

Oversampled Cosine Modulated Filter Banks with Perfect Reconstruction

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Abstract—Oversampled filter banks (FB's) offer more design freedom and better noise immunity than critically sampled FB's. Due to the increased computational complexity caused by oversampling, oversampled FB's allowing an efficient implementation, such as cosine modulated filter banks (CMFB's), are of particular interest.

So far, only critically sampled CMFB's have been considered. In this paper, we introduce *oversampled* CMFB's with perfect reconstruction (PR). Extending a classification of CMFB's recently proposed by Gopinath, we consider two types of oversampled CMFB's with PR. One of these types allows linear phase filters in all channels, and comprises CMFB's recently introduced by Lin and Vaidyanathan as well as Wilson-type CMFB's. For both types of oversampled CMFB's, we formulate PR conditions in the time, frequency, and polyphase domains. It is shown that any PR CMFB corresponds to a PR DFT FB with twice the oversampling factor and that (under a specific condition) the same PR prototype can be used for both CMFB types. We also show that the frame-theoretic properties of a CMFB and of the corresponding DFT FB are closely related. In particular, it is demonstrated that the minimum-norm synthesis prototype in an oversampled PR CMFB equals that in the corresponding DFT FB. Finally, we briefly address design methods and the efficient DCT/DST-based implementation of oversampled CMFB's.

Index Terms—Cosine modulated filter banks, DFT filter banks, frame theory, oversampled filter banks.

I. INTRODUCTION

RECENTLY *oversampled* filter banks (FB's) [1]–[9] have received increased attention, which is mainly due to two advantages over critically sampled FB's: *increased design freedom*—corresponding to the nonuniqueness of the synthesis FB satisfying perfect reconstruction (PR) for a given oversampled analysis FB [1], [2], [8], [9] and to the fact that PR can be satisfied with fewer side constraints; and *improved noise immunity*—corresponding to the fact that oversampled FB's tend to have a better frame bound ratio [2], [8], [9] and permit the application of noise shaping techniques [8], [9]. The last property makes oversampled FB's interesting for source coding applications using low-resolution quantizers. The practical advantages of using low-resolution quantizers at the cost of increased sample rate are indicated by the popular sigma-delta techniques [10]–[12].

These advantages of oversampled FB's come at the expense of increased computational cost caused by the need to process

a larger number of subband signal samples per unit of time. Thus, oversampled FB's allowing an efficient implementation, such as oversampled DFT FB's (or complex modulated FB's) [13], [14], [3], [2], [5]–[8] and oversampled *cosine modulated* FB's (CMFB's) [7], [8], [15], [16], are of particular interest. Here, CMFB's are advantageous as their subband signals are real-valued for real-valued input signal.

It seems that so far only critically sampled CMFB's have been considered in the literature [17]–[28]. This paper introduces and studies *oversampled* CMFB's with PR. Section II introduces two types of oversampled CMFB's, thereby extending a CMFB classification recently proposed for critical sampling by Gopinath [25]. The “odd-stacked” CMFB's extend the traditional CMFB type [17]–[27] (“class B CMFB's” [25]), and the “even-stacked” CMFB's extend the “class A CMFB's” recently introduced for critical sampling by Gopinath [25], to the oversampled case. The latter class contains CMFB's previously proposed (for critical sampling) by Lin and Vaidyanathan [28] and the Wilson-type CMFB's introduced by the authors [15]. Even-stacked CMFB's are attractive since they allow both PR/paraunitarity and linear phase filters in all channels.

Section III shows that odd- and even-stacked, oversampled CMFB's are closely related to a corresponding oversampled DFT FB of the same stacking type but with twice the oversampling factor. A fundamental decomposition theorem for a CMFB's input–output operator is derived, and it is shown that even-stacked CMFB's are related to MDFT FB's [29]–[32]. Section IV provides PR/paraunitarity conditions for odd- and even-stacked, oversampled CMFB's. It is shown that any PR CMFB corresponds to a PR DFT FB and that (under a specific condition) the same PR prototype can be used for both CMFB types. Section V formulates conditions on oversampled CMFB's to provide a frame decomposition. We show that the frame operator and frame bound ratio of an oversampled CMFB [1], [2], [5], [9], [33] (characterizing important numerical properties of the CMFB) are related to those of the corresponding DFT FB, and that the minimum-norm synthesis prototype in an oversampled PR CMFB equals that in the corresponding DFT FB. Finally, Section VI briefly addresses the design and the efficient implementation of oversampled CMFB's.

II. OVERSAMPLED COSINE MODULATED FILTER BANKS

In spite of the advantages of oversampled FB's mentioned in Section I, it appears that so far only CMFB's with critical sampling have been considered. In this section,

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therefore, we introduce oversampled odd-stacked and even-stacked CMFB's.

A. Oversampled Odd-Stacked CMFB's

We first extend the conventional type of CMFB's [17]–[27] (termed “class B” in [25]) to the oversampled case. This CMFB type will be called *odd-stacked* due to its close relation to odd-stacked DFT FB's (cf. Section III-A). We define the analysis and synthesis filters of an odd-stacked CMFB with N channels and decimation factor M (note that the CMFB is oversampled for $N > M$) as¹

$$h_k^{C-o}[n] = \sqrt{2} h[n] \cos \left[\frac{(k+1/2)\pi}{N} n + \phi_k^o \right] \quad (1)$$

$$f_k^{C-o}[n] = \sqrt{2} f[n] \cos \left[\frac{(k+1/2)\pi}{N} n - \phi_k^o \right] \quad (2)$$

for $k = 0, 1, \dots, N-1$. Here, $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, which may be FIR or IIR filters. Extending the phase definition given for critical sampling ($N = M$) by Gopinath and Burrus [22] to the oversampled case, we define the phases ϕ_k^o as

$$\phi_k^o = -\alpha \frac{\pi}{2N} \left(k + \frac{1}{2} \right) + r \frac{\pi}{2} \quad \text{with } \alpha \in \mathbb{Z}, r \in \{0, 1\}.$$

The choice $r = 1$ corresponds to replacing the cos in (1) and (2) by $-\sin$ and \sin , respectively. The above phase expression contains the phases proposed in [18]–[21], [34] as special cases. Note that the channel frequencies in an odd-stacked CMFB are $\theta_k = (k+1/2)/(2N)$ ($k = 0, 1, \dots, N-1$), as depicted in Fig. 1(a). In particular, the channel with index $k = 0$ is centered at frequency $\theta_0 = 1/(4N)$.

An important disadvantage of odd-stacked CMFB's (especially in image coding applications [35], [36]) is that the channel filters do not have linear phase even if the prototypes have linear phase [22], [25]. Other important aspects of odd-stacked CMFB's, such as their relation to DFT FB's, PR

¹The superscripts C-o and C-e indicate that the respective quantity belongs to an odd- and even-stacked CMFB, respectively.

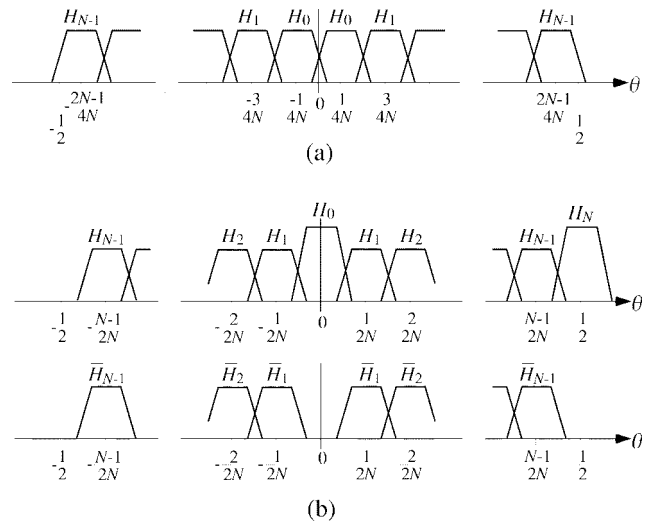


Fig. 1. Transfer functions of the channel filters in (a) an N -channel odd-stacked CMFB, and (b) a $2N$ -channel even-stacked CMFB.

conditions, and frame-theoretic properties, will be considered in Sections III–VI.

B. Oversampled Even-Stacked CMFB's

Next, we generalize the “class A” CMFB's recently proposed for critical sampling by Gopinath [25] to the oversampled case. We call this CMFB type *even-stacked* due to its close relation to even-stacked DFT FB's (cf. Section III-B). The analysis FB in an even-stacked CMFB with $2N$ channels and decimation factor $2M$ (note that the CMFB is oversampled for $N > M$) consists of two partial FB's $\{h_k^{C-e}[n]\}_{k=0,1,\dots,N}$ and $\{\bar{h}_k^{C-e}[n]\}_{k=1,2,\dots,N-1}$ derived from an analysis prototype $h[n]$ as shown in (3) and (4) at the bottom of the page. Similarly, the synthesis FB is defined in terms of a synthesis prototype $f[n]$ as shown at the bottom of the page. Extending the phase definition given for critical sampling in [25], we define the phases as

$$\phi_k^e = -\alpha \frac{\pi}{2N} k + r \frac{\pi}{2} \quad \text{with } \alpha \in \mathbb{Z}, r \in \{0, 1\}.$$

$$h_k^{C-e}[n] = \begin{cases} h[n - rM], & k = 0 \\ \sqrt{2} h[n] \cos \left(\frac{k\pi}{N} n + \phi_k^e \right), & k = 1, 2, \dots, N-1 \\ h[n - sM] (-1)^{n-sM}, & k = N \end{cases} \quad (3)$$

$$\bar{h}_k^{C-e}[n] = \sqrt{2} h[n - M] \sin \left[\frac{k\pi}{N} (n - M) + \phi_k^e \right], \quad k = 1, 2, \dots, N-1 \quad (4)$$

$$f_k^{C-e}[n] = \begin{cases} f[n + rM], & k = 0 \\ \sqrt{2} f[n] \cos \left(\frac{k\pi}{N} n - \phi_k^e \right), & k = 1, 2, \dots, N-1 \\ f[n + sM] (-1)^{n+sM}, & k = N \end{cases}$$

$$\bar{f}_k^{C-e}[n] = -\sqrt{2} f[n + M] \sin \left[\frac{k\pi}{N} (n + M) - \phi_k^e \right], \quad k = 1, 2, \dots, N-1$$

Furthermore, $s \in \{0, 1\}$ with $s = r$ for α even and $s = 1 - r$ for α odd. Note that an even-stacked CMFB has $2N$ channels but there are only $N+1$ different channel frequencies $\theta_k = k/(2N)$ ($k = 0, 1, \dots, N$), as depicted in Fig. 1(b). In particular, the $k = 0$ channel is centered at frequency $\theta_0 = 0$.

For *any* choice of the parameters $\alpha \in \mathbb{Z}$ and $r \in \{0, 1\}$, all analysis filters have linear phase if the analysis prototype $h[n]$ satisfies the linear phase (symmetry) property $h[\alpha + (2l + 1)N - n] = h[n]$ for some $l \in \mathbb{Z}$. Similarly, all synthesis filters have linear phase if $f[-\alpha - (2l + 1)N - n] = f[n]$ for some $l \in \mathbb{Z}$. This is an important advantage over the odd-stacked CMFB's considered in the previous subsection. For the special case of critical sampling, this linear phase property has first been recognized by Gopinath [25].

