

TIME-VARYING SPECTRA FOR UNDERSPREAD AND OVERSPREAD NONSTATIONARY PROCESSES*

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ABSTRACT

We introduce an extended concept of underspread and overspread nonstationary random processes and show its importance for time-varying spectral analysis. We consider two classes of time-varying power spectra that comprise most existing spectra, including Wigner-Ville and evolutionary spectra. For *underspread* processes, all spectra are shown to yield similar results. For *overspread* processes, the spectra may yield very different results and contain oscillatory “cross-terms” indicating time-frequency correlations. These cross-terms can be attenuated via smoothing techniques.

1 INTRODUCTION

This paper discusses the time-varying spectral analysis of nonstationary random processes. In Section 2, we introduce an extended notion of *underspread* and *overspread* processes, and we show that in the case of overspread processes two fundamental families of time-varying power spectra contain oscillatory cross-terms that indicate time-frequency (TF) correlations. In Section 3, we define two generalized classes of time-varying power spectra that allow an attenuation of cross-terms by means of a TF smoothing. Section 4 provides bounds which show that the spectra of underspread processes are approximately equivalent and satisfy other desirable properties. Finally, simulation results are presented in Section 5.

Much of our theory will be based on the *generalized Weyl symbol* [1] and the *generalized spreading function* [1] of a linear operator \mathbf{H} that are defined as

$$L_{\mathbf{H}}^{(\alpha)}(t, f) \triangleq \int_{\tau} H\left(t + \left(\frac{1}{2} - \alpha\right)\tau, t - \left(\frac{1}{2} + \alpha\right)\tau\right) e^{-j2\pi f\tau} d\tau,$$

$$S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) \triangleq \int_t H\left(t + \left(\frac{1}{2} - \alpha\right)\tau, t - \left(\frac{1}{2} + \alpha\right)\tau\right) e^{-j2\pi\nu t} dt,$$

respectively, where $H(t_1, t_2)$ is the kernel of \mathbf{H} and α is a real-valued parameter. $L_{\mathbf{H}}^{(\alpha)}(t, f)$ and $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ are related by a 2-D (symplectic) Fourier transform.

Two Families of Time-varying Power Spectra. The second-order statistics of a nonstationary process $x(t)$ are characterized by the 2-D correlation function $R_x(t_1, t_2) = E\{x(t_1)x^*(t_2)\}$ or the correlation operator \mathbf{R}_x (whose kernel is $R_x(t_1, t_2)$). Alternatively, we may consider an innovations system \mathbf{H} of $x(t)$ which is obtained as a (non-unique) solution to $\mathbf{H}\mathbf{H}^+ = \mathbf{R}_x$ [2, 3], with \mathbf{H}^+ the adjoint of \mathbf{H} .

The following two families of time-varying power spectra of nonstationary processes have been defined previously:

- The *generalized Wigner-Ville spectrum (GWVS)* [4],

$$\overline{W}_x^{(\alpha)}(t, f) \triangleq L_{\mathbf{R}_x}^{(\alpha)}(t, f).$$

Under mild conditions [4], the GWVS is the expectation of the generalized Wigner distribution [5, 6]. The *Wigner-Ville spectrum* [4, 7] and the *Rihaczek spectrum* [8] are obtained for $\alpha = 0$ and $\alpha = 1/2$, respectively.

- The *generalized evolutionary spectrum (GES)* [3],

$$G_x^{(\alpha)}(t, f) \triangleq |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2, \quad (1)$$

with \mathbf{H} an innovations system of $x(t)$. For $\alpha = 1/2$, $\alpha = 0$, and $\alpha = -1/2$, one obtains the *evolutionary spectrum* [9], the *Weyl spectrum* [3], and the *transitory evolutionary spectrum* [3, 10], respectively.

2 UNDERSPREAD AND OVERSPREAD PROCESSES

Whereas stationary processes feature only temporal correlations and (nonstationary) white processes feature only spectral correlations, general nonstationary processes feature both temporal and spectral correlations. These “TF correlations” can be described by the *generalized expected ambiguity function (GEAF)* [11, 12] defined as

$$\overline{A}_x^{(\alpha)}(\tau, \nu) \triangleq S_{\mathbf{R}_x}^{(\alpha)}(\tau, \nu).$$

