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MODULATION AND WARPING OPERATORS IN JOINT SIGNAL ANALYSIS*

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Abstract—We define two broad classes of unitary operator families termed modulation and warping operators. Some fundamental relations connecting these operators are shown. Application of the characteristic function method and the covariance method to modulation and warping operators yields joint (a,b), (α,β), and time-frequency signal representations that satisfy marginal and covariance properties.

1 INTRODUCTION AND OUTLINE

The *characteristic function method* [1]–[4] and *covariance method* [5]–[9] for constructing quadratic time-frequency (TF) signal representations can be based on two unitary operator families $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, where α and β belong to two 1-D locally compact abelian (LCA) groups. This paper studies two special types of unitary operator families termed modulation and warping operators. These include nearly all operators considered so far in the context of TF analysis and yet allow a closed-form calculation of their TF geometry (i.e., localization and displacement function [5, 6, 8, 9, 10]).

The paper is organized as follows. Section 2 provides some basic results on modulation and warping operators. In Sections 3 and 4, we apply the characteristic function method and the covariance method to dual modulation and warping operators to obtain joint (a,b), (α,β), and TF representations. Section 5 discusses the equivalence of these representations, and Section 6 presents an example. Finally, some extensions are pointed out in the concluding Section 7.

For the sake of simplicity, we consider only LCA groups $(G,\circ)$ isomorphic to $(\mathbb{R},+)$ with a differentiable isomorphism $\psi: G\to\mathbb{R}$ satisfying $\psi(g_1\circ g_2)=\psi(g_1)+\psi(g_2)$. The dual group [11, 3] $(\Gamma,\bullet)$ is also isomorphic to $\mathbb{R}$, with a differentiable isomorphism $\varphi:\Gamma\to\mathbb{R}$ satisfying $\varphi(\gamma_1\bullet\gamma_2)=\varphi(\gamma_1)+\varphi(\gamma_2)$. We note, however, that our results can be extended to general LCA groups (see Section 7).

2 MODULATION AND WARPING OPERATORS

Modulation Operators. A family of linear operators $\mathbf{M}_\beta$ defined on the Hilbert space $L^2(\Omega)$ $(\Omega\subseteq\mathbb{R})$ as

$$(\mathbf{M}_\beta x)(t)=e^{2\pi i\psi(\beta)\varphi(m(t))}\,x(t)\,,\quad t\in\Omega,\ \beta\in(G,\circ)$$

will be called a *modulation operator (family)*. Here, the *modulation function* $m(t)$ is assumed to be invertible with domain Ω and range $m(\Omega)=\Gamma$, where $(\Gamma,\bullet)$ is the group dual to $(G,\circ)$. It is easily checked that $\{\mathbf{M}_\beta\}_{\beta\in(G,\circ)}$ is a unitary representation [12] of the LCA group $(G,\circ)$. The generalized [13] eigenfunctions and eigenvalues of $\mathbf{M}_\beta$ are given by

$$u_a^{\mathbf{M}}(t)=c(t)\,\delta\big(\varphi(m(t))-\varphi(a)\big),\quad t\in\Omega,\ a\in(\Gamma,\bullet)\quad(1)$$
$$\lambda_{\beta,a}^{\mathbf{M}}=e^{2\pi i\psi(\beta)\varphi(a)},\quad \beta\in(G,\circ),\ a\in(\Gamma,\bullet).$$

If $\varphi(m(t))$ is differentiable, the choice $c(t)=\sqrt{|[\varphi(m(t))]'|}$ guarantees that the eigenfunctions $\{u_a^{\mathbf{M}}(t)\}$ are complete and orthonormal in the generalized functions sense [13].

The modulation operator $\mathbf{M}_\beta$ is unitarily equivalent [10, 14, 3] to the frequency shift operator $\mathbf{F}_\nu$ defined as $(\mathbf{F}_\nu x)(t)=e^{2\pi i\nu t}x(t)$ with $\nu\in\mathbb{R}$, $x\in L^2(\mathbb{R})$:

$$\mathbf{M}_\beta=\mathbf{U}\mathbf{F}_{\psi(\beta)}\mathbf{U}^{-1}\ \text{ with }\ (\mathbf{U}x)(t)=c(t)\,x\big(\varphi(m(t))\big).\quad(2)$$