Two specific even-stacked CMFB's have been proposed previously. The first one, corresponding to $\alpha = 0$ and $r = 0$ (hence $s = 0$), is the CMFB recently introduced (for critical sampling) by Lin and Vaidyanathan [28]. A second special even-stacked CMFB, obtained for $\alpha = N$ and $r = 0$ (hence $s = 0$ for N even and $s = 1$ for N odd), is the Wilson-type CMFB recently introduced by the authors [15].

An interesting interpretation of the subband signals can be given for real-valued input signal $x[n]$ and real-valued $h[n]$. Let $v_k[n]$ and $\bar{v}_k[n]$ denote the outputs (subband signals, before subsampling) of the filters $h_k^{C-e}[n]$ and $\bar{h}_k^{C-e}[n]$, respectively. It can be shown that the output of the filter $\sqrt{2}H(zW_{2N}^k)e^{j\phi_k}$ (with $W_{2N} = e^{-j2\pi/(2N)}$) has real part $v_k[n]$ and imaginary part $\bar{v}_k[n + M]$. This extends the interpretation given in [28] for the subband signals of a critically sampled Lin–Vaidyanathan FB. The consequences of this result are discussed in [28] and [8].

III. REPRESENTATION OF CMFB'S VIA DFT FILTER BANKS

In this section, we shall show that an oversampled, odd- or even-stacked CMFB is related to a corresponding oversampled DFT FB of the same stacking type but with twice the CMFB's oversampling factor. In Sections IV and V, this relation will be seen to yield a unified and simplified framework for the analysis and design of odd- and even-stacked CMFB's. We note that the idea of constructing CMFB's from DFT FB's has been previously used in the special case of critical sampling and near-PR (see, for example, [24]).

A. Odd-Stacked CMFB's

To each odd-stacked CMFB with N channels and decimation factor M , we consider a *corresponding odd-stacked DFT FB* (or complex modulated FB) with the same prototypes $h[n]$ and $f[n]$, $2N$ channels, and decimation factor M . Note that the oversampling factor of this DFT FB, $2N/M$, is twice that of the CMFB, N/M . The analysis and synthesis filters of this corresponding DFT FB are derived from the prototypes $h[n]$ and $f[n]$, respectively, by complex modulation² [13], [7],

$$\begin{aligned} h_k^{D-o}[n] &= h[n]W_{2N}^{-(k+1/2)n} \\ f_k^{D-o}[n] &= f[n]W_{2N}^{-(k+1/2)n}, \quad k = 0, 1, \dots, 2N - 1 \end{aligned}$$

²The superscripts D-o and D-e indicate that the respective quantity belongs to an odd- and even-stacked DFT FB, respectively.

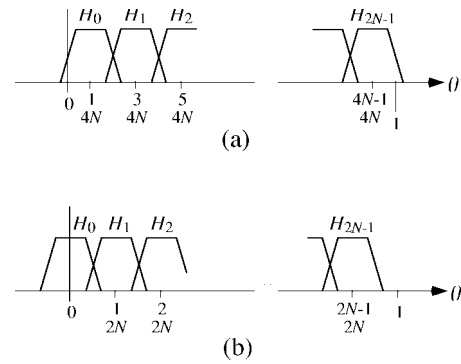


Fig. 2. Transfer functions of the channel filters in (a) a $2N$ -channel odd-stacked DFT FB and (b) a $2N$ -channel even-stacked DFT FB.

where $W_{2N} = e^{-j2\pi/(2N)}$. The associated transfer functions are $H_k^{D-o}(z) = H(zW_{2N}^{k+1/2})$ and $F_k^{D-o}(z) = F(zW_{2N}^{k+1/2})$ with $k = 0, 1, \dots, 2N - 1$ [see Fig. 2(a)]. With (1), it easily follows that the CMFB's analysis filters can be expressed in terms of the analysis filters of the corresponding DFT FB as

$$\begin{aligned} H_k^{C-o}(z) &= \frac{1}{\sqrt{2}} \left[H_k^{D-o}(z)e^{j\phi_k} + H_{2N-k-1}^{D-o}(z)e^{-j\phi_k} \right], \\ k &= 0, 1, \dots, N - 1. \end{aligned}$$

The filters $H_k^{D-o}(z)$ and $H_{2N-k-1}^{D-o}(z)$ correspond to channel frequencies $\theta = (k + 1/2)/(2N)$ and $\theta = -(k + 1/2)/(2N)$, respectively. Thus, the CMFB's analysis filters are obtained by combining the analysis filters of a DFT FB corresponding to positive and negative frequencies. A similar relation exists for the synthesis filters.

The relation between odd-stacked CMFB's and DFT FB's can also be characterized in a different manner that will be important subsequently. The input–output relation in an odd-stacked CMFB with N channels and decimation factor M is

$$\hat{x}_{C-o}[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{C-o} \rangle f_{k,m}^{C-o}[n] \quad (5)$$

with $\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n]$ and the analysis and synthesis functions

$$\begin{aligned} h_{k,m}^{C-o}[n] &= h_k^{C-o*}[mM - n] \\ &= \sqrt{2} h^*[mM - n] \cos \left[\frac{(k + 1/2)\pi}{N} (mM - n) + \phi_k^o \right] \end{aligned} \quad (6)$$

$$\begin{aligned} f_{k,m}^{C-o}[n] &= f_k^{C-o}[n - mM] \\ &= \sqrt{2} f[n - mM] \cos \left[\frac{(k + 1/2)\pi}{N} (n - mM) - \phi_k^o \right]. \end{aligned} \quad (7)$$

Let us compare this to the input–output relation of the corresponding odd-stacked DFT FB. This DFT FB has analysis functions $h_{k,m}^{D-o}[n] = h_k^{D-o*}[mM - n] = h^*[mM - n]W_{2N}^{(k+1/2)(mM-n)}$ and synthesis functions $f_{k,m}^{D-o}[n] =$

$f_k^{\text{D-o}}[n - mM] = f[n - mM]W_{2N}^{-(k+1/2)(n-mM)}$. It follows that the CMFB analysis functions can be expressed in terms of the corresponding DFT FB analysis functions as

$$h_{k,m}^{\text{C-o}}[n] = \frac{1}{\sqrt{2}} \left[h_{k,m}^{\text{D-o}}[n] e^{-j\phi_k^c} + h_{2N-k-1,m}^{\text{D-o}}[n] e^{j\phi_k^c} \right]. \quad (8)$$

The CMFB synthesis functions $f_{k,m}^{\text{C-o}}[n]$ can be written in a similar manner in terms of the $f_{k,m}^{\text{D-o}}[n]$.

B. Even-Stacked CMFB's

Similarly, to an even-stacked CMFB with $2N$ channels and decimation factor $2M$, we consider a *corresponding even-stacked DFT FB* with the same prototypes $h[n]$ and $f[n]$, $2N$ channels, and decimation factor M (and thus twice the CMFB's oversampling factor). The analysis filters of this DFT FB are [13]

$$\begin{aligned} h_k^{\text{D-e}}[n] &= h[n]W_{2N}^{-kn} \\ f_k^{\text{D-e}}[n] &= f[n]W_{2N}^{-kn}, \quad k = 0, 1, \dots, 2N - 1 \end{aligned}$$

with transfer functions $H_k^{\text{D-e}}(z) = H(zW_{2N}^k)$ and $F_k^{\text{D-e}}(z) = F(zW_{2N}^k)$ with $k = 0, 1, \dots, 2N - 1$ [see Fig. 2(b)]. With (3) and (4), the CMFB's analysis filters can be expressed in terms of the DFT FB's analysis filters as shown in the first set of equations at the bottom of the page. Note that the filters $H_k^{\text{D-e}}(z)$ and $H_{2N-k}^{\text{D-e}}(z)$ correspond to channel frequencies $\theta = k/(2N)$ and $\theta = -k/(2N)$, respectively. Similar relations exist for the synthesis filters.

This correspondence to a DFT FB can also be characterized using the analysis and synthesis functions. The input-output

relation in an even-stacked CMFB with $2N$ channels and decimation factor $2M$ is

$$\hat{x}_{\text{C-e}}[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{\text{C-e}} \rangle f_{k,m}^{\text{C-e}}[n] \quad (9)$$

with the analysis and synthesis functions as shown in (10) and (11) at the bottom of the page. The corresponding even-stacked DFT FB has analysis functions $h_{k,m}^{\text{D-e}}[n] = h^*[mM - n]W_{2N}^{k(mM-n)}$ and synthesis functions $f_{k,m}^{\text{D-e}}[n] = f[n - mM]W_{2N}^{-k(n-mM)}$. It follows (after simple manipulations) that the CMFB's analysis functions can be expressed in terms of the DFT FB's analysis functions according to (12), shown at the bottom of the page. The synthesis functions can be expressed in a similar manner.

C. A Fundamental Decomposition

In Sections III-A and B, we showed that the analysis (synthesis) functions of a CMFB can be expressed in terms of the analysis (synthesis) functions of a corresponding DFT FB. This implies the following important decomposition which will be the basis for our subsequent analysis of CMFB's, in particular, for formulating PR and paraunitarity conditions in Section IV.

Theorem 1: The reconstructed signal of an odd- or even-stacked CMFB can be decomposed as

$$\hat{x}[n] = \frac{1}{2}[(\mathbf{S}_D^{(h,f)}x)[n] + (\mathbf{T}_D^{(h,f)}x)[n]] \quad (13)$$

$$\begin{aligned} H_k^{\text{C-e}}(z) &= \begin{cases} z^{-rM} H_0^{\text{D-e}}(z), & k = 0 \\ \frac{1}{\sqrt{2}} \left[H_k^{\text{D-e}}(z) e^{j\phi_k^c} + H_{2N-k}^{\text{D-e}}(z) e^{-j\phi_k^c} \right], & k = 1, 2, \dots, N - 1 \\ z^{-sM} H_N^{\text{D-e}}(z) = z^{-sM} H_0^{\text{D-e}}(-z), & k = N \end{cases} \\ \bar{H}_k^{\text{C-e}}(z) &= \frac{1}{j\sqrt{2}} z^{-M} \left[H_k^{\text{D-e}}(z) e^{j\phi_k^c} - H_{2N-k}^{\text{D-e}}(z) e^{-j\phi_k^c} \right], \quad k = 1, 2, \dots, N - 1 \end{aligned}$$

$$h_{k,m}^{\text{C-e}}[n] = \begin{cases} h_k^{\text{C-e}*}[2mM - n], & k = 0, 1, \dots, N \\ \bar{h}_{k-N}^{\text{C-e}*}[2mM - n], & k = N + 1, N + 2, \dots, 2N - 1 \end{cases} \quad (10)$$

$$f_{k,m}^{\text{C-e}}[n] = \begin{cases} f_k^{\text{C-e}}[n - 2mM], & k = 0, 1, \dots, N \\ \bar{f}_{k-N}^{\text{C-e}}[n - 2mM], & k = N + 1, N + 2, \dots, 2N - 1 \end{cases} \quad (11)$$

$$h_{k,m}^{\text{C-e}}[n] = \begin{cases} \frac{1}{\sqrt{2}} \left[h_{k,2m}^{\text{D-e}}[n] e^{-j\phi_k^c} + h_{2N-k,2m}^{\text{D-e}}[n] e^{j\phi_k^c} \right], & k = 1, 2, \dots, N - 1 \\ \frac{j}{\sqrt{2}} \left[h_{k-N,2m-1}^{\text{D-e}}[n] e^{-j\phi_{k-N}^c} - h_{N-k,2m-1}^{\text{D-e}}[n] e^{j\phi_{k-N}^c} \right], & k = N + 1, N + 2, \dots, 2N - 1 \\ h_{0,2m-r}^{\text{D-e}}[n], & k = 0 \\ h_{N,2m-s}^{\text{D-e}}[n], & k = N \end{cases} \quad (12)$$

where the operators $\mathbf{S}_D^{(h,f)}$ and $\mathbf{T}_D^{(h,f)}$ are defined as

$$(\mathbf{S}_D^{(h,f)}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^D \rangle f_{k,m}^D[n] \quad (14)$$

$$(\mathbf{T}_D^{(h,f)}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \langle x, h_{k,m}^D \rangle \hat{f}_{k,m}^D[n]. \quad (15)$$

