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Oversampled modulated filter banks

Helmut Bölcskei and Franz Hlawatsch

ABSTRACT – Oversampled filter banks (FBs) offer increased design freedom and noise immunity as compared to critically sampled FBs. Since these advantages come at the cost of greater computational complexity, oversampled FBs allowing an efficient implementation are of particular interest. In this chapter, we discuss oversampled DFT FBs and oversampled cosine modulated FBs (CMFBs) which allow efficient FFT- or DCT/DST-based implementations. We provide conditions for perfect reconstruction and a frame-theoretic analysis. We show that, concerning both perfect reconstruction properties and frame-theoretic properties, oversampled cosine modulated filter banks are closely related to DFT filter banks with twice the oversampling factor.

9.1 Introduction and outline

Oversampled filter banks (FBs) have recently been found to be attractive due to their increased design freedom and improved noise immunity as compared to critically sampled FBs [BHF96b, CV, BHF96a, BHF96c, Jan95a, Vel93]. The *increased design freedom* corresponds to the nonuniqueness of the synthesis FB satisfying perfect reconstruction (PR) for a given oversampled analysis FB [BHF96b, BH97b]. The *improved noise immunity* corresponds to the fact that oversampled FBs tend to have better frame bounds [BHF96b, BH97b]. Furthermore, oversampled FBs permit the application of *noise shaping techniques* by which considerable noise reduction can be achieved [BH97b]. This makes oversampled FBs interesting for source coding applications with low-resolution quantizers in the subbands. The benefits obtained from using low-resolution quantizers at the cost of increased sample rate are indicated by the popular sigma-delta techniques [Gra87].

These advantages of oversampling come at the cost of increased computational complexity caused by the need to process more subband signal samples per unit of time. Therefore, oversampled FBs allowing efficient implementations are of particular interest. Oversampled *modulated* FBs such as DFT FBs (also known as complex modulated FBs) [CR83, BHF96c, Cve95b, BHF96b, CV] and cosine modulated FBs (CMFBs)[BH96c, BH97a] allow efficient FFT- or DCT/DST-based implementations [CR83, BH96b]. Here, CMFBs are advantageous as their subband signals are real-valued if the input signal and the analysis prototype are real-valued.

In this chapter, we discuss oversampled DFT FBs and CMFBs using results from the theory of frames [DS52, HW89, Dau92]. The application of frame theory is based on the close relations (or even equivalences) between modulated FBs on the one hand and Gabor expansions (Weyl–Heisenberg frames) [Bas80b, Jan81, WR90, Jan95b, DLL95] and Wilson expansions [DJJ91, FGW92, Aus94, BFGH96, BFGH97] on the other hand.

This chapter is organized as follows. In Section 2, we discuss oversampled uniform FBs and their relation to frame theory, thereby establishing a basis for our study of modulated FBs in subsequent sections. For a given oversampled analysis FB, we parameterize all synthesis FBs providing PR. We discuss a relation between the FB polyphase matrices and the frame operator, and we find conditions for a FB to provide a frame expansion.

Section 3 considers oversampled DFT FBs and their relation to Weyl-Heisenberg frames and Gabor expansions. We discuss PR conditions and frame-theoretic properties, and we show that the theory of DFT FBs simplifies considerably in the case of integer oversampling.

In Section 4, extending a recent classification of critically sampled CMFBs [Gop96], we consider two classes of oversampled CMFBs. In particular, the class of even-stacked CMFBs allows both PR/paraunitarity and linear phase filters in all channels. We finally show that CMFBs are closely related to PR DFT FBs with twice the oversampling factor.

9.2 Oversampled filter banks and frames

In this section we discuss (uniform) FBs in general, thereby establishing a theoretical basis for our study of modulated FBs in Sections 3 and 4. We extend the polyphase approach proposed in [Vet87, Vai87, Vai93, VK95] for critically sampled (maximally decimated) FBs to the oversampled case [BHF96a, BHF96b, CV, BHF96c, Bön]. We furthermore introduce *uniform filter bank frames* and establish their relation to FBs. Our discussion emphasizes PR and frame-theoretic properties of uniform FBs.

9.2.1 Uniform filter banks

We consider a FB with N channels (or subbands) and subsampling by the integer factor M in each channel, as depicted in Fig. 9.2.1. The FB is as-

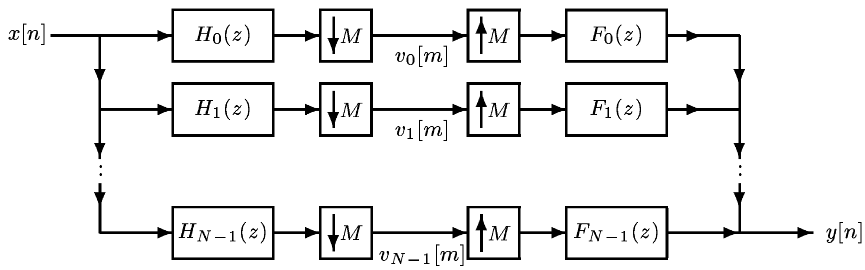

FIGURE 9.2.1. N-channel uniform FB.

sumed to have PR with zero delay[1], so that $y[n] = x[n]$ where $x[n]$ and $y[n]$ denote the input and reconstructed signal[2], respectively. The impulse responses of the analysis and synthesis filters are respectively $h_k[n]$ and $f_k[n]$ ($k = 0, 1, ..., N-1$), with corresponding transfer functions (z-transforms[3]) $H_k(z)$ and $F_k(z)$. The subband signals are given by

$$v_k[m] = \sum_{n=-\infty}^{\infty} x[n]\, h_k[mM - n]\,, \qquad k = 0, 1, ..., N-1\,, \tag{9.2.1}$$

and the reconstructed signal is

$$y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} v_k[m]\, f_k[n - mM]\,. \tag{9.2.2}$$

In a *critically sampled* (or *maximally decimated*) FB we have $N = M$ and thus the subband signals $v_k[m]$ contain exactly as many samples (per unit of time) as the input signal $x[n]$. In the *oversampled* case $N > M$, the subband signals are redundant in that they contain more samples (per unit of time) than the input signal $x[n]$. Finally, the *undersampled* case $N < M$ excludes PR.