Warping Operators. A family of linear operators $\mathbf{W}_\alpha$ defined on $L^2(\Theta)$ $(\Theta\subseteq\mathbb{R})$ as[1]

$$(\mathbf{W}_\alpha x)(t)=\sqrt{|w_\alpha'(t)|}\,x\big(w_\alpha(t)\big),\quad t\in\Theta,\ \alpha\in(\Gamma,\bullet)$$

will be called a *warping operator (family)*. For any fixed $\alpha\in(\Gamma,\bullet)$, the *warping function* $w_\alpha(t)$ is assumed to be a differentiable and invertible function mapping Θ onto Θ, and to satisfy the composition property

$$w_{\alpha_1}\big(w_{\alpha_2}(t)\big)=w_{\alpha_1\bullet\alpha_2}(t).\quad(3)$$

$\{\mathbf{W}_\alpha\}_{\alpha\in(\Gamma,\bullet)}$ is a unitary representation of $(\Gamma,\bullet)$. Warping operators have been previously introduced in [15].

The next theorem (proved in the Appendix) states an important relation between warping and modulation functions. For the special case $(\Gamma,\bullet)=(\mathbb{R},+)$, a similar result has been proved in [16] and (using a different argument) in [15].

Theorem 1. *If $m(t)$ is a modulation function on Ω, i.e., it is invertible with range $m(\Omega)=\Gamma$, then*

$$w_\alpha(t)\ \triangleq\ m^{-1}\big(m(t)\bullet\alpha^{-1}\big),\quad t\in\Omega,\ \alpha\in(\Gamma,\bullet)$$

(where $(\Gamma,\bullet)$ is some LCA group and α^{-1} denotes the group inverse of α) is a warping function on Ω, i.e., it is invertible with range $w_\alpha(\Omega)=\Omega$ and it satisfies the composition property (3). If $\varphi(m(t))$ is differentiable, then so is $w_\alpha(t)$.

Conversely, let $w_\alpha(t)$ $(t\in\Theta,\ \alpha\in(\Gamma,\bullet))$ satisfy (3). If $f(\alpha)\triangleq w_{\alpha^{-1}}(t_0)$ $(\alpha\in\Gamma;\ t_0\in\Theta$ arbitrary but fixed$)$ is invertible with range $f(\Gamma)=\Omega\subseteq\Theta$, then $w_\alpha(t)=f\big(f^{-1}(t)\bullet\alpha^{-1}\big)$ for $t\in\Omega$, $\alpha\in(\Gamma,\bullet)$, or equivalently, setting $m(t)\triangleq f^{-1}(t)$,

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[1]At this point, $(\Gamma,\bullet)$ is any LCA group that is isomorphic to $(\mathbb{R},+)$ with isomorphism $\varphi(\cdot)$. The reason why we call this group $(\Gamma,\bullet)$ will be clear presently.

$$w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big)\,, \qquad t \in \Omega,\ \alpha \in (\Gamma, \bullet)\,. \tag{4}$$

Here, $m(t)$ is unique up to group translations, i.e., $m_1^{-1}\big(m_1(t) \bullet \alpha^{-1}\big) \equiv m_2^{-1}\big(m_2(t) \bullet \alpha^{-1}\big)$ implies $m_2(t) \equiv m_1(t) \bullet \gamma$ with some $\gamma \in \Gamma$. Furthermore, there is $w_\alpha(\Omega) = \Omega$.

If $\Omega \subset \Theta$, i.e., $\Omega \neq \Theta$, then Θ (the domain of $w_\alpha(t)$) can be partitioned into disjoint subintervals Ω_i with associated modulation functions $m_i(t)$, $t \in \Omega_i$, such that $w_\alpha(t) = m_i^{-1}\big(m_i(t) \bullet \alpha^{-1}\big)$ for $t \in \Omega_i$.

Using the modulation function $m(t)$ associated to $w_\alpha(t)$ according to (4), we can give the following expressions for the generalized eigenfunctions and eigenvalues of $\mathbf{W}_\alpha$ on Ω:

$$u_b^{\mathbf{W}}(t) = c(t)\, e^{2\pi i \psi(b) \varphi(m(t))}, \qquad \lambda_{\alpha,b}^{\mathbf{W}} = e^{-2\pi i \varphi(\alpha)\psi(b)}, \tag{5}$$

where $t \in \Omega$, $b \in (G, \circ)$ (the group dual to $(\Gamma, \bullet)$), $\alpha \in (\Gamma, \bullet)$, and $c(t) = \sqrt{|[\varphi(m(t))]'|}$ as before. The eigenfunctions $u_b^{\mathbf{W}}(t)$ are complete and orthonormal.