Here, $h_{k,m}^D[n] = h_{k,m}^{D-o}[n]$, $f_{k,m}^D[n] = f_{k,m}^{D-o}[n]$, $\hat{f}_{k,m}^D[n] = f_{2N-k-1,m}^{D-o}[n]$, $\phi_k = \phi_k^o$, and $c_m = 1$ for an odd-stacked CMFB and $h_{k,m}^D[n] = h_{k,m}^{D-e}[n]$, $f_{k,m}^D[n] = f_{k,m}^{D-e}[n]$, $\hat{f}_{k,m}^D[n] = f_{2N-k,m}^{D-e}[n]$, $\phi_k = \phi_k^e$, and $c_m = (-1)^m$ for an even-stacked CMFB.

A proof of this theorem is given in the Appendix. We note that the component $(\mathbf{S}_D^{(h,f)}x)[n]$ is the output signal of the corresponding DFT FB that has the same stacking type but twice the oversampling factor.

D. Relation with MDFT Filter Banks

Modified DFT FB's (MDFT FB's), proposed for critical sampling by Fliege and Karp [29]–[32], were the first modulated FB's allowing real processing and linear phase filters in all channels. We now demonstrate a close relation between MDFT FB's and even-stacked CMFB's of the Wilson type. It can be shown that the reconstructed signal in a (possibly oversampled) MDFT FB can be decomposed as

$$\hat{x}[n] = \frac{1}{2} \left[(\mathbf{S}_D^{(h,f)}x)[n] + (\mathbf{T}_D^{(h,f)}x^*)[n] \right] \quad (16)$$

where $(\mathbf{S}_D^{(h,f)}x)[n]$ and $(\mathbf{T}_D^{(h,f)}x)[n]$ are defined as in (14) and (15) for the even-stacked case with parameters $\alpha = N$ and $r = 0$ (the parameters of Wilson FB's). Here, $\mathbf{T}_D^{(h,f)}$ acts on the *conjugate* of the input signal $x[n]$. For real-valued $x[n]$, the input–output relation in an MDFT FB equals that in a Wilson FB.

For general (complex-valued) $x[n]$, it can be shown (similarly to the proof of Theorem 2 in Section IV-A) that an MDFT FB satisfies the PR property if and only if $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$ and $\mathbf{T}_D^{(h,f)} = \mathbf{O}$. This equals the PR condition in a Wilson FB (see Section IV-A). Thus, *PR MDFT FB's are equivalent to PR Wilson FB's*. Furthermore, it follows from the decomposition (16) that the PR and paraunitarity conditions to be formulated in Section IV apply to MDFT FB's as well.

IV. PERFECT RECONSTRUCTION AND PARAUNITARITY CONDITIONS

In this section, we provide PR conditions and paraunitarity conditions for odd- and even-stacked CMFB's with arbitrary oversampling. For the special case of critical sampling ($N = M$), our conditions simplify to those derived in [17]–[28].

A. PR Conditions Using the CMFB Operators

We start with a fundamental PR condition that follows from the decomposition in Theorem 1.

Theorem 2: A CMFB (even-stacked or odd-stacked, oversampled or critically sampled) satisfies the PR property $\hat{x}[n] = x[n]$ if and only if

$$\begin{aligned} \mathbf{S}_D^{(h,f)} &= 2\mathbf{I} \\ \mathbf{T}_D^{(h,f)} &= \mathbf{O} \end{aligned} \quad (17)$$

where \mathbf{I} and \mathbf{O} denote the identity and zero operator, respectively, on $\ell^2(\mathbb{Z})$ (the space of square-summable sequences).

Proof: Inserting (17) into (13) gives $\hat{x}[n] = x[n]$, so that (17) is sufficient for PR. We now show that (17) is also necessary for PR. From (13), there is $\hat{x}[n] = \frac{1}{2}[(\mathbf{S}_D^{(h,f)}x)[n] + (\mathbf{T}_D^{(h,f)}x)[n]]$. It can be shown that

$$\begin{aligned} (\mathbf{S}_D^{(h,f)}x)[n] &= 2N \sum_{l=-\infty}^{\infty} d_l x[n - 2lN] \sum_{m=-\infty}^{\infty} f[n - mM] \\ &\quad \cdot h[-n + mM + 2lN] \end{aligned} \quad (18)$$

$$\begin{aligned} (\mathbf{T}_D^{(h,f)}x)[n] &= (-1)^r 2N \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} d_l a_m \\ &\quad \cdot x[-n + 2mM - 2lN - \alpha] f[n - mM] \\ &\quad \cdot h[n - mM + \alpha + 2lN] \end{aligned} \quad (19)$$

with $d_l = (-1)^l$ and $a_m = 1$ in the odd-stacked case and $d_l = 1$ and $a_m = (-1)^m$ in the even-stacked case. Hence, $(\mathbf{S}_D^{(h,f)}x)[n]$ is a linear combination of shifted signal versions $x[n - 2lN]$, whereas $(\mathbf{T}_D^{(h,f)}x)[n]$ is a linear combination of *time-reversed* and shifted signal versions $x[-n + 2mM - 2lN - \alpha]$. For $\hat{x}[n] = x[n]$, all time-reversed signal versions $x[-n + 2mM - 2lN - \alpha]$ (due to $\mathbf{T}_D^{(h,f)}$) must be weighted by zero since, for general signal $x[n]$, they cannot be cancelled by the signal versions $x[n - 2lN]$ (due to $\mathbf{S}_D^{(h,f)}$). Hence, there must be $\mathbf{T}_D^{(h,f)} = \mathbf{O}$, which entails $\hat{x}[n] = \frac{1}{2}(\mathbf{S}_D^{(h,f)}x)[n]$. This finally implies $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$. \square

If the second PR condition, $\mathbf{T}_D^{(h,f)} = \mathbf{O}$, is satisfied, the CMFB's input–output relation (13) reduces to $\hat{x}[n] = \frac{1}{2}(\mathbf{S}_D^{(h,f)}x)[n]$, which is (up to a constant factor) the input–output relation of the corresponding DFT FB. This explains why the first PR condition, $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$, is (up to a constant factor) the PR condition for the corresponding DFT FB [13]; this PR condition is *the same* for odd-stacked and even-stacked CMFB's. We conclude that *CMFB's with PR correspond to PR DFT FB's of the same stacking type and with twice the oversampling factor*. In the special case of critical sampling, a similar relation has previously been shown to exist between MDFT FB's and DFT FB's [31]. As a consequence of Theorem 2, CMFB prototypes designed to satisfy the PR property can be used without modifications in a PR CMFB of the respective other stacking type (always assuming that $\mathbf{T}_D^{(h,f)} = \mathbf{O}$ is satisfied for both stacking types) or in a PR MDFT FB.

We next reformulate the two PR conditions (17) in the time, frequency, and polyphase domains; this allows us to formulate the PR side constraints arising in the design of PR CMFB's in the domain that is most appropriate. The PR conditions given below hold for arbitrary oversampling factors unless indicated

otherwise. Their derivations are partly tedious and will not be provided due to space restrictions.

B. PR Conditions in the Time Domain

With (18), it follows that the first PR condition, $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$, is satisfied if and only if

$$\sum_{m=-\infty}^{\infty} f[n-mM]h[-n+mM+2lN] = \frac{1}{N}\delta[l] \quad (20)$$

where $\delta[0] = 1$ and $\delta[l] = 0$ for $l \neq 0$. This condition is independent of the stacking type. Similarly, it follows with (19) that the second PR condition, $\mathbf{T}_D^{(h,f)} = \mathbf{0}$, holds if and only if

$$\sum_{i=-\infty}^{\infty} b_i f[n-iPM]h[n+iPM+\alpha+2lN] = 0, \quad l = 0, 1, \dots, Q-1$$

where $N/M = P/Q$ with P, Q relatively prime and $b_i = (-1)^{iQ}$ in the odd-stacked case and $b_i = (-1)^{iP}$ in the even-stacked case. In the special case of *integer oversampling*, i.e., $Q = 1$ or $N = PM$ with $P \in \mathbb{N}$, the latter condition simplifies to

$$\sum_{i=-\infty}^{\infty} b_i f[n-iPM]h[n+iPM+\alpha] = 0 \quad (21)$$

where $b_i = (-1)^i$ in the odd-stacked case and $b_i = (-1)^{iP}$ in the even-stacked case. Critical sampling, $N = M$, is a special case with $P = 1$. It can be shown that if prototypes $h[n]$, $f[n]$ satisfy the first PR property (20) in the critical case, the scaled prototypes $\sqrt{P}h[n]$, $\sqrt{P}f[n]$ will satisfy this property in the case of integer oversampling with oversampling factor P (this has also been observed for the paraunitary case in [37]). However, we caution that a similar rule does *not* hold for the second PR condition (21). Thus, prototypes providing PR in the critical case do not automatically provide PR for integer oversampling.

C. PR Conditions in the Frequency Domain

For either stacking type, $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$ if and only if

$$\sum_{k=0}^{2N-1} F(zW_{2N}^k)H(zW_{2N}^k W_M^l) = 2M \delta[l], \quad l = 0, 1, \dots, M-1.$$

Similarly, it can be shown that $\mathbf{T}_D^{(h,f)} = \mathbf{0}$ if and only if

$$\sum_{k=0}^{2N-1} \hat{F}_k^D(z) H_k^D(zW_M^{l+\beta}) W_{2N}^{k\alpha} = 0, \quad l = 0, 1, \dots, M-1$$

where $\hat{F}_k^D(z) = F_{2N-k-1}^{D-o}(z)$, $H_k^D(z) = H_k^{D-o}(z)$, and $\beta = 0$ in the odd-stacked case and $\hat{F}_k^D(z) = F_{2N-k}^{D-e}(z)$, $H_k^D(z) = H_k^{D-e}(z)$, and $\beta = 1/2$ in the even-stacked case.