The GEAF magnitude is independent of α , and hence we write $|\overline{A}_x^{(\alpha)}(\tau, \nu)| = |\overline{A}_x(\tau, \nu)|$. The value of $|\overline{A}_x(\tau, \nu)|$ at a specific point (τ, ν) is a global measure of the statistical correlation of process components separated in time by τ and in frequency by ν [11, 12]. The GEAF is related to the GWVS by a 2-D (symplectic) Fourier transform,

$$\overline{W}_x^{(\alpha)}(t, f) = \int_{\tau} \int_{\nu} \overline{A}_x^{(\alpha)}(\tau, \nu) e^{-j2\pi(f\tau - t\nu)} d\tau d\nu. \quad (2)$$

An Extended Concept of Underspread Processes. In [11, 12], a process has been called *underspread* if its GEAF has compact support with area ≤ 1 . This implies a sharp limitation of the process’ TF correlations. Since in practice GEAFs are almost never compactly supported, we now formulate an extended concept of underspread processes which merely requires the GEAF to be concentrated about the origin of the (τ, ν) -plane. We propose to measure the GEAF’s concentration via the weighted integrals

$$m_x^{(\phi)} \triangleq \frac{1}{\|\overline{A}_x\|_1} \int_{\tau} \int_{\nu} \phi(\tau, \nu) |\overline{A}_x(\tau, \nu)| d\tau d\nu \geq 0,$$

$$M_x^{(\phi)} \triangleq \frac{1}{\|\overline{A}_x\|_2} \left[\int_{\tau} \int_{\nu} \phi^2(\tau, \nu) |\overline{A}_x(\tau, \nu)|^2 d\tau d\nu \right]^{1/2} \geq 0.$$

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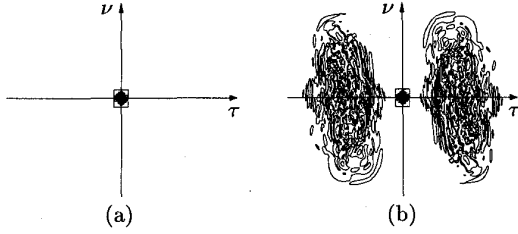


Figure 1. Contour-line plots of the GEAF magnitude of (a) an underspread process and (b) an overspread process. The small rectangles shown have area 1 and thus permit to assess the respective process' underspread or overspread property. The lowest contour line is 20 dB below the maximum value, $|\bar{A}_x(0,0)|$.

Here, $\phi(\tau, \nu)$ with $\phi(\tau, \nu) \geq \phi(0,0) = 0$ is a weighting function penalizing GEAF contributions away from the origin. Special cases of $m_x^{(\phi)}$ and $M_x^{(\phi)}$ are the GEAF moments $m_x^{(k,l)}$ and $M_x^{(k,l)}$ (with $k, l \in \mathbb{N}_0$) which are defined using the weighting function $\phi(\tau, \nu) = |\tau|^k |\nu|^l$. We note that $m_x^{(k,l)} \leq \sqrt{m_x^{(2k,0)} m_x^{(0,2l)}}$ and $M_x^{(k,l)} \leq \sqrt{M_x^{(2k,0)} M_x^{(0,2l)}}$.

The above defined weighted GEAF integrals/moments characterize the TF correlations of nonstationary processes in a flexible manner. In particular, a process will be called *underspread* if the weighted integrals/moments are small; otherwise, it will be called *overspread*. Fig. 1 contrasts the GEAFs of an underspread and an overspread process.

The innovations system \mathbf{H} provides an alternative characterization of TF correlations. \mathbf{H} is called underspread if $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ is concentrated about the origin, which indicates that \mathbf{H} introduces only small TF displacements [12–14]. The concentration of $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ can be measured by weighted integrals $m_{\mathbf{H}}^{(\phi)}$, $M_{\mathbf{H}}^{(\phi)}$ and moments $m_{\mathbf{H}}^{(k,l)}$, $M_{\mathbf{H}}^{(k,l)}$ of $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ whose definitions are analogous to the GEAF integrals/moments [14]. The relation $\mathbf{R}_x = \mathbf{H}\mathbf{H}^+$ implies [12, 14]

$$|\bar{A}_x(\tau, \nu)| \leq |S_{\mathbf{H}}(\tau, \nu)| ** |S_{\mathbf{H}}(-\tau, -\nu)|, \quad (3)$$

where $|S_{\mathbf{H}}(\tau, \nu)|$ denotes the (α -independent) magnitude of $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ and $**$ denotes 2-D convolution. Hence, if \mathbf{H} is underspread, $|\bar{A}_x(\tau, \nu)|$ will be concentrated and thus $x(t)$ will be underspread too. Conversely, if $x(t)$ is underspread, one can always find an underspread innovations system \mathbf{H} .