The warping operator $\mathbf{W}_\alpha$ is unitarily equivalent to the time shift operator $\mathbf{T}_\tau$ defined as $(\mathbf{T}_\tau x)(t) = x(t-\tau)$ with $\tau \in \mathbb{R}$, $x \in L^2(\mathbb{R})$, i.e., $\mathbf{W}_\alpha = \mathbf{U}\mathbf{T}_{\varphi(\alpha)}\mathbf{U}^{-1}$ with $\mathbf{U}$ as in (2).

Dual Modulation and Warping Operators. The concept of *dual operators* has been introduced in [8, 9, 6] (there called *conjugate operators*) and independently in [17, 18]. The duality property has several important implications for joint signal analysis [6, 8, 9, 17, 18, 19] (cf. Section 5). The next theorem can be verified in a straightforward manner.

Theorem 2. *The modulation operator $\mathbf{M}_\beta$ with modulation function $m(t)$ and the warping operator $\mathbf{W}_\alpha$ with associated warping function $w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big)$ are dual, i.e.,*

$$\mathbf{M}_\beta \mathbf{W}_\alpha = e^{2\pi i \psi(\beta)\varphi(\alpha)}\, \mathbf{W}_\alpha \mathbf{M}_\beta \tag{6}$$

and

$$\big(\mathbf{M}_\beta\, u_b^{\mathbf{W}}(t)\big) = u_{b\circ\beta}^{\mathbf{W}}(t)\,, \qquad \big(\mathbf{W}_\alpha\, u_a^{\mathbf{M}}(t)\big) = u_{a\bullet\alpha}^{\mathbf{M}}(t)\,.$$

Example 1. Consider the modulation function $m(t) = t$ on $\Omega = \mathbb{R}$. Since $m(\Omega) = \mathbb{R}$, we must choose $(\Gamma, \bullet) = (\mathbb{R}, +)$, whence $\varphi(a) = a$. Setting $(G, \circ) = (\mathbb{R}, +)$, whence $\psi(b) = b$, the modulation operator is obtained as the frequency shift operator, $(\mathbf{M}_\beta x)(t) = e^{2\pi i \beta t} x(t)$ with $t \in \Omega = \mathbb{R}$, $\beta \in (G, \circ) = (\mathbb{R}, +)$. The associated warping function is $w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big) = t - \alpha$ with $t \in \Omega = \mathbb{R}$, $\alpha \in (\Gamma, \bullet) = (\mathbb{R}, +)$, so that the associated warping operator is the time shift operator, $(\mathbf{W}_\alpha x)(t) = x(t - \alpha)$.

Example 2. Next, consider $m(t) = t$ on $\Omega = \mathbb{R}^+$. Here, $m(\Omega) = \mathbb{R}^+$ so that we must choose $(\Gamma, \bullet) = (\mathbb{R}^+, \cdot)$, isomorphic to $(\mathbb{R}, +)$ by $\varphi(a) = \ln a$. Setting $(G, \circ) = (\mathbb{R}, +)$, whence $\psi(b) = b$, the modulation operator is obtained as $(\mathbf{M}_\beta x)(t) = e^{2\pi i \beta \ln t} x(t)$, $t \in \mathbb{R}^+$, $\beta \in (\mathbb{R}, +)$. The associated warping function is $w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big) = t/\alpha$ with $t \in \mathbb{R}^+$, $\alpha \in (\mathbb{R}^+, \cdot)$, so that the associated warping operator is $(\mathbf{W}_\alpha x)(t) = \sqrt{1/\alpha}\, x(t/\alpha)$, $t \in \mathbb{R}^+$, $\alpha \in (\mathbb{R}^+, \cdot)$.

These and some more examples are listed in Table 1.

$m(t)$	Ω or Ω_i	$(\Gamma, \bullet)$	$w_\alpha(t)$	Θ
t	$\mathbb{R}$	$(\mathbb{R}, +)$	$t - \alpha$	$\mathbb{R}$
$\lvert t\rvert$	$\mathbb{R}^+$, $\mathbb{R}^-$	$(\mathbb{R}^+, \cdot)$	t/α	$\mathbb{R}$
$\lvert t\rvert^\kappa$	$\mathbb{R}^+, \mathbb{R}^-$	$(\mathbb{R}^+, \cdot)$	$\alpha^{-1/\kappa}\, t$	$\mathbb{R}$
$\operatorname{sgn}(t)\lvert t\rvert^\kappa$	$\mathbb{R}\backslash\{0\}$	$(\mathbb{R}, +)$	$\operatorname{sgn}\big(\operatorname{sgn}(t)\lvert t\rvert^\kappa - \alpha\big) \cdot \big\lvert \operatorname{sgn}(t)\lvert t\rvert^\kappa - \alpha \big\rvert^{1/\kappa}$	$\mathbb{R}$
$\ln\lvert t\rvert$	$\mathbb{R}^+$, $\mathbb{R}^-$	$(\mathbb{R}, +)$	$e^{-\alpha} t$	$\mathbb{R}$
$\lvert\ln t\rvert$	$(0,1), (1,\infty)$	$(\mathbb{R}^+, \cdot)$	$t^{1/\alpha}$	$\mathbb{R}^+$
e^t	$\mathbb{R}$	$(\mathbb{R}^+, \cdot)$	$t - \ln\alpha$	$\mathbb{R}$
$\tan t$	$\big((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\big)$	$(\mathbb{R}, +)$	$\arctan(\tan t - \alpha)$	$\mathbb{R}$