D. PR Conditions in the Polyphase Domain

For Q odd, we have $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$ if and only if

$$\begin{aligned} A_{l,n}(z) &\triangleq \sum_{i=0}^{P-1} \left[R_n(zW_{2P}^i) E_{n-2lN}(zW_{2P}^i) \right. \\ &\quad \left. + R_n(-zW_{2P}^i) E_{n-2lN}(-zW_{2P}^i) \right] \\ &= \frac{2Q}{M} \delta[l] \end{aligned} \quad (22)$$

for $l = 0, 1, \dots, Q-1$, $n = 0, 1, \dots, M-1$, where $E_n(z) = \sum_{m=-\infty}^{\infty} h[mM-n]z^{-m}$ and $R_n(z) = \sum_{m=-\infty}^{\infty} f[mM+n]z^{-m}$ are the polyphase components of $h[n]$ and $f[n]$, respectively. For Q even, $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$ if and only if

$$\begin{aligned} d_l A_{l,n}(z) + z^{-P} d_{l+Q/2} A_{l+Q/2,n}(z) &= \frac{2Q}{M} \delta[l], \\ l = 0, 1, \dots, \frac{Q}{2} - 1; \quad n = 0, 1, \dots, M-1 \end{aligned}$$

where, again, $d_l = (-1)^l$ in the odd-stacked case and $d_l = 1$ in the even-stacked case.

Similarly, for Q odd we have $\mathbf{T}_D^{(h,f)} = \mathbf{0}$ if and only if

$$\begin{aligned} B_{l,n}(z) &\triangleq \sum_{i=0}^{P-1} \left[R_n(zW_{2P}^{-i}) E_{-n-\alpha-2lN}(-\sigma zW_{2P}^i) \right. \\ &\quad \left. + R_n(-zW_{2P}^i) E_{-n-\alpha-2lN}(\sigma zW_{2P}^i) \right] \\ &= 0 \end{aligned} \quad (23)$$

for $l = 0, 1, \dots, Q-1$, $n = 0, 1, \dots, M-1$, where $\sigma = -1$ in the odd-stacked case and $\sigma = 1$ in the even-stacked case. For Q even, $\mathbf{T}_D^{(h,f)} = \mathbf{0}$ if and only if

$$\begin{aligned} d_l B_{l,n}(z) + (-1)^a z^{-P} d_{l+Q/2} B_{l+Q/2,n}(z) &= 0, \\ l = 0, 1, \dots, \frac{Q}{2} - 1; \quad n = 0, 1, \dots, M-1 \end{aligned}$$

with $a = 0$ in the odd-stacked case and $a = 1$ in the even-stacked case.

For integer oversampling ($Q = 1$), (22) and (23) simplify, respectively, to

$$\begin{aligned} A_{0,n}(z) &= \sum_{i=0}^{P-1} \left[R_n(zW_{2P}^i) E_n(zW_{2P}^i) \right. \\ &\quad \left. + R_n(-zW_{2P}^i) E_n(-zW_{2P}^i) \right] \\ &= \frac{2}{M}, \quad n = 0, 1, \dots, M-1 \\ B_{0,n}(z) &= \sum_{i=0}^{P-1} \left[R_n(zW_{2P}^{-i}) E_{-n-\alpha}(-\sigma zW_{2P}^i) \right. \\ &\quad \left. + R_n(-zW_{2P}^{-i}) E_{-n-\alpha}(\sigma zW_{2P}^i) \right] \\ &= 0, \quad n = 0, 1, \dots, M-1. \end{aligned}$$

E. PR Conditions in the Dual Polyphase Domain

Alternatively, $\mathbf{S}_D^{(h,f)} = 2\mathbf{I}$ if and only if

$$\sum_{i=0}^{2P-1} R'_{n-iM}(z)E'_{n-iM}(zW_Q^l) = \frac{Q}{N}\delta[l],$$

$$n = 0, 1, \dots, 2N - 1$$

and $\mathbf{T}_D^{(h,f)} = \mathbf{0}$ if and only if

$$R'_{n-iM}(z)E'_{-n-\alpha+iM}(zW_Q^l) + a_P z^Q R'_{n-iM-NQ}(z)E'_{-n-\alpha+iM+NQ}(zW_Q^l) = 0$$

for $l = 0, 1, \dots, Q - 1, i = 0, 1, \dots, P - 1$, and $n = 0, 1, \dots, 2N - 1$. Here, $a_P = 1$ in the odd-stacked case and $a_P = (-1)^P$ in the even-stacked case, and $E'_n(z) = \sum_{m=-\infty}^{\infty} h[m2N - n]z^{-m}$ and $R'_n(z) = \sum_{m=-\infty}^{\infty} f[m2N + n]z^{-m}$ ($n = 0, 1, \dots, 2N - 1$) are the *dual* polyphase components [8] of $h[n]$ and $f[n]$, respectively. For integer oversampling ($Q = 1$ or, equivalently, $N = PM$), these conditions simplify, respectively, to

$$\sum_{i=0}^{2P-1} R'_{n-iM}(z)E'_{n-iM}(z) = \frac{1}{PM},$$

$$n = 0, 1, \dots, 2PM - 1$$

$$R'_{n-iM}(z)E'_{-n-\alpha+iM}(z) + a_P z R'_{n-iM-PM}(z) \cdot E'_{-n-\alpha+iM+PM}(z) = 0 \quad (24)$$

for $i = 0, 1, \dots, P - 1$ and $n = 0, 1, \dots, 2PM - 1$.

F. Paraunitarity Conditions

Paraunitary FB's, i.e., PR FB's with $f_k[n] = h_k^*[-n]$, have a number of desirable properties [24]. It can be shown that a CMFB (even-stacked or odd-stacked, oversampled or critically sampled) is paraunitary if and only if it satisfies PR and its prototypes are related as $f[n] = h^*[-n]$. This implies that the two PR conditions of Theorem 2, with $f[n]$ replaced by $h^*[-n]$, are necessary and sufficient for paraunitarity. Thus, an odd- or even-stacked CMFB is paraunitary if and only if

$$\mathbf{S}_D^{(h,\tilde{h})} = 2\mathbf{I}$$

$$\mathbf{T}_D^{(h,\tilde{h})} = \mathbf{0} \quad \text{with } \tilde{h}[n] \triangleq h^*[-n]. \quad (25)$$

Furthermore, all PR conditions formulated in Subsections B–E become paraunitarity conditions if $f[n]$, $F(z)$, $R_n(z)$, and $R'_n(z)$ are replaced by $h^*[-n]$, $\tilde{H}(z)$, $\tilde{E}_n(z)$, and $\tilde{E}'_n(z)$, respectively. Here, for example, $\tilde{H}(z) = H^*(1/z^*)$ denotes the paraconjugate of $H(z)$.

For integer oversampling, a simple sufficient condition for the second paraunitarity property, $\mathbf{T}_D^{(h,\tilde{h})} = \mathbf{0}$, is stated in the following theorem. We note that this condition implies that $h[n]$ has linear phase.

Theorem 3: For odd-stacked CMFB's with arbitrary integer oversampling factor P , and for even-stacked CMFB's with odd P , the symmetry property

$$h^*[\alpha + (2l + 1)PM - n] = h[n] \quad \text{with some } l \in \mathbb{Z} \quad (26)$$

is sufficient for $\mathbf{T}_D^{(h,\tilde{h})} = \mathbf{0}$.

Proof: Using (21) with $f[n] = h^*[-n]$ shows that $\mathbf{T}_D^{(h,\tilde{h})} = \mathbf{0}$ if and only if

$$\sum_{i=-\infty}^{\infty} b_i h^*[-n + iPM] h[n + iPM + \alpha] = 0$$

where $b_i = (-1)^i$ for odd-stacked CMFB's and $b_i = (-1)^{iP}$ for even-stacked CMFB's. Now, for odd-stacked CMFB's with arbitrary integer P and for even-stacked CMFB's with odd P , we have $b_i = (-1)^i$ and thus we have to show that

$$A[n] \triangleq \sum_{i=-\infty}^{\infty} (-1)^i h^*[-n + iPM] h[n + iPM + \alpha] = 0. \quad (27)$$

Inserting (26), i.e., $h[n] = h^*[\alpha + (2l + 1)PM - n]$, and subsequently substituting $2l + 1 - i = j$, we obtain

$$A[n] = \sum_{i=-\infty}^{\infty} (-1)^i h[n + (2l + 1 - i)PM + \alpha] \cdot h^*[-n + (2l + 1 - i)PM]$$

$$= - \sum_{j=-\infty}^{\infty} (-1)^j h^*[-n + jPM] h[n + jPM + \alpha]$$

which when compared to (27) is seen to be $-A[n]$. Thus $A[n] = -A[n]$, which implies $A[n] = 0$. \square

It can be shown that if $h[n]$ is the prototype of a critically sampled, paraunitary CMFB with parameter α and, more specifically, $h[n]$ satisfies the symmetry property (26) with a certain l , then the scaled prototype $\sqrt{P}h[n]$ induces a paraunitary CMFB in the case of integer oversampling (with oversampling factor P as restricted in Theorem 3), provided that in the oversampled case α is replaced by $\alpha' = \alpha - (2l + 1)(P - 1)M$.

V. FRAME-THEORETIC ANALYSIS

Inserting the PR relation $\hat{x}[n] = x[n]$ in the CMFB input–output relation (5) or (9) yields

$$x[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^C \rangle f_{k,m}^C[n] \quad (28)$$

where $N' = N$, $h_{k,m}^C[n] = h_{k,m}^{C-o}[n]$, and $f_{k,m}^C[n] = f_{k,m}^{C-o}[n]$ [see (6)–(8)] for an odd-stacked CMFB and $N' = 2N$, $h_{k,m}^C[n] = h_{k,m}^{C-e}[n]$, and $f_{k,m}^C[n] = f_{k,m}^{C-e}[n]$ [see (10)–(12)] for an even-stacked CMFB. Thus, a CMFB with PR provides an expansion of the input signal $x[n]$ into a set of synthesis functions $f_{k,m}^C[n]$, and the expansion coefficients (subband signals) $\langle x, h_{k,m}^C \rangle$ are calculated using a set of analysis functions $h_{k,m}^C[n]$ [1]–[8].

In the oversampled case, the synthesis functions are “redundant.” The *theory of frames* [33] is a powerful mathematical tool for redundant signal expansions and, hence, for oversampled uniform FB's [1]–[8]. In particular, frame theory allows the calculation of the minimum-norm synthesis FB, the construction of paraunitary FB's from given nonparaunitary FB's, and an elegant characterization of important numerical properties of an FB [1], [2], [5], [8]. In this section, we will

show that frame theory also provides important results for oversampled CMFB's with PR.

A. Frame-Theoretic Fundamentals

A set of functions $h_{k,m}[n]$ with $k = 0, 1, \dots, N' - 1$ and $m \in \mathbb{Z}$ is called a *frame* for $l^2(\mathbb{Z})$ if

$$A\|x\|^2 \leq \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} |\langle x, h_{k,m} \rangle|^2 \leq B\|x\|^2 \quad \forall x[n] \in l^2(\mathbb{Z}) \quad (29)$$

with the *frame bounds* $A > 0$ and $B < \infty$ [33]. The frame bound ratio B/A characterizes the noise sensitivity of the signal expansion, and thus also of the corresponding FB [9], [33]. A frame is called *tight* if $B/A = 1$; this corresponds to the associated FB being paraunitary. The above frame condition (29) can also be written as $A\|x\|^2 \leq \langle \mathbf{S}x, x \rangle \leq B\|x\|^2$, where \mathbf{S} is the *frame operator* defined as

$$(\mathbf{S}x)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle h_{k,m}[n].$$

The frame bounds A and B are the infimum and supremum, respectively, of the eigenvalues of \mathbf{S} [33].