Since in the underspread case $\bar{A}_x^{(\alpha)}(\tau, \nu)$ and $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ are concentrated about the origin and since $\bar{A}_x^{(\alpha)}(\tau, \nu) \leftrightarrow \bar{W}_x^{(\alpha)}(t, f)$ and $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) \leftrightarrow L_{\mathbf{H}}^{(\alpha)}(t, f)$ are Fourier transform pairs, the GWVS and GES of an underspread process must be smooth functions. In fact, for any process $x(t)$ the partial derivatives of the GWVS and GES are bounded as

$$\left| \frac{\partial^{k+l} \bar{W}_x^{(\alpha)}(t, f)}{\partial t^l \partial f^k} \right| \leq (2\pi)^{k+l} \|\bar{A}_x\|_1 m_x^{(k,l)},$$

$$\left| \frac{\partial^{k+l} G_x^{(\alpha)}(t, f)}{\partial t^l \partial f^k} \right| \leq \beta_{\mathbf{H}},$$

with $\beta_{\mathbf{H}} \triangleq (2\pi)^{k+l} \|S_{\mathbf{H}}\|_1^2 \sum_{i=0}^k \sum_{j=0}^l \binom{k}{i} \binom{l}{j} m_{\mathbf{H}}^{(i,j)} m_{\mathbf{H}}^{(k-i, l-j)}$. Here, \mathbf{H} is the innovations system used in $G_x^{(\alpha)}(t, f)$.

Overspread Processes and Statistical Cross-Terms. If a process is *overspread*, i.e., if its GEAF has significant components far away from the origin, then it follows from (2) that the GWVS must contain oscillatory and partly neg-

ative components. Since (3) implies that the innovations systems of overspread processes are overspread, the GES contains oscillatory components too; these are nonnegative and thus more difficult to identify than in the GWVS case. These oscillatory GWVS and GES components are related to *cross-terms* that indicate the strong TF correlations existing in overspread processes (such “statistical cross-terms” have previously been reported for a special case in [15]).

In order to illustrate the mechanism and geometry of statistical cross-terms, we consider the two-component process $x(t) = x_1(t) + x_2(t)$ with $x_1(t) = a_1 x_0(t - t_1) e^{j2\pi f_1 t}$ and $x_2(t) = a_2 x_0(t - t_2) e^{j2\pi f_2 t}$, where $x_0(t)$ is a process whose GWVS is localized about the origin of the TF plane and a_1, a_2 are random factors uncorrelated with $x_0(t)$. Hence, the components $x_1(t)$ and $x_2(t)$ are localized about the TF points (t_1, f_1) and (t_2, f_2) , respectively, and they are correlated if a_1 and a_2 are correlated. In the overall process $x(t)$, the TF points (t_1, f_1) and (t_2, f_2) will then be correlated. The GWVS of $x(t)$ consists of two “auto-terms,” $\bar{W}_{x_1}^{(\alpha)}(t, f) = E\{|a_1|^2\} \bar{W}_{x_0}^{(\alpha)}(t - t_1, f - f_1)$ and $\bar{W}_{x_2}^{(\alpha)}(t, f) = E\{|a_2|^2\} \bar{W}_{x_0}^{(\alpha)}(t - t_2, f - f_2)$, that are correctly localized about (t_1, f_1) and (t_2, f_2) , respectively, and two oscillatory “cross-terms” which are localized about the TF points $(\frac{t_1+t_2}{2} - \alpha(t_1 - t_2), \frac{f_1+f_2}{2} + \alpha(f_1 - f_2))$ and $(\frac{t_1+t_2}{2} + \alpha(t_1 - t_2), \frac{f_1+f_2}{2} - \alpha(f_1 - f_2))$ (these cross-terms can be calculated using results from [16]). In particular, for $\alpha = 0$ the two cross-terms collapse into a single real-valued cross-term given by $c(t - \frac{t_1+t_2}{2}, f - \frac{f_1+f_2}{2})$ with

$$c(t, f) = 2|\tau| \bar{W}_{x_0}^{(0)}(t, f) \cos[2\pi((f_1 - f_2)t - (t_1 - t_2)f) + \varphi],$$

where $\varphi = 2\pi(f_1 - f_2)(t_1 - t_2) + \arg\{r\}$, with $r = E\{a_1 a_2^*\}$. This cross-term is located about the center point $(\frac{t_1+t_2}{2}, \frac{f_1+f_2}{2})$ and is oscillatory with significant negative components. From this cross-term (whose geometry equals that of the Wigner distribution [16]), it is even possible to infer the correlated TF locations (t_1, f_1) and (t_2, f_2) . One can similarly show the existence of oscillatory (but nonnegative) cross-terms also for the GES.