Table 1. Some modulation and associated warping functions. We note that $m(\Omega) = \Gamma$, $\alpha \in (\Gamma, \bullet)$, $\kappa \in \mathbb{R}\backslash\{0\}$, and $k \in \mathbb{Z}$.

3 JOINT SIGNAL ANALYSIS USING THE CHARACTERISTIC FUNCTION METHOD

Let $\mathbf{W}_\alpha$ with $\alpha \in (\Gamma, \bullet)$ and $\mathbf{M}_\beta$ with $\beta \in (G, \circ)$ (where $(\Gamma, \bullet)$ and $(G, \circ)$ are dual groups) be warping and modulation operators defined on $L^2(\Omega)$ and based on the same modulation function $m(t)$. Hence, $\mathbf{W}_\alpha$ and $\mathbf{M}_\beta$ are dual operators.

Joint (a, b) Distributions. Applying the characteristic function method [1]–[4] to $\mathbf{W}_\alpha$ and $\mathbf{M}_\beta$ yields the joint (a, b) energy distributions

$$P_x(a,b) = \int_G \int_\Gamma S_x(\alpha,\beta)\, e^{2\pi i[\psi(b)\varphi(\alpha) - \varphi(a)\psi(\beta)]}\, d\varphi(\alpha)\, d\psi(\beta), \tag{7}$$

with $a \in (\Gamma, \bullet)$ and $b \in (G, \circ)$, where

$$S_x(\alpha,\beta) = \kappa(\alpha,\beta)\langle \mathbf{S}_{\alpha,\beta} x, x\rangle = \kappa(\alpha,\beta) \int_\Omega (\mathbf{S}_{\alpha,\beta} x)(t)\, x^*(t)\, dt\,.$$

Here, $\kappa(\alpha, \beta)$ is a signal-independent kernel function satisfying $\kappa(\alpha_0, \beta) = \kappa(\alpha, \beta_0) = 1$ and $\mathbf{S}_{\alpha,\beta}$ is any linear operator satisfying $\mathbf{S}_{\alpha,\beta_0} = \mathbf{W}_\alpha$ and $\mathbf{S}_{\alpha_0,\beta} = \mathbf{M}_\beta$, where α_0 and β_0 are the identity elements of $(\Gamma, \bullet)$ and $(G, \circ)$, respectively. For example, for $\mathbf{S}_{\alpha,\beta} = \mathbf{M}_\beta \mathbf{W}_\alpha$ we obtain

$$P_x(a,b) = \int_G \int_\Gamma \int_\Omega x\big(w_\alpha(t)\big)\, x^*(t)\, \kappa(\alpha,\beta) \sqrt{|w_\alpha'(t)|} \cdot e^{2\pi i\{\psi(b)\varphi(\alpha) + [\varphi(m(t)) - \varphi(a)]\psi(\beta)\}}\, dt\, d\varphi(\alpha)\, d\psi(\beta)\,. \tag{8}$$

$\mathbf{W}_\alpha$ and $\mathbf{M}_\beta$ being dual operators, it follows from (6) that using any other ordering of $\mathbf{W}_\alpha$ and $\mathbf{M}_\beta$ to define $\mathbf{S}_{\alpha,\beta}$ yields the same class of distributions $P_x(a, b)$ [19].

Any (a, b) distribution $P_x(a, b)$ in (7) or (8) satisfies the marginal properties [1]–[4]

$$\int_G P_x(a,b)\, d\psi(b) = |\tilde{x}(a)|^2, \qquad \int_\Gamma P_x(a,b)\, d\varphi(a) = |\tilde{X}(b)|^2,$$

where $\tilde{x}(a) = \langle x, u_a^{\mathbf{M}}\rangle = x\big(m^{-1}(a)\big)/\sqrt{|m'(m^{-1}(a))\, \varphi'(a)|}$ and $\tilde{X}(b) = \langle x, u_b^{\mathbf{W}}\rangle = \int_\Omega x(t)\, e^{-2\pi i \psi(b)\, \varphi(m(t))} \sqrt{|[\varphi(m(t))]'|}\, dt = \int_\Gamma \tilde{x}(a)\, e^{-2\pi i \psi(b)\varphi(a)}\, d\varphi(a)$.

