processing: A survey," *Proc. IEEE*, vol. 73, pp. 433–481, March 1985.

[5] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces.* Alphen aan de Rijn (The Netherlands): Sijthoff & Noordhoff, 1978.

[6] G. Matz and F. Hlawatsch, "Minimax robust time-frequency filters for nonstationary signal estimation." Submitted to IEEE ICASSP-99, (Phoenix, AZ), Mar. 1999.

[7] W. Kozek, "Time-frequency signal processing based on the Wigner-Weyl framework," *Signal Processing*, vol. 29, pp. 77–92, Oct. 1992.

[8] R. G. Shenoy and T. W. Parks, "The Weyl correspondence and time-frequency analysis," *IEEE Trans. Signal Processing*, vol. 42, pp. 318–331, Feb. 1994.

[9] G. Matz and F. Hlawatsch, "Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory," *J. Math. Phys., Special Issue on Wavelet and Time-Frequency Analysis*, vol. 39, pp. 4041–4071, Aug. 1998.

[10] W. Martin and P. Flandrin, "Wigner-Ville spectral analysis of nonstationary processes," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 33, pp. 1461–1470, Dec. 1985.

[11] P. Flandrin and W. Martin, "The Wigner-Ville spectrum of nonstationary random signals," in *The Wigner Distribution — Theory and Applications in Signal Processing* (W. Mecklenbräuker and F. Hlawatsch, eds.), pp. 211–267, Amsterdam (The Netherlands): Elsevier, 1997.

[12] F. Hlawatsch, *Time-Frequency Analysis and Synthesis of Linear Signal Spaces: Time-Frequency Filters, Signal Detection and Estimation, and Range-Doppler Estimation.* Boston: Kluwer, 1998.

[13] F. Hlawatsch and W. Kozek, "Time-frequency projection filters and time-frequency signal expansions," *IEEE Trans. Signal Processing*, vol. 42, pp. 3321–3334, Dec. 1994.

[14] W. Kozek and F. Hlawatsch, "Time-frequency filter banks with perfect reconstruction," in *Proc. IEEE ICASSP-91*, (Toronto, Canada), pp. 2049–2052, May 1991.

[15] W. Kozek, F. Hlawatsch, H. Kirchauer, and U. Trautwein, "Correlative time-frequency analysis and classification of nonstationary random processes," in *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, (Philadelphia, PA), pp. 417–420, Oct. 1994.

[16] H. Kirchauer, F. Hlawatsch, and W. Kozek, "Time-frequency formulation and design of nonstationary Wiener filters," in *Proc. IEEE ICASSP-95*, (Detroit, MI), pp. 1549–1552, May 1995.

[17] F. Hlawatsch and W. Kozek, "Second-order time-frequency synthesis of nonstationary random processes," *IEEE Trans. Inf. Theory*, vol. 41, pp. 255–267, Jan. 1995.