3 SMOOTHED TIME-VARYING SPECTRA

The “statistical cross-terms” discussed above may be useful as indicators of TF correlations. On the other hand, they tend to mask the “auto-terms” that characterize the mean TF energy distribution of the process and thus indicate the TF locations of the process' components. An attenuation or suppression of cross-terms can be achieved by a TF smoothing, as explained in the following.

Type I Spectra. It can be shown [4] that all TF shift-covariant spectra with linear dependence on the correlation function $R_x(t_1, t_2)$ can be written as

$$\bar{C}_x(t, f) \triangleq \psi(t, f) ** \bar{W}_x^{(0)}(t, f), \quad (4)$$

where $\psi(t, f)$ is some function that does not depend on $R_x(t_1, t_2)$. Here, without loss of generality, we have used the GWVS with $\alpha = 0$ (using a different α value in (4) would result in the same class of spectra). The 2-D Fourier transform of $\bar{C}_x(t, f)$ is given by $\Psi(\tau, \nu) \bar{A}_x^{(0)}(\tau, \nu)$ with $\Psi(\tau, \nu) = \int_t \int_f \psi(t, f) e^{j2\pi(\tau f - \nu t)} dt df$. Hence, if $\Psi(\tau, \nu)$ is concentrated about the origin of the (τ, ν) -plane, which implies that $\psi(t, f)$ is a smooth function, then components of $\bar{A}_x^{(0)}(\tau, \nu)$ located away from the origin and, in turn, oscillatory cross-

terms in $\overline{W}_x^{(0)}(t, f)$ will be attenuated. Note that the convolution in (4) here becomes a *smoothing*. We shall assume $\Psi(0, 0) = 1$ which guarantees that $\int_t \int_f \overline{C}_x(t, f) dt df$ yields the mean signal energy $\overline{E}_x = \int_t R_x(t, t) dt$.

The class of time-varying spectra defined by (4), hereafter called *type I spectra*, has previously been considered in [4, 17, 18]. Under mild conditions [4], the type I spectra equal the expectation of TF signal representations of Cohen's class [19, 20]. The GWVS with $\alpha \neq 0$, Page's instantaneous power spectrum [21], Levin's spectrum [22], and the physical spectrum [23] are obtained for specific choices of $\Psi(\tau, \nu)$.

Type II Spectra. We define the novel class of *type II spectra* as the GES (again with $\alpha = 0$, without loss of generality) augmented by a convolution that is implemented prior to taking the squared magnitude in (1), i.e.,

$$\tilde{G}_x(t, f) \triangleq |\psi(t, f) ** L_{\mathbf{H}}^{(0)}(t, f)|^2,$$

where $\psi(t, f)$ is some function that does not depend on the innovations system \mathbf{H} . The 2-D Fourier transform of $\psi(t, f) ** L_{\mathbf{H}}^{(0)}(t, f)$ is $\Psi(\tau, \nu) S_{\mathbf{H}}^{(0)}(\tau, \nu)$ with $\Psi(\tau, \nu)$ the 2-D Fourier transform of $\psi(t, f)$. Hence, if $\Psi(\tau, \nu)$ is concentrated about the origin of the (τ, ν) -plane, i.e., $\psi(t, f)$ is a smooth function, then components of $S_{\mathbf{H}}^{(0)}(\tau, \nu)$ located away from the origin and, equivalently, oscillatory cross-terms in $L_{\mathbf{H}}^{(0)}(t, f)$ (and, hence, in $G_x^{(0)}(t, f)$) will be attenuated. We note that the GES is obtained with $\Psi(\tau, \nu) = e^{-j2\pi\alpha\nu\tau}$. Hereafter, we assume $|\Psi(\tau, \nu)| \leq \Psi(0, 0) = 1$.

4 UNDERSPREAD APPROXIMATIONS

We will now show that for underspread processes, different type I and type II spectra are approximately equal and approximately satisfy some desirable properties. As a mathematical underpinning of these approximations, we present bounds on the approximation errors that are formulated in terms of the weighted integrals/moments from Section 2.