Joint TF Distributions. The joint (a, b) distributions $P_x(a, b)$ can be converted into joint TF distributions $\tilde{P}_x(t, f)$ by a mapping $(t, f) = l(a, b)$. Here, the *localization function* $l(\cdot, \cdot)$ is obtained by solving the system of equations [10, 8] $\tau_a^{\mathbf{M}}(f) = t$, $\nu_b^{\mathbf{W}}(t) = f$ for t, f, where $\tau_a^{\mathbf{M}}(f) = m^{-1}(a)$ is the group delay of $u_a^{\mathbf{M}}(t)$ in (1) and $\nu_b^{\mathbf{W}}(t) = \psi(b)\, [\varphi(m(t))]'$ is the instantaneous frequency of $u_b^{\mathbf{W}}(t)$ in (5). Hence, the localization function mapping $(t, f) = l(a, b)$ is obtained as

$$t = m^{-1}(a)\,, \qquad f = \psi(b)\, m'\big(m^{-1}(a)\big)\, \varphi'(a)\,, \tag{9}$$

and the inverse mapping $(a, b) = l^{-1}(t, f)$ is given by

$$a = m(t)\,, \qquad b = \psi^{-1}\Big(\frac{f}{[\varphi(m(t))]'}\Big). \tag{10}$$

Inserting into (7) and (8) yields the class of TF distributions

$$\begin{aligned}\tilde{P}_x(t,f) &\triangleq P_x(a,b)\big|_{(a,b)=l^{-1}(t,f)} \\ &= \int_G\int_\Gamma S_x(\alpha,\beta)\, e^{2\pi i\left\{\frac{f\,\varphi(\alpha)}{[\varphi(m(t))]'} - \varphi(m(t))\psi(\beta)\right\}} d\varphi(\alpha)\, d\psi(\beta) \\ &= \int_G\int_\Gamma\int_\Omega x\big(w_\alpha(t')\big)\, x^*(t')\, \kappa(\alpha,\beta)\sqrt{|w'_\alpha(t')|} \\ &\quad\cdot e^{2\pi i\left\{\frac{f\,\varphi(\alpha)}{[\varphi(m(t))]'} + [\varphi(m(t'))-\varphi(m(t))]\psi(\beta)\right\}} dt'\, d\varphi(\alpha)\, d\psi(\beta)\,. \end{aligned} \tag{11}$$

These TF distributions satisfy the marginal properties [8, 9]

$$\begin{aligned}\int_{\mathbb{R}} \tilde{P}_x\big(m^{-1}(a),f\big)\, \frac{df}{|m'(m^{-1}(a))\,\varphi'(a)|} &= |\tilde{x}(a)|^2 \\ \int_\Omega \tilde{P}_x\big(t, \psi(b)\,[\varphi(m(t))]'\big)\, |[\varphi(m(t))]'|\, dt &= |\tilde{X}(b)|^2. \end{aligned}$$

4 JOINT SIGNAL ANALYSIS USING THE COVARIANCE METHOD

As before, let $\mathbf{W}_\alpha$ with $\alpha \in (\Gamma, \bullet)$ and $\mathbf{M}_\beta$ with $\beta \in (G, \circ)$ be dual warping and modulation operators defined on $L^2(\Omega)$.

Covariant Joint (α,β) Representations. With (6), the composite operator $\mathbf{D}_{\alpha,\beta} \triangleq \mathbf{M}_\beta \mathbf{W}_\alpha$ is easily checked to satisfy $\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-2\pi i\psi(\beta_1)\varphi(\alpha_2)}\,\mathbf{D}_{\alpha_1\bullet\alpha_2,\beta_1\circ\beta_2}$. It is thus a *TF displacement operator* as defined in [5, 6, 8]. Hence, the covariance method [5]–[9] can be applied to $\mathbf{D}_{\alpha,\beta} = \mathbf{M}_\beta\mathbf{W}_\alpha$, yielding the joint (α,β) representations