If the analysis functions $\{h_{k,m}[n]\}$ are a frame for $l^2(\mathbb{Z})$, the PR synthesis functions with minimum norm are given by [33], [38]

$$f_{k,m}[n] = (\mathbf{S}^{-1}h_{k,m})[n] \quad (30)$$

where \mathbf{S}^{-1} is the inverse of the frame operator. These synthesis functions $\{f_{k,m}[n]\}$ are also a frame (the “dual” frame), with frame bounds $A' = 1/B$ and $B' = 1/A$.

While the frame operator \mathbf{S} corresponds to a matrix of infinite size, it is shown in [1], [2], [8] that \mathbf{S} can be represented in the polyphase domain by the $M \times M$ matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\mathbf{E}(z)$, where $\mathbf{E}(z)$ is the FB's analysis polyphase matrix³ [24], [26] and $\tilde{\mathbf{E}}(z) = \mathbf{E}^H(1/z^*)$ (with superscript H denoting conjugate transposition) is its paraconjugate. Similarly, the inverse frame operator \mathbf{S}^{-1} is represented in the polyphase domain by the $M \times M$ matrix $\mathbf{S}^{-1}(z) = \mathbf{R}(z)\tilde{\mathbf{R}}(z)$ defined in terms of the synthesis polyphase matrix $\mathbf{R}(z)$. The eigenvalues of the frame operator \mathbf{S} equal the eigenvalues $\lambda_n(\theta)$ of the matrix $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\mathbf{E}(e^{j2\pi\theta})$. Hence, the frame bounds A and B are the infimum and supremum, respectively, of the eigenvalues $\lambda_n(\theta)$.

B. Decomposition of the CMFB Frame Operator

In the remainder of this section, we shall discuss oversampled, odd- or even-stacked CMFB's providing frame decompositions, i.e., $h_{k,m}^C[n]$ is a frame for $l^2(\mathbb{Z})$. The frame operator of a CMFB is given by

$$(\mathbf{S}_C x)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^C \rangle h_{k,m}^C[n]$$

³The analysis polyphase matrix is the $N \times M$ matrix defined as $[\mathbf{E}(z)]_{k,n} = \sum_{m=-\infty}^{\infty} h_k[mM - n]z^{-m}$, and the synthesis polyphase matrix is the $M \times N$ matrix defined as $[\mathbf{R}(z)]_{n,k} = \sum_{m=-\infty}^{\infty} f_k[mM + n]z^{-m}$.

with N' and $h_{k,m}^C[n]$ as in (28). Our frame-theoretic analysis of CMFB's will be based on the following fundamental decomposition of the CMFB frame operator. A similar decomposition has first been given for orthogonal continuous-time Wilson expansions [39], and has subsequently been extended to oversampled continuous-time Wilson expansions [40] and to discrete-time Wilson expansions [41].

Theorem 4: The frame operator of both odd- and even-stacked CMFB's can be decomposed as

$$\mathbf{S}_C = \frac{1}{2}(\mathbf{S}_D + \mathbf{T}_D). \quad (31)$$

Here, \mathbf{S}_D is the frame operator of a DFT FB with $2N$ channels and decimation factor M ,

$$(\mathbf{S}_D x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^D \rangle h_{k,m}^D[n]$$

and \mathbf{T}_D is given by

$$(\mathbf{T}_D x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \langle x, h_{k,m}^D \rangle \hat{h}_{k,m}^D[n]$$

with $h_{k,m}^D[n]$, ϕ_k , and c_m as defined in Theorem 1 and $\hat{h}_{k,m}^D[n] = h_{2N-k-1,m}^D[n]$ in the odd-stacked case and $\hat{h}_{k,m}^D[n] = h_{2N-k,m}^D[n]$ in the even-stacked case.

The proof of this theorem is similar to that of Theorem 1 and will therefore be omitted. We emphasize that \mathbf{S}_D is the frame operator of the DFT FB corresponding to the CMFB as defined in Section III. Furthermore, the operators \mathbf{S}_D and \mathbf{T}_D are obtained from the CMFB operators $\mathbf{S}_D^{(h,f)}$ in (14) and $\mathbf{T}_D^{(h,f)}$ in (15), respectively, by replacing $f[n]$ with $\tilde{h}[n] = h^*[-n]$:

$$\begin{aligned} \mathbf{S}_D &= \mathbf{S}_D^{(h,\tilde{h})} \\ \mathbf{T}_D &= \mathbf{T}_D^{(h,\tilde{h})} \quad \text{with } \tilde{h}[n] = h^*[-n]. \end{aligned}$$

C. Relations Between CMFB's and DFT FB's

Based on the above decomposition, we now show that, under a specific condition, the frame-theoretic properties of a CMFB are closely related to those of the corresponding DFT FB.

Theorem 5: Let $h[n]$ and $f[n]$ denote the prototypes in an odd-stacked CMFB with N channels and decimation factor M , or in an even-stacked CMFB with $2N$ channels and decimation factor $2M$. Let $h[n]$ be such that $\{h_{k,m}^D[n]\}$ is a frame⁴ for $l^2(\mathbb{Z})$, i.e.,

$$A_D\|x\|^2 \leq \langle \mathbf{S}_D x, x \rangle \leq B_D\|x\|^2 \quad \forall x[n] \in l^2(\mathbb{Z}). \quad (32)$$

Furthermore, let $h[n]$ be such that $\mathbf{T}_D = \mathbf{0}$ for the given stacking type. Then,

- i) the CMFB analysis functions $\{h_{k,m}^C[n]\}$ are a frame in $l^2(\mathbb{Z})$ with frame bounds $A_C = A_D/2$ and $B_C = B_D/2$,

⁴It can be shown that $\{h_{k,m}^{D-o}[n]\}$ is a frame if and only if $\{h_{k,m}^{D-e}[n]\}$ is a frame [8]. In other words, if a prototype $h[n]$ is such that the odd-stacked DFT FB constructed from it corresponds to a frame decomposition, the associated even-stacked DFT FB corresponds to a frame decomposition as well, and vice versa.

i.e.,

$$\frac{A_D}{2} \|x\|^2 \leq \langle \mathbf{S}_C x, x \rangle \leq \frac{B_D}{2} \|x\|^2 \quad \forall x[n] \in \ell^2(\mathbb{Z}) \quad (33)$$

- ii) for $f[n] = 2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$ with $\tilde{h}[n] = h^*[-n]$, the synthesis CMFB $\{f_k^C[n]\}$ constructed from $f[n]$ is the PR synthesis CMFB with minimum-norm filters.

Proof: Statement i) follows easily from Theorem 4: for $\mathbf{T}_D = \mathbf{0}$, (31) implies that the CMFB frame operator reduces to $\mathbf{S}_C = \frac{1}{2} \mathbf{S}_D$. Thus, using $\mathbf{S}_D = 2\mathbf{S}_C$ in (32) we obtain (33).

Statement ii) will be proved for the odd-stacked case; the proof for the even-stacked case is similar. For $\mathbf{T}_{D-o} = \mathbf{0}$, we have $\mathbf{S}_{C-o} = \frac{1}{2} \mathbf{S}_{D-o}$ and thus $\mathbf{S}_{C-o}^{-1} = 2\mathbf{S}_{D-o}^{-1}$. With (30), the minimum-norm PR synthesis functions are given by $f_{k,m}^{C-o}[n] = (\mathbf{S}_{C-o}^{-1} h_{k,m}^{C-o})[n] = 2(\mathbf{S}_{D-o}^{-1} h_{k,m}^{C-o})[n]$. With (1) and (6), it follows that the minimum-norm PR synthesis filters are obtained from the analysis filters as

$$\begin{aligned} f_k^{C-o}[n] &= f_{k,0}^{C-o}[n] \\ &= 2(\mathbf{S}_{D-o}^{-1} h_{k,0}^{C-o})[n] \\ &= 2\mathbf{S}_{D-o}^{-1} \left\{ \sqrt{2} h^*[-n] \cos \left[-\frac{(k+1/2)\pi}{N} n + \phi_k^o \right] \right\} \\ &= \sqrt{2} \left[\mathbf{S}_{D-o}^{-1} \left\{ \tilde{h}[n] W_{2N}^{-(k+1/2)n} \right\} e^{-j\phi_k^o} \right. \\ &\quad \left. + \mathbf{S}_{D-o}^{-1} \left\{ \tilde{h}[n] W_{2N}^{(k+1/2)n} \right\} e^{j\phi_k^o} \right]. \end{aligned}$$

Now with $\mathbf{S}_{D-o}^{-1} \left\{ \tilde{h}[n] W_{2N}^{\pm(k+1/2)n} \right\} = W_{2N}^{\pm(k+1/2)n} (\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$, this becomes further

$$\begin{aligned} f_k^{C-o}[n] &= \sqrt{2} \left[(\mathbf{S}_{D-e}^{-1} \tilde{h})[n] W_{2N}^{-(k+1/2)n} e^{-j\phi_k^o} \right. \\ &\quad \left. + (\mathbf{S}_{D-e}^{-1} \tilde{h})[n] W_{2N}^{(k+1/2)n} e^{j\phi_k^o} \right] \\ &= \sqrt{2} 2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n] \cos \left[\frac{(k+1/2)\pi}{N} n - \phi_k^o \right]. \end{aligned}$$

Comparing with (2) we see that $f[n] = 2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$, which completes the proof. \square

Note that for $\mathbf{T}_D = \mathbf{0}$, $2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$ is the minimum-norm synthesis prototype in *both* the odd- and even-stacked case. Assuming that $\mathbf{T}_D = \mathbf{0}$, the following conclusions can be derived from Theorem 5.

- Equation (31) implies $\mathbf{S}_C = \frac{1}{2} \mathbf{S}_D$, which means that the CMFB frame operator reduces to the frame operator of the corresponding DFT FB. Since $(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$ is the minimum-norm synthesis prototype of the corresponding DFT FB (even-stacked or odd-stacked) [8], the minimum-norm PR synthesis prototype in the CMFB, $f[n] = 2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$, is equal (up to a constant factor) to the minimum-norm PR synthesis prototype in the DFT FB. (The minimum-norm synthesis FB is of great practical interest since it maximizes the FB's noise immunity [8], [9].)
- If a DFT FB provides a frame decomposition, this is also true for the corresponding CMFB. The CMFB frame bounds $A_C = A_D/2$ and $B_C = B_D/2$ are trivially related to the frame bounds A_D and B_D of the corresponding

DFT FB. Since $B_C/A_C = B_D/A_D$, the CMFB inherits important numerical properties of the DFT FB even though it has just half the oversampling factor of the DFT FB. This is remarkable, since usually a decrease of redundancy leads to a deterioration of the numerical properties.

- In particular, if the DFT FB is paraunitary (which means $A_D = B_D$ or, equivalently, $\mathbf{S}_D = A_D \mathbf{I}$ [1]–[5], [8]), then the corresponding CMFB is paraunitary as well ($A_C = B_C$, $\mathbf{S}_C = (A_D/2) \mathbf{I}$).