Equivalence of Type I Spectra. It can be shown that the difference between any two type I spectra $\overline{C}_x^{(1)}(t, f)$ and $\overline{C}_x^{(2)}(t, f)$ is bounded as

$$\begin{aligned} |\overline{C}_x^{(1)}(t, f) - \overline{C}_x^{(2)}(t, f)| &\leq \|\overline{A}_x\|_1 m_x^{(\phi)}, \\ \|\overline{C}_x^{(1)} - \overline{C}_x^{(2)}\|_2 &\leq \|\overline{A}_x\|_2 M_x^{(\phi)}, \end{aligned} \quad (5)$$

with $\phi(\tau, \nu) = |\Psi^{(1)}(\tau, \nu) - \Psi^{(2)}(\tau, \nu)|$. Thus, if $x(t)$ is underspread with small $m_x^{(\phi)}$ and $M_x^{(\phi)}$, then we have

$$\overline{C}_x^{(1)}(t, f) \approx \overline{C}_x^{(2)}(t, f).$$

As a special case, the difference between two GWVS with different α parameters is bounded as

$$\begin{aligned} |\overline{W}_x^{(\alpha_1)}(t, f) - \overline{W}_x^{(\alpha_2)}(t, f)| &\leq 2\pi|\alpha_1 - \alpha_2| \|\overline{A}_x\|_1 m_x^{(1,1)}, \\ \|\overline{W}_x^{(\alpha_1)} - \overline{W}_x^{(\alpha_2)}\|_2 &\leq 2\pi|\alpha_1 - \alpha_2| \|\overline{A}_x\|_2 M_x^{(1,1)}. \end{aligned}$$

(These bounds extend a result in [11].) Hence, for an underspread process $x(t)$ with small $m_x^{(1,1)}$ and $M_x^{(1,1)}$, we have

$$\overline{W}_x^{(\alpha_1)}(t, f) \approx \overline{W}_x^{(\alpha_2)}(t, f).$$

Note that small $m_x^{(1,1)}$ and $M_x^{(1,1)}$ imply that $\overline{A}_x^{(\alpha)}(\tau, \nu)$ is concentrated along the τ axis and/or the ν axis.

Equivalence of Type II Spectra. The difference between two type II spectra $\tilde{G}_x^{(1)}(t, f)$ and $\tilde{G}_x^{(2)}(t, f)$ that are

based on the same innovations system \mathbf{H} is bounded as

$$|\tilde{G}_x^{(1)}(t, f) - \tilde{G}_x^{(2)}(t, f)| \leq 2\|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(\phi)}, \quad (6)$$

with $\phi(\tau, \nu) = |\Psi^{(1)}(\tau, \nu) - \Psi^{(2)}(\tau, \nu)|$. Hence, if \mathbf{H} is underspread such that $m_{\mathbf{H}}^{(\phi)}$ is small (which is only possible for $x(t)$ underspread), then

$$\tilde{G}_x^{(1)}(t, f) \approx \tilde{G}_x^{(2)}(t, f).$$

In particular, the difference between two GES with different α parameters is bounded as

$$|G_x^{(\alpha_1)}(t, f) - G_x^{(\alpha_2)}(t, f)| \leq 4\pi|\alpha_1 - \alpha_2| \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(1,1)}.$$

(This bound extends a result in [3].) Hence, for an underspread innovations system \mathbf{H} with small $m_{\mathbf{H}}^{(1,1)}$, we have

$$G_x^{(\alpha_1)}(t, f) \approx G_x^{(\alpha_2)}(t, f).$$

Equivalence of Type I and Type II Spectra. Extending a result in [11, 12], the difference between the GWVS and the GES with the same α can be shown to be bounded as

$$|\overline{W}_x^{(\alpha)}(t, f) - G_x^{(\alpha)}(t, f)| \leq 2\pi\|S_{\mathbf{H}}\|_1^2 \beta_{\mathbf{H}}^{(\alpha)} \quad (7)$$

with

$$\beta_{\mathbf{H}}^{(\alpha)} \triangleq \left| \left| \alpha + \frac{1}{2} \right| + \left| \alpha - \frac{1}{2} \right| \right] m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + 2|\alpha| m_{\mathbf{H}}^{(1,1)}, \quad (8)$$

where \mathbf{H} is the innovations system used in $G_x^{(\alpha)}(t, f)$. Hence, for an underspread process with small moments of \mathbf{H} ,

$$\overline{W}_x^{(\alpha)}(t, f) \approx G_x^{(\alpha)}(t, f).$$

Note that $\beta_{\mathbf{H}}^{(\alpha)}$ in (8) is smallest for $\alpha = 0$. Here, the bound (7) can be tightened by using the metaplectic covariance of the generalized Weyl symbol with $\alpha = 0$ [1, 14, 24]:

$$|\overline{W}_x^{(0)}(t, f) - G_x^{(0)}(t, f)| \leq 2\pi\|S_{\mathbf{H}}\|_1^2 \inf_{\mathbf{U}} \{m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}\}, \quad (9)$$

where the infimum is over all unitary metaplectic operators \mathbf{U} (corresponding to linear, area-preserving TF coordinate transforms) [1, 6, 14, 24].