$$\begin{aligned} C_x(\alpha,\beta) &= \big\langle x, \mathbf{D}_{\alpha,\beta}\mathbf{H}\mathbf{D}^{-1}_{\alpha,\beta}x\big\rangle \qquad (12) \\ &= \int_\Omega\int_\Omega x(t_1)\,x^*(t_2)\,H^*\big(w_\alpha(t_1), w_\alpha(t_2)\big)\sqrt{|w'_\alpha(t_1)\,w'_\alpha(t_2)|} \\ &\quad\cdot e^{-2\pi i\psi(\beta)[\varphi(m(t_1))-\varphi(m(t_2))]}\,dt_1\,dt_2\,, \qquad (13) \end{aligned}$$

with $\alpha \in (\Gamma,\bullet)$, $\beta \in (G,\circ)$. Here, $\mathbf{H}$ is an arbitrary linear operator on $L^2(\Omega)$ and $H(t_1,t_2)$ is its kernel. Any (α,β) representation $C_x(\alpha,\beta)$ in (13) satisfies the covariance property

$$C_{\mathbf{D}_{\alpha_1,\beta_1}x}(\alpha,\beta) = C_x(\alpha\bullet\alpha_1^{-1}, \beta\circ\beta_1^{-1})\,. \tag{14}$$

Conversely, any quadratic (α,β) representation that satisfies the covariance property (14) is of the form (13) [5]–[9].

Covariant Joint TF Representations. The covariant (α,β) representations $C_x(\alpha,\beta)$ can be converted into covariant TF representations $\tilde{C}_x(t,f)$ by a mapping $(t,f) = d(\alpha,\beta)$. Here, the *displacement function* $d(\cdot,\cdot)$ is constructed as follows [6]. Let $x_{\alpha,\beta}(t) \triangleq D_{\alpha,\beta}(t,\hat{t})$, with $D_{\alpha,\beta}(t,t')$ the kernel of $\mathbf{D}_{\alpha,\beta}$ and $\hat{t} \in \Omega$ some fixed time. Similarly, define $y_{\alpha,\beta}(t)$ via its Fourier transform by $Y_{\alpha,\beta}(f) \triangleq \tilde{D}_{\alpha,\beta}(f,\hat{f})$, with $\tilde{D}_{\alpha,\beta}(f,f')$ the frequency-domain kernel of $\mathbf{D}_{\alpha,\beta}$ and $\hat{f} \in \mathbb{R}$ some fixed frequency. With the simplifying choice $\hat{t} = m^{-1}(\alpha_0)$ and $\hat{f} = 0$, the group delay of $x_{\alpha,\beta}(t)$ is $\tau_{\alpha,\beta}(f) = m^{-1}(\alpha)$ and the instantaneous frequency of $y_{\alpha,\beta}(t)$ is $\nu_{\alpha,\beta}(t) = \psi(\beta)\,[\varphi(m(t))]'$. The displacement function mapping $(t,f) = d(\alpha,\beta)$ is then obtained by solving the system of equations $\tau_{\alpha,\beta}(f) = t$, $\nu_{\alpha,\beta}(t) = f$ for t, f, which yields $t = m^{-1}(\alpha)$, $f = \psi(\beta)\,m'\big(m^{-1}(\alpha)\big)\,\varphi'(\alpha)$. This is identical to the localization function mapping in (9), thus verifying a conjecture [8, 6] about the identity of localization and displacement function in the case of dual operators.

Inserting the inverse displacement function mapping $\alpha = m(t)$, $\beta = \psi^{-1}\big(\frac{f}{[\varphi(m(t))]'}\big)$ (cf. (10)) into (13), we obtain the class of TF representations

$$\begin{aligned}\tilde{C}_x(t,f) &\triangleq C_x(\alpha,\beta)\big|_{(\alpha,\beta)=d^{-1}(t,f)} = \int_\Omega\int_\Omega x(t_1)\,x^*(t_2) \\ &\quad\cdot H^*\big(w_{m(t)}(t_1), w_{m(t)}(t_2)\big)\sqrt{\big|w'_{m(t)}(t_1)\,w'_{m(t)}(t_2)\big|} \\ &\quad\cdot e^{-2\pi i\frac{f}{[\varphi(m(t))]'}[\varphi(m(t_1))-\varphi(m(t_2))]}\,dt_1 dt_2\,. \end{aligned} \tag{15}$$

These TF representations satisfy the covariance property

$$\tilde{C}_{\mathbf{D}_{\alpha_1,\beta_1}x}(t,f) = \tilde{C}_x\left(w_{\alpha_1}(t),\, \frac{f-\psi(\beta_1)\,[\varphi(m(t))]'}{w'_{\alpha_1}(t)}\right). \tag{16}$$

Conversely, any quadratic TF representation that satisfies the covariance property (16) is of the form (15) [6].