All these results hinge on the condition $\mathbf{T}_D = \mathbf{0}$. As noted further above, $\mathbf{T}_D = \mathbf{0}$ is equivalent to the second PR property in Theorem 2, $\mathbf{T}_D^{(h,f)} = \mathbf{0}$, with $f[n]$ replaced by $h^*[-n]$. Hence, all conditions for $\mathbf{T}_D^{(h,f)} = \mathbf{0}$ formulated in Sections IV-B–E in the time, frequency, and polyphase domains can immediately be reformulated as conditions for $\mathbf{T}_D = \mathbf{0}$. For integer oversampling with oversampling factor P as constrained in Theorem 3, a simple sufficient condition for $\mathbf{T}_D = \mathbf{0}$ is given in (26); this condition implies that $h[n]$ has linear phase. Thus, PR (with linear phase filters in the case of an even-stacked CMFB) is achieved by choosing $h[n]$ according to (26) and using $f[n] = 2(\mathbf{S}_{D-e}^{-1} \tilde{h})[n]$; it can here be shown that $f[n]$ will have linear phase as well. In particular, the CMFB will be *paraunitary* if $\mathbf{S}_D = 2\mathbf{I}$ [cf. (25)].

D. Special Cases

In this section, we consider special cases where the expressions for \mathbf{S}_D and the calculation of the minimum-norm synthesis prototype using \mathbf{S}_{D-e}^{-1} simplify considerably.

1) *Time-Limited Prototype (Length $\leq 2N$):* If $h[n]$ has finite length $\leq 2N$, it follows from (18) (with $f[n]$ replaced by $h^*[-n]$) that the frame operator \mathbf{S}_D is a multiplication operator, i.e.,

$$(\mathbf{S}_D x)[n] = 2g_h[n] x[n], \quad (\mathbf{S}_D^{-1} x)[n] = \frac{x[n]}{2g_h[n]}$$

where the factor $g_h[n]$ is a periodized version of $|h[-n]|^2$,

$$g_h[n] \triangleq N \sum_{m=-\infty}^{\infty} |h[-n + mM]|^2. \quad (34)$$

For $\mathbf{T}_D = \mathbf{0}$, the frame bounds are given (independently of the stacking type) by $A_C = \min_{n=0,1,\dots,M-1} g_h[n]$ and $B_C = \max_{n=0,1,\dots,M-1} g_h[n]$; for $g_h[n] \equiv A_C$ the CMFB is paraunitary with frame bound A_C .

2) *Band-Limited Prototype (Bandwidth $\leq 1/M$):* If $h[n]$ is band-limited to a frequency interval of length $\leq 1/M$, \mathbf{S}_D is a multiplication operator in the frequency domain, i.e.,

$$\begin{aligned} (\hat{\mathbf{S}}_D X)(e^{j2\pi\theta}) &= 2G_h(\theta) X(e^{j2\pi\theta}) \\ (\hat{\mathbf{S}}_D^{-1} X)(e^{j2\pi\theta}) &= \frac{X(e^{j2\pi\theta})}{2G_h(\theta)}. \end{aligned}$$

Here, $\hat{\mathbf{S}}_D = \mathcal{F} \mathbf{S}_D \mathcal{F}^{-1}$ (with \mathcal{F} denoting the Fourier transform operator) and the factor $G_h(\theta)$ is a periodized version of $|H(e^{j2\pi\theta})|^2$,

$$G_h(\theta) \triangleq \frac{1}{2M} \sum_{k=0}^{2N-1} \left| H \left(e^{j2\pi(\theta - k/(2N))\theta_0} \right) \right|^2 \quad (35)$$

with $\theta_0 = 1/(4N)$ in the oddstacked case and $\theta_0 = 0$ in the even-stacked case. For $\mathbf{T}_D = \mathbf{0}$, the frame bounds are given by $A_C = \inf_{\theta \in [0,1]} G_h(\theta)$ and $B_C = \sup_{\theta \in [0,1]} G_h(\theta)$; they are independent of the stacking type since for the two stacking types $G_h(\theta)$ differs merely by a frequency shift. For $\mathbf{T}_D = \mathbf{0}$ and $G_h(\theta) \equiv A_C$, the CMFB is paraunitary with frame bound A_C .

3) *Integer Oversampling*: For integer oversampling, $N = PM$, \mathbf{S}_D is a multiplication operator in the polyphase domain, i.e., the polyphase components $S_n(z) = \sum_{m=-\infty}^{\infty} (\mathbf{S}_D x)[n + mM]z^{-m}$ are given by

$$S_n(z) = 2G_n(z)X_n(z)$$

with

$$G_n(z) = \frac{M}{2} \sum_{i=0}^{P-1} [\tilde{E}_n(zW_{2P}^{i+v})E_n(zW_{2P}^{i+v}) + \tilde{E}_n(-zW_{2P}^{i+v})E_n(-zW_{2P}^{i+v})] \quad (36)$$

where $v = 1/2$ in the odd-stacked case and $v = 0$ in the even-stacked case. For $\mathbf{T}_D = \mathbf{0}$, the frame bounds are $A_C = \inf_{\theta \in [0,1], n=0,1,\dots,M-1} G_n(e^{j2\pi\theta})$ and $B_C = \sup_{\theta \in [0,1], n=0,1,\dots,M-1} G_n(e^{j2\pi\theta})$; they are independent of the stacking type. Still assuming $\mathbf{T}_D = \mathbf{0}$, the CMFB is paraunitary with frame bound A_C if $G_n(z) \equiv A_C$ ($n = 0, 1, \dots, M-1$); this condition is the same for both stacking types. With (21), $\mathbf{T}_D = \mathbf{0}$ if and only if

$$\sum_{i=-\infty}^{\infty} b_i h^*[-n + iPM]h[n + iPM + \alpha] = 0$$

where $b_i = (-1)^i$ in the odd-stacked case and $b_i = (-1)^{iP}$ in the even-stacked case. We recall that a sufficient condition for $\mathbf{T}_D = \mathbf{0}$ has been stated in Theorem 3.

E. Construction of Paraunitary Prototypes

Paraunitary FB's correspond to tight frames for $l^2(\mathbb{Z})$ [2], [4], which implies $A = B$ and $\mathbf{S} = A\mathbf{I}$. For a given frame with frame operator \mathbf{S} , a tight frame can be constructed by applying the positive operator square root $\mathbf{S}^{-1/2}$ to each of the frame functions [33]. In [2] and [8], this method has been specialized to the design of paraunitary FB's from given nonparaunitary FB's. The application to CMFB's is considered in the next theorem.

Theorem 6: Let $h[n]$ be the analysis prototype of a (non-paraunitary) even- or odd-stacked CMFB providing a frame expansion with $\mathbf{T}_D = \mathbf{0}$. Then the CMFB with analysis prototype

$$h_p[n] = \sqrt{2}(\mathbf{S}_{D-e}^{-1/2}\tilde{h})^*[-n]$$

and synthesis prototype $f[n] = h_p^*[-n]$ is paraunitary with frame bound $A_C = 1$.

Proof: Applying $\mathbf{S}_C^{-1/2}$ to each of the frame functions $h_{k,m}^C[n]$ yields a tight frame with $A = 1$ [33]. Let us first consider the odd-stacked case and define the time-frequency shift operator $\mathbf{W}_{k,m}^o$ as $(\mathbf{W}_{k,m}^o x)[n] = x[n - mM]W_{2N}^{-(k+1/2)(n-mM)}$. With $\mathbf{S}_{C-o} = \frac{1}{2}\mathbf{S}_{D-o}$

and (6), we then get $h_{p_{k,m}}^{C-o}[n] = \sqrt{2}(\mathbf{S}_{D-o}^{-1/2}h_{k,m}^{C-o})[n] = (\mathbf{S}_{D-o}^{-1/2}\mathbf{W}_{k,m}^o\tilde{h})[n]e^{-j\phi_k^o} + (\mathbf{S}_{D-o}^{-1/2}\mathbf{W}_{2N-k-1,m}^o\tilde{h})[n]e^{j\phi_k^o}$. Now, using $\mathbf{S}_{D-o}^{-1/2}\mathbf{W}_{k,m}^o = \mathbf{W}_{k,m}^o\mathbf{S}_{D-e}^{-1/2}$ [8] we get $h_{p_{k,m}}^{C-o}[n] = 2(\mathbf{S}_{D-e}^{-1/2}\tilde{h})[n - mM]\cos[(k+1/2)\pi/N](mM - n) + \phi_k^o]$ and hence $\tilde{h}_p[n] = \sqrt{2}(\mathbf{S}_{D-e}^{-1/2}\tilde{h})[n]$, which completes the proof for the odd-stacked case. In the even-stacked case, the proof is similar using $\mathbf{S}_{D-e}^{-1/2}\mathbf{W}_{k,m}^e = \mathbf{W}_{k,m}^e\mathbf{S}_{D-e}^{-1/2}$ where $(\mathbf{W}_{k,m}^e x)[n] = x[n - mM]W_{2N}^{-k(n-mM)}$. \square

The above construction of $h_p[n]$ simplifies in the special situations considered in Subsection D. If $h[n]$ has length $\leq 2N$, then

$$h_p[n] = \sqrt{2}(\mathbf{S}_{D-e}^{-1/2}\tilde{h})^*[-n] = \frac{h[n]}{\sqrt{g_h[-n]}}$$

with $g_h[n]$ defined in (34). If $h[n]$ is band-limited with bandwidth $\leq 1/M$, then the Fourier transform of $h_p[n]$ is given by

$$H_p(e^{j2\pi\theta}) = \sqrt{2}(\hat{\mathbf{S}}_{D-e}^{-1/2}H^*)^*(e^{j2\pi\theta}) = \frac{H(e^{j2\pi\theta})}{\sqrt{G_h(\theta)}}$$

with $G_h(\theta)$ defined in (35). Finally, for integer oversampling the polyphase components of $h_p[n]$, $E_{p_n}(z) = \sum_{m=-\infty}^{\infty} h_p[mM - n]z^{-m}$, are obtained from the polyphase components $E_n(z)$ of $h[n]$ as

$$E_{p_n}(z) = \frac{E_n(z)}{B_n(z)}$$

where $B_n(z)$ is a ‘‘positive square root’’ of $G_n(z)$ [see (36)], i.e., $B_n(z)\hat{B}_n(z) = G_n(z)$ with $B_n(e^{j2\pi\theta}) > 0$ and $\hat{B}_n(z) = B_n(z)$.

VI. DESIGN AND EFFICIENT IMPLEMENTATION

In this section, we briefly discuss two design methods for paraunitary CMFB's with integer oversampling, and we propose efficient DCT/DST-based implementations of oversampled CMFB's.

A. CMFB Design

For a paraunitary CMFB, there is $f[n] = h^*[-n]$ and hence the CMFB design reduces to the design of the analysis prototype $h[n]$. We here assume $h[n]$ to be real-valued.