By combining the bounds in (5), (6), and (9) using the triangle inequality, we finally obtain the following bound on the difference between any type I and type II spectrum,

$$\begin{aligned} |\overline{C}_x^{(1)}(t, f) - \tilde{G}_x^{(2)}(t, f)| \\ \leq \|S_{\mathbf{H}}\|_1^2 \left[m_x^{(\phi_1)} + 2m_{\mathbf{H}}^{(\phi_2)} + 2\pi \inf_{\mathbf{U}} \{m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}\} \right], \end{aligned}$$

with $\phi_1(\tau, \nu) = |1 - \Psi^{(1)}(\tau, \nu)|$ and $\phi_2(\tau, \nu) = |1 - \Psi^{(2)}(\tau, \nu)|$. Hence, if the process $x(t)$ and the innovations system \mathbf{H} are underspread such that the respective weighted integrals and moments are small, we have

$$\overline{C}_x^{(1)}(t, f) \approx \tilde{G}_x^{(2)}(t, f).$$

Properties of Type I Spectra. Next, we discuss some desirable properties of type I spectra.

Real-valuedness. A type I spectrum $\overline{C}_x(t, f)$ is real-valued if and only if $\Psi^*(-\tau, -\nu) = \Psi(\tau, \nu)$. In particular, the GWVS is real-valued only for $\alpha = 0$. In general, the imaginary part of $\overline{C}_x(t, f)$ satisfies

$$|\text{Im}\{\overline{C}_x(t, f)\}| \leq \frac{1}{2} \|\overline{A}_x\|_1 m_x^{(\phi)}, \quad \|\text{Im}\{\overline{C}_x\}\|_2 \leq \frac{1}{2} \|\overline{A}_x\|_2 M_x^{(\phi)}$$

with $\phi(\tau, \nu) = |\Psi(\tau, \nu) - \Psi^*(-\tau, -\nu)|$. Hence, for an underspread process with small $m_x^{(\phi)}$ and $M_x^{(\phi)}$, $\overline{C}_x(t, f)$ will be

nearly real-valued. For the GWVS with $\alpha \neq 0$, there also exist simpler but coarser bounds since here $m_x^{(\phi)} \leq 4\pi|\alpha|m_x^{(1,1)}$ and $M_x^{(\phi)} \leq 4\pi|\alpha|M_x^{(1,1)}$.

Marginal properties. A type I spectrum $\bar{C}_x(t, f)$ satisfies the marginal properties

$$\int_f \bar{C}_x(t, f) df = R_x(t, t), \quad \int_t \bar{C}_x(t, f) dt = R_X(f, f),$$

with $R_X(f, f) = E\{|X(f)|^2\}$, if and only if $\Psi(0, \nu) = 1$ and $\Psi(\tau, 0) = 1$, respectively. In the general case, the error $\Delta_1(t) \triangleq \int_f \bar{C}_x(t, f) df - R_x(t, t)$ can be shown to satisfy

$$|\Delta_1(t)| \leq \lambda_x, \quad \|\Delta_1\|_2 = \Lambda_x,$$

where $\lambda_x = \int_\nu |1 - \Psi(0, \nu)| |\bar{A}_x(0, \nu)| d\nu$ and $\Lambda_x = \int_\nu |1 - \Psi(0, \nu)|^2 |\bar{A}_x(0, \nu)|^2 d\nu$. Similarly, $\Delta_2(f) \triangleq \int_t \bar{C}_x(t, f) dt - R_X(f, f)$ satisfies

$$|\Delta_2(f)| \leq \theta_x, \quad \|\Delta_2\|_2 = \Theta_x,$$

where $\theta_x = \int_\tau |1 - \Psi(\tau, 0)| |\bar{A}_x(\tau, 0)| d\tau$ and $\Theta_x = \int_\tau |1 - \Psi(\tau, 0)|^2 |\bar{A}_x(\tau, 0)|^2 d\tau$. (Note that the quantities λ_x , θ_x and Λ_x , Θ_x can formally be viewed as degenerate weighted integrals $m_x^{(\phi)}$ and $M_x^{(\phi)}$, respectively.) Thus, for underspread processes where these quantities are small, the marginal properties will be satisfied at least approximately.