5 EQUIVALENCE OF THE TWO METHODS

For the case of dual operators considered here, it has been shown in [6, 8, 9] that the class of (a,b) distributions $P_x(a,b)$ equals the class of covariant (α,β) representations $C_x(\alpha,\beta)$. Since moreover $d(\cdot,\cdot) \equiv l(\cdot,\cdot)$, this equivalence holds also for the TF representations $\tilde{P}_x(t,f)$ and $\tilde{C}_x(t,f)$. In fact, one can show the following result.

Theorem 3. *The covariant (α,β) representation in (13) equals the (a,b) distribution in (8), i.e., $C_x(\alpha,\beta) \equiv P_x(\alpha,\beta)$, and the covariant TF representation in (15) equals the TF distribution in (11), i.e., $\tilde{C}_x(t,f) \equiv \tilde{P}_x(t,f)$, if the kernel $H(t_1,t_2)$ defining $C_x(\alpha,\beta)$ and $\tilde{C}_x(t,f)$ is related to the kernel $\kappa(\alpha,\beta)$ defining $P_x(a,b)$ and $\tilde{P}_x(t,f)$ as*

$$\begin{aligned} H^*(t_1,t_2) &= \sqrt{|[\varphi(m(t_1))]'\,[\varphi(m(t_2))]'|} \\ &\quad\cdot \int_G \kappa\big(m(t_2)\bullet[m(t_1)]^{-1}, \beta\big)\, e^{2\pi i\varphi(m(t_2))\psi(\beta)}\, d\psi(\beta)\,. \end{aligned}$$

Furthermore, it can be shown that the TF representations $\tilde{C}_x(t,f) \equiv \tilde{P}_x(t,f)$ can be expressed as [10, 9]

$$\tilde{C}_x(t,f) \equiv \tilde{P}_x(t,f) = T_{\mathbf{U}^{-1}x}\left(\varphi(m(t)),\, \frac{f}{[\varphi(m(t))]'}\right),$$

with the Cohen's class TF representation [1] $T_x(t,f) = \int_\Omega\int_\Omega x(t_1)\,x^*(t_2)\,\tilde{H}^*(t_1-t, t_2-t)\,e^{-2\pi i f(t_1-t_2)}\,dt_1 dt_2$. Here, $\tilde{H}(t_1,t_2)$ is the kernel of the operator $\tilde{\mathbf{H}} = \mathbf{U}^{-1}\mathbf{H}\mathbf{U}$ with $\mathbf{H}$ the operator underlying $\tilde{C}_x(t,f)$ (cf. (12)) and $\mathbf{U}$ as in (2). We note that $T_x(t,f)$ is a special case of $\tilde{C}_x(t,f) \equiv \tilde{P}_x(t,f)$ with $(\Gamma,\bullet) = (G,\circ) = (\mathbb{R},+)$ and $\mathbf{W}_\alpha = \mathbf{T}_\alpha$, $\mathbf{M}_\beta = \mathbf{F}_\beta$.

6 EXAMPLE: THE HYPERBOLIC CLASS

We continue Example 2 from Section 2, i.e., $(\mathbf{M}_\beta x)(t) = e^{2\pi i\beta\ln t}\,x(t)$, $t>0$, $\beta\in\mathbb{R}$ and $(\mathbf{W}_\alpha x)(t) = \sqrt{1/\alpha}\;x(t/\alpha)$, $t>0$, $\alpha>0$. With (11), we obtain the TF distributions

$$\begin{aligned}\tilde{P}_x(t,f) &= \int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty x(\alpha t')\,x^*(t')\,\frac{\kappa(\alpha,\beta)}{\sqrt{\alpha}} \\ &\quad\cdot e^{2\pi i\left[tf\ln\alpha + \left(\ln\frac{t'}{t}\right)\beta\right]}\,dt'd\alpha d\beta\,. \end{aligned}$$

With (15), we obtain the covariant TF representations

$$\tilde{C}_x(t,f) = \frac{1}{t}\int_0^\infty\!\!\int_0^\infty x(t_1)\,x^*(t_2)\,H^*\!\left(\frac{t_1}{t},\frac{t_2}{t}\right) e^{-2\pi i t f \ln\frac{t_1}{t_2}}\,dt_1 dt_2 \,.$$

These TF representations are identical, i.e., $\tilde{C}_x(t,f) = \tilde{P}_x(t,f)$, if $H^*(t_1,t_2) = \frac{1}{\sqrt{t_1 t_2}}\int_{-\infty}^{\infty}\kappa\big(\frac{t_2}{t_1},\beta\big)\,e^{2\pi i (\ln t_2)\beta}\,d\beta$. They form a time-domain version of the *hyperbolic class* [20].