Design via Constrained Optimization: Our first design method (extending the methods presented, for example, in [24] and [27] for critical sampling) minimizes the stopband energy

$$C_h \triangleq \int_{\theta_s}^{1/2} |H(e^{j2\pi\theta})|^2 d\theta \quad (37)$$

subject to the paraunitarity side constraints $\mathbf{S}_D = 2\mathbf{I}$ and $\mathbf{T}_D = \mathbf{0}$. For integer oversampling with oversampling factor P as constrained in Theorem 3, the symmetry property (26) is sufficient for $\mathbf{T}_D = \mathbf{0}$, and it remains to satisfy $\mathbf{S}_D = 2\mathbf{I}$. This is a quadratic side constraint that is obtained by substituting $f[n] = h[-n]$ in the PR condition (20). Assuming an FIR prototype $h[n]$, the cost function (37) can be written as a

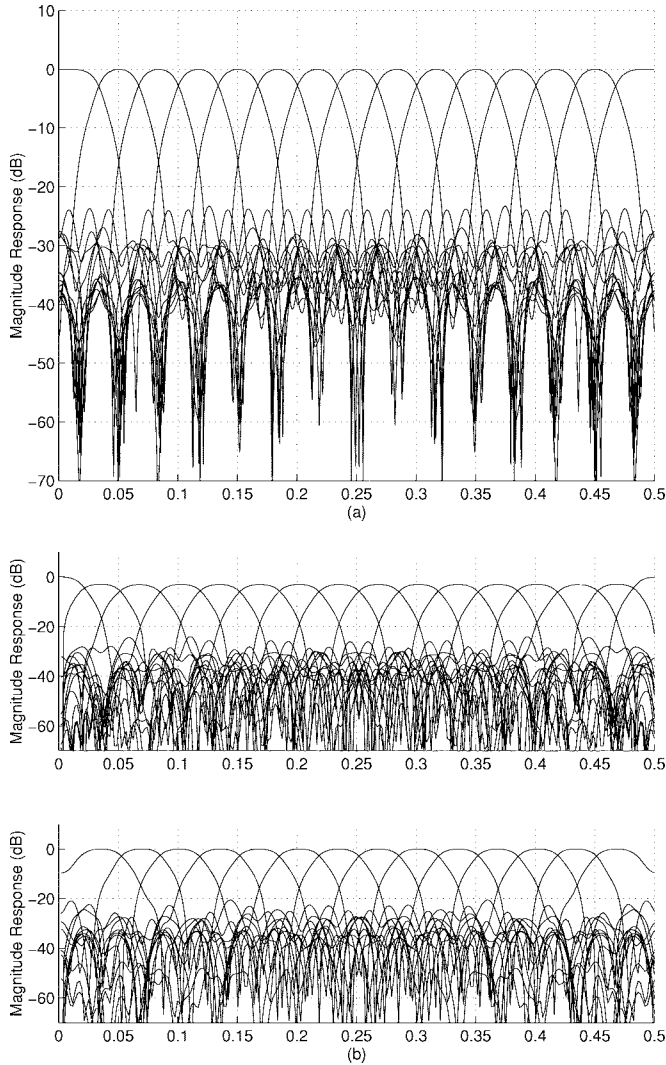


Fig. 3. Design of critically sampled paraunitary CMFB's via constrained minimization of stopband energy: (a) odd-stacked CMFB with $N = 15$ and $M = 15$; (b) even-stacked CMFB with $2N = 30$ and $2M = 30$ [upper plot: filters $H_k(z)$ ($k = 0, 1, \dots, N$); lower plot: filters $\bar{H}_k(z)$ ($k = 1, 2, \dots, N - 1$)].

quadratic form $C_h = \mathbf{h}^T \mathbf{P} \mathbf{h}$ where \mathbf{h} is the vector of filter coefficients and the superscript T denotes transposition [24]. Hence, the design amounts to minimizing a quadratic form subject to a quadratic side constraint. An efficient iterative design algorithm using a linearized side constraint is described in [8] and [36].

We now present design examples that were obtained using the above constrained optimization method, with nominal bandwidth $\theta_s = 1/15$ and prototype length $L = 136$. Fig. 3 shows critically sampled CMFB's with $N = M = 15$, while Fig. 4 shows oversampled CMFB's with $N = 15$, $M = 5$ (i.e., oversampling factor $P = 3$). In both cases, the odd- and even-stacked CMFB's use the same prototype $h[n]$ since the PR side constraint $\mathbf{S}_D = 2\mathbf{I}$ is independent of the stacking type. In the oversampled case, the filters are observed to have better stopband attenuation and frequency selectivity. This can be attributed to the reduced effective number of PR side constraints and, hence, the increased design freedom in the oversampled case [8].

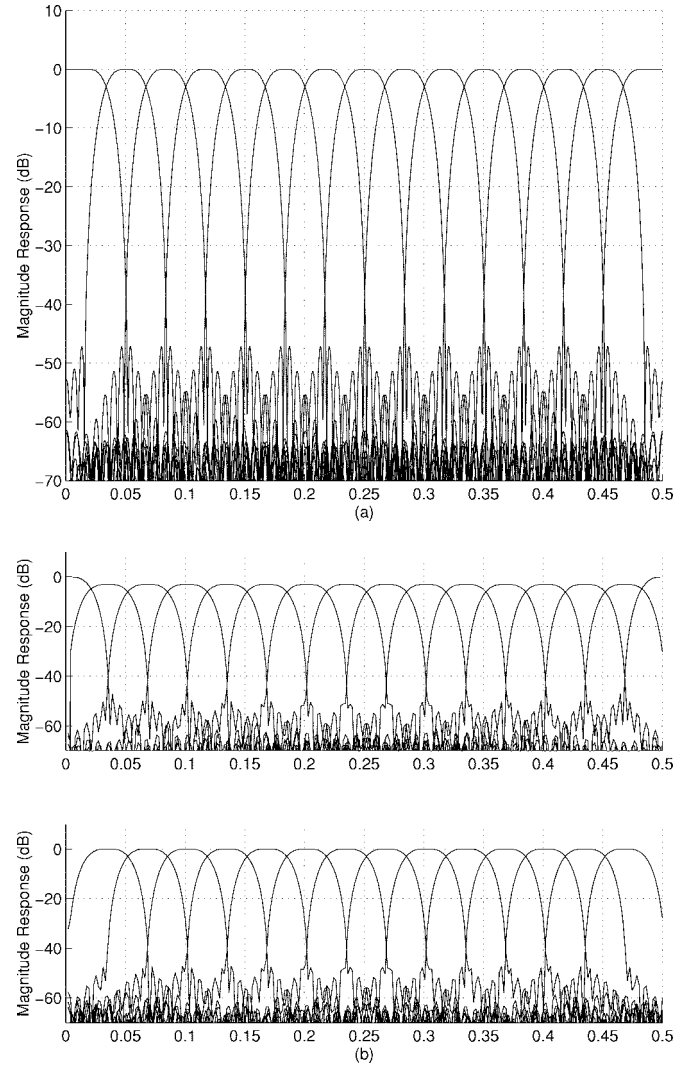


Fig. 4. Design of oversampled paraunitary CMFB's via constrained minimization of stopband energy: (a) odd-stacked CMFB with $N = 15$ and $M = 5$; (b) even-stacked CMFB with $2N = 30$ and $2M = 10$ [upper plot: filters $H_k(z)$ ($k = 0, 1, \dots, N$); lower plot: filters $\bar{H}_k(z)$ ($k = 1, 2, \dots, N - 1$)].

Lattice Design: The lattice design method for CMFB's was introduced in [20] for critically sampled, paraunitary, odd-stacked CMFB's. We next outline an extension of this method to odd- and even-stacked, paraunitary CMFB's with integer oversampling. As before, we assume that $\mathbf{T}_D = \mathbf{0}$. For integer oversampling, the paraunitarity condition $\mathbf{S}_D = 2\mathbf{I}$ can be formulated in the dual polyphase domain as [cf. (24) with $R'_n(z)$ replaced by $\tilde{E}'_n(z)$ and $i = l + 2j$ where $l = 0, 1$ and $j = 0, 1, \dots, P - 1$]

$$\sum_{j=0}^{P-1} \left[\tilde{E}'_{n-j2M}(z) E'_{n-j2M}(z) + \tilde{E}'_{n-j2M-M}(z) E'_{n-j2M-M}(z) \right] = \frac{1}{PM}, \quad n = 0, 1, \dots, 2PM - 1.$$

This condition can be satisfied by using the original lattice method [20] to design, for each n , P polyphase filter pairs $\{E'_{n-j2M}(z), E'_{n-j2M-M}(z)\}$ ($j = 0, 1, \dots, P - 1$) that

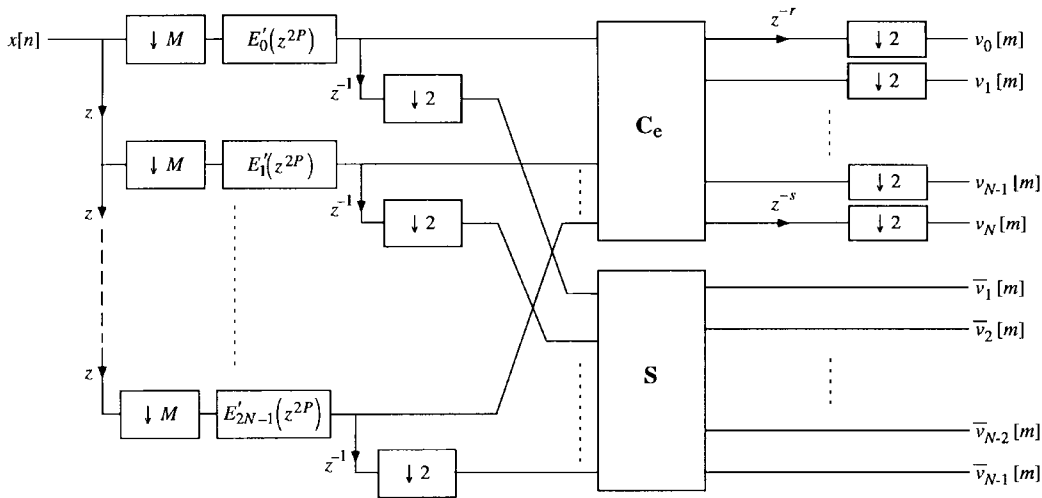


Fig. 5. Polyphase structure for even-stacked CMFB's (analysis part) with integer oversampling.

satisfy the power complementarity condition

$$\begin{aligned} & \tilde{E}'_{n-j2M}(z)E'_{n-j2M}(z) + \tilde{E}'_{n-j2M-M}(z)E'_{n-j2M-M}(z) \\ &= \frac{1}{P^2M}. \end{aligned}$$

Imposing the symmetry property (26) to achieve $\mathbf{T}_D = \mathbf{0}$ results in a symmetry relation of the polyphase components, which reduces the effective number of polyphase filters to be designed. The advantage of this lattice design method is that PR is inherently satisfied and need not be included as a side constraint [20].

B. Efficient Implementation of CMFB's

This Subsection proposes efficient DCT/DST-based implementations of oversampled CMFB's.