Moyal-type relation. The Moyal-type relation

$$\iint_f \bar{C}_x(t, f) \bar{C}_y^*(t, f) dt df = \iint_{t_1 t_2} R_x(t_1, t_2) R_y^*(t_1, t_2) dt_1 dt_2$$

is satisfied if and only if $|\Psi(\tau, \nu)| \equiv 1$. In the general case, the difference $\Delta_3 \triangleq \iint_f \bar{C}_x(t, f) \bar{C}_y^*(t, f) dt df - \iint_{t_1 t_2} R_x(t_1, t_2) R_y^*(t_1, t_2) dt_1 dt_2$ is bounded as

$$|\Delta_3| \leq \|\bar{A}_x\|_2 \|\bar{A}_y\|_2 M_x^{(\phi)} M_y^{(\phi)},$$

with $\phi(\tau, \nu) = |1 - \Psi(\tau, \nu)|$. Hence, for underspread processes $x(t)$ and $y(t)$ where $M_x^{(\phi)}$ and $M_y^{(\phi)}$ are small, the Moyal-type relation will be approximately satisfied.

Positivity. For underspread processes, the type I spectra have been shown to be approximately equal to the (non-negative) GES, and hence they are approximately nonnegative. Specifically, we will consider the Wigner-Ville spectrum $\bar{W}_x^{(0)}(t, f)$ which satisfies many desirable properties. Whereas $\bar{W}_x^{(0)}(t, f)$ is not strictly nonnegative, its negative part, $\bar{W}_x^{(0)-}(t, f) \triangleq [|\bar{W}_x^{(0)}(t, f)| - \bar{W}_x^{(0)}(t, f)]/2$, can be shown to be bounded as

$$|\bar{W}_x^{(0)-}(t, f)| \leq \min\{\beta_1, \beta_2\},$$

$$\|\bar{W}_x^{(0)-}\|_2 \leq 2 \|\bar{A}_x\|_2 \inf_g \{M_x^{(\phi_g)}\},$$

where $\beta_1 = 4\pi \inf_{\mathbf{H}} \{ \|\mathbf{S}_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} \}$ (with \mathbf{H} innovations systems of $x(t)$) and $\beta_2 = 2 \|\bar{A}_x\|_1 \inf_g \{ m_x^{(\phi_g)} \}$. Here, $\phi_g(\tau, \nu) = |1 - A_g^{(0)}(\tau, \nu)|$ where $A_g^{(0)}(\tau, \nu) = \int_t g(t + \frac{\tau}{2}) g^*(t - \frac{\tau}{2}) e^{-j2\pi\nu t} dt$ with $\|g\| = 1$. Hence, the Wigner-Ville spectrum of an underspread process is approximately nonnegative, which conforms to observations made in [4, 25].

Properties of Type II Spectra. Since type II spectra are always real-valued and nonnegative, we shall only consider the marginal properties and a mean energy property.

Mean energy property. A type II spectrum $\tilde{G}_x(t, f)$ satisfies

$\int_t \int_f \tilde{G}_x(t, f) dt df = \bar{E}_x$ if and only if $|\Psi(\tau, \nu)| \equiv 1$. This is satisfied by all GES but not necessarily by other type II spectra. However, it can be shown that the difference $\Delta_4 \triangleq \int_t \int_f \tilde{G}_x(t, f) dt df - \bar{E}_x$ is bounded as

$$|\Delta_4| \leq \|\mathbf{S}_{\mathbf{H}}\|_2^2 [M_{\mathbf{H}}^{(\phi)}]^2,$$

where \mathbf{H} is the innovations system underlying $\tilde{G}_x(t, f)$ and $\phi(\tau, \nu) = \sqrt{|1 - |\Psi(\tau, \nu)|^2|}$. Thus, as long as $|\Psi(\tau, \nu)| \approx 1$ on the effective support of $|\mathbf{S}_{\mathbf{H}}(\tau, \nu)|$, the mean energy property will be approximately satisfied.

Marginal properties. Bounds for the errors $\int_f \tilde{G}_x(t, f) df - R_x(t, t)$ and $\int_t \tilde{G}_x(t, f) dt - R_X(f, f)$ can be derived but they are somewhat involved in general. Therefore, we shall only consider the special case of the GES $G_x^{(\alpha)}(t, f)$. Extending previous bounds in [3], it can be shown that the differences $\Delta_5(t) \triangleq \int_f G_x^{(\alpha)}(t, f) df - R_x(t, t)$ and $\Delta_6(f) \triangleq \int_t G_x^{(\alpha)}(t, f) dt - R_X(f, f)$ are bounded as

$$|\Delta_5(t)| \leq 2\pi|1 - 2\alpha| \max_{\nu} \{ \int_{\nu} |\mathbf{S}_{\mathbf{H}}(\tau, \nu)| d\nu \} \|\mathbf{S}_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(1,1)},$$

$$|\Delta_6(f)| \leq 2\pi|1 + 2\alpha| \max_{\tau} \{ \int_{\tau} |\mathbf{S}_{\mathbf{H}}(\tau, \nu)| d\tau \} \|\mathbf{S}_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(1,1)}.$$

Thus, if $m_{\mathbf{H}}^{(1,1)}$ is small, i.e., $|\mathbf{S}_{\mathbf{H}}(\tau, \nu)|$ is concentrated along the τ axis and/or the ν axis, the marginal properties will be approximately satisfied.