7 CONCLUSION

We have applied the characteristic function method and the covariance method to dual pairs of modulation and warping operators. Due to duality, the two methods are equivalent; they yield identical classes of TF representations that satisfy marginal and covariance properties.

Our results can be extended in the following directions:

- Modulation and warping operators can be defined on LCA groups not isomorphic to $(\mathbb{R},+)$, such as the group $(\mathbb{Z},+)$ and the torus group. All results remain valid if the exponentials $e^{2\pi i\varphi(\alpha)\psi(\beta)}$ are replaced by the group characters $\lambda_{\alpha,\beta}$.
- A structurally analogous but nonequivalent definition of modulation and warping operators can be given in the frequency domain. Our theory can easily be reformulated for this case, yielding different but structurally analogous classes of TF representations (cf. the usual definition of the hyperbolic class [20]).
- For *non-dual* modulation and warping operators defined on the same time interval, the characteristic function method can always be applied whereas the covariance method can be applied only if the operator $\mathbf{D}_{\alpha,\beta} = \mathbf{M}_\beta \mathbf{W}_\alpha$ is a unitary representation of the affine group [21]. Note that in the non-dual case the two methods are no longer equivalent.
- Our results can be extended to signal spaces $L^2(\Omega, d\mu)$. Here, the definition of $[\varphi(m(t))]'$, $w'_\alpha(t)$, etc. must be adjusted and dt must be replaced with $d\mu(t)$.

APPENDIX: PROOF OF THEOREM 1

Since $m(t)$ is invertible so is $m^{-1}(\alpha)$, and hence $w_\alpha(t) = m^{-1}\big(m(t)\bullet\alpha^{-1}\big)$ is invertible for fixed α. From $m(\Omega)=\Gamma$ and $m^{-1}(\Gamma)=\Omega$, it follows that $w_\alpha(\Omega)=\Omega$. It is easily verified that $w_\alpha(t) = m^{-1}\big(m(t)\bullet\alpha^{-1}\big)$ satisfies (3). Finally, if $\varphi(m(t))$ is differentiable, we have $[\varphi(m(t))]' \neq 0$ since $\varphi(m(t))$ is invertible, and thus the inverse function $m^{-1}(\varphi^{-1}(r))$ is differentiable as well. Hence, $w_\alpha(t) = m^{-1}\big(m(t)\bullet\alpha^{-1}\big) = m^{-1}\big[\varphi^{-1}\big(\varphi(m(t)) - \varphi(\alpha)\big)\big]$ is differentiable for fixed α.

Our proof of the theorem's second part generalizes a proof in [16]. Let $f(\alpha) \triangleq w_{\alpha^{-1}}(t_0)$ with fixed $t_0\in\Theta$. Then $w_\alpha\big(f(\alpha_1)\big) = w_\alpha\big(w_{\alpha_1^{-1}}(t_0)\big) = w_{\alpha\bullet\alpha_1^{-1}}(t_0) = f(\alpha_1\bullet\alpha^{-1})$. Setting $f(\alpha_1)=t$, we obtain $w_\alpha(t) = f\big(f^{-1}(t)\bullet\alpha^{-1}\big)$. Since by assumption $f(\alpha)$ is invertible on its range $f(\Gamma)=\Omega\subseteq\Theta$, this relation holds for all $t\in\Omega\subseteq\Theta$ and $\alpha\in\Gamma$. Furthermore, the relation implies that $w_\alpha(\Omega) = f(\Gamma) = \Omega$.

To show that $f(\alpha)$ is unique up to group translations, assume there is a function $g(\alpha)$ such that $w_\alpha(t) = g\big(g^{-1}(t)\bullet\alpha^{-1}\big)$. Then $w_{\alpha_1}(g(\alpha_2)) = g(\alpha_2\bullet\alpha_1^{-1})$, but at the same time, $w_{\alpha_1}(g(\alpha_2)) = f(f^{-1}(g(\alpha_2))\bullet\alpha_1^{-1})$. Hence, setting $\alpha_1^{-1}=\alpha$ and $\alpha_2=\alpha_0$, we obtain $g(\alpha\bullet\alpha_0) = f(f^{-1}(g(\alpha_0))\bullet\alpha)$, i.e., $g(\alpha) = f(\alpha\bullet\gamma)$ with $\gamma = f^{-1}(g(\alpha_0))\in\Gamma$ fixed.

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