Even-Stacked CMFB's: We first extend the DCT-based implementation of critically sampled Lin-Vaidyanathan FB's [28] to arbitrary even-stacked CMFB's with integer oversampling. Let us define the $2N$ -dimensional analysis filter vector $\mathbf{h}(z) \triangleq [H_0(z) H_1(z) \cdots H_N(z) \bar{H}_1(z) \cdots \bar{H}_{N-1}(z)]^T$. Using the dual polyphase decomposition (cf. Section IV-E)

$$H(z) = \sum_{n=0}^{2N-1} E'_n(z^{2N})z^n \tag{38}$$

we obtain after simple manipulations

$$\mathbf{h}(z) = \mathbf{D}(z) \begin{pmatrix} \mathbf{C}_e \\ \mathbf{S} \end{pmatrix} \mathbf{G}(z^{2N}) \mathbf{e}(z). \tag{39}$$

Here, $\mathbf{D}(z)$ is a $2N \times 2N$ diagonal matrix defined as

$$\mathbf{D}(z) \triangleq \text{diag}\{z^{-rM}, \underbrace{1 \ 1 \ \cdots \ 1}_{N-1 \text{ times}}, z^{-sM}, \underbrace{z^{-M} z^{-M} \ \cdots \ z^{-M}}_{N-1 \text{ times}}\}$$

\mathbf{C}_e is an $(N + 1) \times 2N$ matrix defined as

$$\mathbf{C}_e = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ c_{1,0} & c_{1,1} & c_{1,2} & \cdots & c_{1,2N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{N-1,0} & c_{N-1,1} & c_{N-1,2} & \cdots & c_{N-1,2N-1} \\ 1 & -1 & 1 & \cdots & -1 \end{pmatrix}$$

with $c_{k,l} = \sqrt{2} \cos\left(\frac{k\pi}{N}l - \phi_k^e\right)$

\mathbf{S} is an $(N - 1) \times 2N$ matrix defined as

$$\mathbf{S} = \begin{pmatrix} s_{1,0} & s_{1,1} & \cdots & s_{1,2N-1} \\ s_{2,0} & s_{2,1} & \cdots & s_{2,2N-1} \\ \vdots & \vdots & \vdots & \vdots \\ s_{N-1,0} & s_{N-1,1} & \cdots & s_{N-1,2N-1} \end{pmatrix}$$

with $s_{k,l} = -\sqrt{2} \sin\left(\frac{k\pi}{N}l - \phi_k^e\right)$

$\mathbf{G}(z) = \text{diag}\{E'_n(z)\}_{n=0}^{2N-1}$, and $\mathbf{e}(z) = (1 \ z \ \cdots \ z^{2N-1})^T$. Straightforward manipulation of (39) (using noble identities [24]) leads to an efficient polyphase structure of the analysis part of the CMFB that is shown in Fig. 5. This implementation requires $2N$ polyphase filters plus the matrix multiplications by \mathbf{C}_e and \mathbf{S} ; these matrix multiplications can be implemented using fast DCT and DST algorithms [42]. A similarly efficient polyphase implementation can be derived for the CMFB's synthesis part.

For $\alpha = 0$ and $r = s = 0$ (Lin-Vaidyanathan CMFB), further simplifications result in the matrix multiplication by \mathbf{C}_e operating at a rate reduced by a factor of 2, thus extending the implementation given in [28] for the critically sampled case. (This extension is obtained by substituting $E'_n(z^{2P})$ ($n = 0, 1, \dots, 2PM - 1$) for the polyphase filters $G_n(z)$ in [28, Fig. 5].)

Odd-Stacked CMFB's: An efficient implementation of critically sampled odd-stacked CMFB's has been proposed,

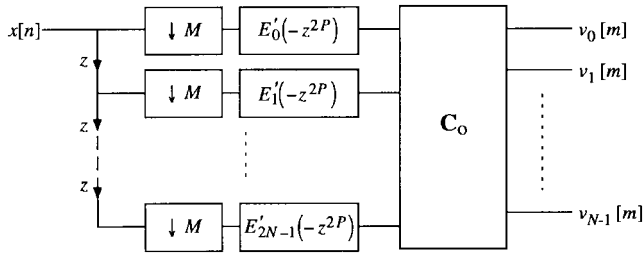


Fig. 6. Polyphase structure for odd-stacked CMFB's (analysis part) with integer oversampling.

e.g., in [17]–[20]. This implementation will now be extended to integer oversampling ($N = PM$). Using the dual polyphase decomposition (38) and proceeding similarly as in the even-stacked case, the analysis filter vector $\mathbf{h}(z) = [H_0(z) H_1(z) \cdots H_{N-1}(z)]^T$ of an odd-stacked CMFB can be written as

$$\mathbf{h}(z) = \mathbf{C}_o \mathbf{G}(-z^{2N}) \mathbf{e}(z)$$

where \mathbf{C}_o is an $N \times 2N$ matrix defined as $[\mathbf{C}_o]_{k,l} = \sqrt{2} \cos[(k + 1/2)\pi/N]l - \phi_k^o$ and $\mathbf{G}(z) = \text{diag}\{E'_n(z)\}_{n=0}^{2N-1}$. The use of noble identities and other manipulations yield the efficient polyphase structure shown in Fig. 6. This implementation requires $2N$ polyphase filters plus the matrix multiplication by \mathbf{C}_o ; this matrix multiplication can be implemented using fast DCT algorithms [42]. Again, a similarly efficient polyphase implementation can be derived for the CMFB's synthesis part.

General Oversampling: The particularly efficient polyphase structures proposed above can be used only in the case of integer oversampling. For general oversampling, CMFB's still allow a fairly efficient DCT/DST-based implementation if FIR prototypes are used. For example, the subband signals in an oversampled, odd-stacked CMFB can be written as

$$\begin{aligned} v_k[m] &= \langle x, h_{k,m}^{C-o} \rangle \\ &= \sqrt{2} \sum_{n=0}^{L-1} x[mM - n] h[n] \cos \left[\frac{(k + 1/2)\pi}{N} n + \phi_k^o \right] \end{aligned}$$

where L is the length of the FIR analysis prototype $h[n]$. Hence, the subband signals can be efficiently calculated by applying a fast DCT algorithm to $x[n]$ windowed by $h[n]$ about the time point $n = mM$. A DCT-based implementation exists also for the synthesis part. Furthermore, a similar DCT- and DST-based implementation can be developed for even-stacked CMFB's with arbitrary oversampling. The blockwise operation of the overall algorithm is described in [13].

VII. CONCLUSION

We introduced and studied two classes of *oversampled* cosine modulated filter banks (CMFB's), thus extending a classification of CMFB's recently proposed by Gopinath for critical sampling. The class of oversampled *even-stacked* CMFB's is especially attractive as it allows both perfect

reconstruction (PR) and linear phase filters in all channels. The CMFB recently introduced for critical sampling by Lin and Vaidyanathan has been extended to the oversampled case and shown to be a special case of even-stacked CMFB's.

We derived PR conditions for oversampled odd- and even-stacked CMFB's and demonstrated that, concerning both PR and frame-theoretic properties, CMFB's are closely related to DFT filter banks of the same stacking type and with twice the oversampling factor. In particular, it has been shown that any oversampled PR CMFB corresponds to an oversampled PR DFT filter bank, and that under an additional condition the frame bound ratio of a CMFB equals that of the corresponding DFT filter bank. Finally, we briefly discussed the design and efficient DCT/DST-based implementation of oversampled CMFB's.

APPENDIX

PROOF OF THEOREM 1

We first consider the odd-stacked case. Inserting (8) and the analogous expression for $f_{k,m}^{C-o}[n]$ in (5), we get

$$\begin{aligned} \hat{x}_{C-o}[n] &= \frac{1}{2} \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{D-o} e^{-j\phi_k^o} + h_{2N-k-1,m}^{D-o} e^{j\phi_k^o} \rangle \\ &\quad \cdot \left[f_{k,m}^{D-o}[n] e^{-j\phi_k^o} + f_{2N-k-1,m}^{D-o}[n] e^{j\phi_k^o} \right] \\ &= \frac{1}{2} \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left[\langle x, h_{k,m}^{D-o} \rangle f_{k,m}^{D-o}[n] \right. \\ &\quad + \langle x, h_{2N-k-1,m}^{D-o} \rangle f_{2N-k-1,m}^{D-o}[n] \\ &\quad + \langle x, h_{k,m}^{D-o} \rangle f_{2N-k-1,m}^{D-o}[n] e^{j2\phi_k^o} \\ &\quad \left. + \langle x, h_{2N-k-1,m}^{D-o} \rangle f_{k,m}^{D-o}[n] e^{-j2\phi_k^o} \right] \\ &= \frac{1}{2} \left[\sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{D-o} \rangle f_{k,m}^{D-o}[n] \right. \\ &\quad \left. + \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k^o} \langle x, h_{k,m}^{D-o} \rangle f_{2N-k-1,m}^{D-o}[n] \right] \end{aligned}$$

which is (13). Note that in the last step we have used $e^{-j2\phi_{2N-k-1}^o} = e^{j2\phi_k^o}$.

Next, we consider the even-stacked case. Inserting (12) and the analogous expression for $f_{k,m}^{C-e}[n]$ in (9) yields after manipulations similar to those in the odd-stacked case above

$$\begin{aligned} \hat{x}_{C-e}[n] &= \frac{1}{2} \sum_{k=1}^{N-1} \sum_{m=-\infty}^{\infty} \left[\langle x, h_{k,m}^{D-e} \rangle f_{k,m}^{D-e}[n] \right. \\ &\quad + \langle x, h_{2N-k,m}^{D-e} \rangle f_{2N-k,m}^{D-e}[n] \\ &\quad + (-1)^m \langle x, h_{k,m}^{D-e} \rangle f_{2N-k,m}^{D-e}[n] e^{j2\phi_k^e} \\ &\quad \left. + (-1)^m \langle x, h_{2N-k,m}^{D-e} \rangle f_{k,m}^{D-e}[n] e^{-j2\phi_k^e} \right] \\ &\quad + \sum_{m=-\infty}^{\infty} \left[\langle x, h_{0,2m-r}^{D-e} \rangle f_{0,2m-r}^{D-e}[n] \right. \\ &\quad \left. + \langle x, h_{N,2m-s}^{D-e} \rangle f_{N,2m-s}^{D-e}[n] \right]. \end{aligned}$$

Using $e^{-j2\phi_{2N-k}^c} = e^{j2\phi_k^c}$, it follows that

$$\begin{aligned} \hat{x}_{C-e}[n] = & \frac{1}{2} \sum_{\substack{k=1 \\ k \neq N}}^{2N-1} \sum_{m=-\infty}^{\infty} [\langle x, h_{k,m}^{D-e} \rangle f_{k,m}^{D-e}[n] \\ & + (-1)^m e^{j2\phi_k^c} \langle x, h_{k,m}^{D-e} \rangle f_{2N-k,m}^{D-e}[n]] \\ & + \sum_{m=-\infty}^{\infty} [\langle x, h_{0,2m-r}^{D-e} \rangle f_{0,2m-r}^{D-e}[n] \\ & + \langle x, h_{N,2m-s}^{D-e} \rangle f_{N,2m-s}^{D-e}[n]]. \end{aligned}$$

Finally, with $e^{j2\phi_0^c} = (-1)^r$, $e^{j2\phi_N^c} = (-1)^{\alpha+r}$, and $s = r$ for α even and $s = 1 - r$ for α odd, we obtain

$$\begin{aligned} \hat{x}_{C-e}[n] = & \frac{1}{2} \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{D-e} \rangle f_{k,m}^{D-e}[n] \\ & + \frac{1}{2} \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k^c} (-1)^m \\ & \cdot \langle x, h_{k,m}^{D-e} \rangle f_{2N-k,m}^{D-e}[n] \end{aligned}$$

which is (13).

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