5 SIMULATION RESULTS

Fig. 2 shows various type I and type II spectra, with and without smoothing, of an underspread process that was generated according to [26] and whose GEAF was shown in Fig. 1(a). All spectra yield practically identical results and correctly characterize the process' mean TF energy distribution.

Fig. 3 shows the results obtained for the overspread process whose GEAF was shown in Fig. 1(b). Some of these results are quite different. The GWVS and GES in parts (a), (b), (d), (e) contain oscillatory cross-terms that indicate strong statistical correlations between the 'T' and 'F' components. In the smoothed spectra in parts (c) and (f), these cross-terms are effectively suppressed and the process' mean TF energy distribution is better visible. However, the smoothed spectra no longer indicate the TF correlations.

6 CONCLUSION

The distinction between *underspread* and *overspread* non-stationary processes is of fundamental importance for time-varying spectral analysis. In the underspread case, all time-varying spectra tend to yield similar and valid results. In the overspread case, different spectra may yield dramatically different results. Here, one can choose between

- non-smoothed spectra containing statistical cross-terms, which indicate the strong time-frequency (TF) correlations inherent in overspread processes but tend to obscure the process' mean TF energy distribution,
- smoothed spectra with attenuated cross-terms, which better represent the process' mean TF energy distribution but fail to indicate TF correlations.

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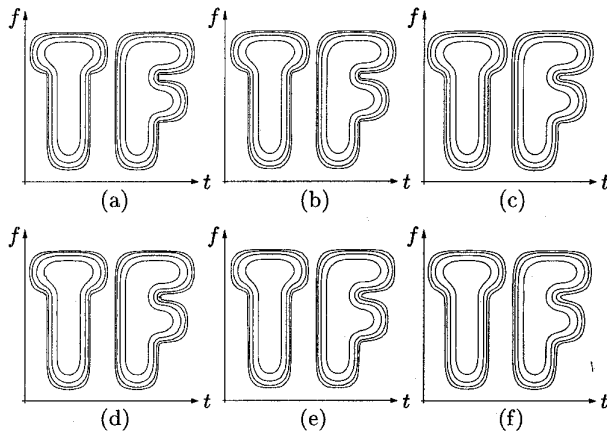


Figure 2. Time-varying spectra of an underspread process: (a) Wigner-Ville spectrum, $\bar{W}_x^{(0)}(t, f)$, (b) real part of Rihaczek spectrum, $\text{Re}\{\bar{W}_x^{(1/2)}(t, f)\}$, (c) type I spectrum incorporating smoothing, (d) Weyl spectrum, $G_x^{(0)}(t, f)$, (e) evolutionary spectrum, $G_x^{(1/2)}(t, f)$ (simultaneously the transitory evolutionary spectrum, $G_x^{(-1/2)}(t, f)$, since the underlying innovations system is positive semi-definite [3]), (f) type II spectrum incorporating smoothing. The signal length is 256 samples.

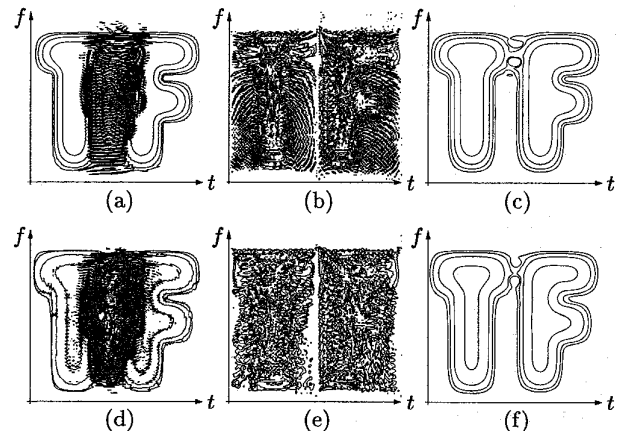


Figure 3. The same time-varying spectra as in Fig. 2, but now for an overspread process.

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