The *polyphase decomposition* of the analysis filters $H_k(z)$ reads

$$H_k(z) = \sum_{n=0}^{M-1} z^n\, E_{k,n}(z^M)\,, \qquad k = 0, 1, ..., N-1\,,$$

[1]We note that our theory can easily be extended to nonzero delay.

[2]Signals are usually assumed to be in $l^2(\mathbb{Z})$, the space of square-summable functions $x[n]$, i.e., $\|x\|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$, with inner product $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$ where * stands for complex conjugation.

[3]For example, $H_k(z) = \sum_{n=-\infty}^{\infty} h_k[n]\, z^{-n}$.

where

$$E_{k,n}(z) = \sum_{m=-\infty}^{\infty} h_k[mM-n]\, z^{-m}, \qquad k=0,1,...,N-1, \;\; n=0,1,...,M-1$$

is the nth polyphase component of the kth analysis filter $H_k(z)$. The $N \times M$ *analysis polyphase matrix* $\mathbf{E}(z)$ is defined as $[\mathbf{E}(z)]_{k,n} = E_{k,n}(z)$. The synthesis filters $F_k(z)$ can be similarly decomposed,

$$F_k(z) = \sum_{n=0}^{M-1} z^{-n}\, R_{k,n}(z^M)\,, \qquad k=0,1,...,N-1\,,$$

with the synthesis polyphase components

$$R_{k,n}(z) = \sum_{m=-\infty}^{\infty} f_k[mM+n]\, z^{-m}, \qquad k=0,1,...,N-1, \;\; n=0,1,...,M-1\,.$$

The $M \times N$ *synthesis polyphase matrix* $\mathbf{R}(z)$ is defined as $[\mathbf{R}(z)]_{n,k} = R_{k,n}(z)$.

9.2.2 Uniform filter bank frames

FB analysis and synthesis can be interpreted as a signal expansion [VK95, CV94, Vai93, BHF95]. The subband signals in (9.2.1) can be written as the inner products

$$v_k[m] = \langle x, h_{k,m} \rangle \qquad \text{with} \;\; h_{k,m}[n] = h_k^*[mM-n]\,, \qquad k=0,1,...,N-1\,.$$

Furthermore, with (9.2.2) and the PR property $y[n]=x[n]$, we have

$$x[n] = y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n] \qquad \text{with} \;\; f_{k,m}[n] = f_k[n-mM]\,.$$

This shows that the FB corresponds to an expansion of the input signal $x[n]$ into the function set $\{f_{k,m}[n]\}$ with $k=0,1,...,N-1$ and $-\infty < m < \infty$. In general the set $\{f_{k,m}[n]\}$ is not orthogonal, so that the expansion coefficients, i.e., the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$, are obtained by projecting the signal $x[n]$ onto a "dual" set of functions $\{h_{k,m}[n]\}$. Critically sampled FBs correspond to orthogonal or biorthogonal signal expansions [VH92], whereas oversampled FBs correspond to redundant (overcomplete) expansions [VK95, BHF96b, CV, BHF96c, BHF96a].

The *theory of frames* [DS52, HW89, Dau92] is a powerful vehicle for the study of redundant signal expansions. We will call the set $\{h_{k,m}[n]\}$ with

$h_{k,m}[n] = h_k^*[mM - n]$ a *uniform filter bank frame* (UFBF) for $l^2(\mathbb{Z})$ if

$$A\|x\|^2 \;\leq\; \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\langle x, h_{k,m}\rangle|^2 \;\leq\; B\|x\|^2 \qquad \forall\; x[n] \in l^2(\mathbb{Z}) \quad (9.2.3)$$

with the *frame bounds* $A > 0$ and $B < \infty$. Note that the UFBF functions $h_{k,m}[n]$ are generated by uniformly time-shifting the N (conjugated and time-reversed) analysis filter impulse responses $h_k^*[-n]$. The above frame condition (9.2.3) can also be written as $A\|x\|^2 \leq \langle \boldsymbol{S}x, x\rangle \leq B\|x\|^2$, where $\boldsymbol{S}$ is the *frame operator* defined as

$$(\boldsymbol{S}x)[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}\rangle\, h_{k,m}[n]\,. \qquad (9.2.4)$$

If (9.2.3) is satisfied, then the frame operator is a positive definite, linear operator that maps $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z})$. The frame bounds A and B are the infimum and supremum, respectively, of the eigenvalues of $\boldsymbol{S}$ [Dau92]; they determine important numerical properties of the FB (see Section 9.2.6).

For analysis filters $h_k[n]$ such that $\{h_{k,m}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, a particular PR synthesis set (the PR synthesis set with minimum norm) is given by [HW89, Dau92]

$$f_{k,m}[n] = (\boldsymbol{S}^{-1} h_{k,m})[n]\,,$$

where $\boldsymbol{S}^{-1}$ is the inverse frame operator. If the analysis set $\{h_{k,m}[n]\}$ is a frame, then the synthesis set $f_{k,m}[n] = (\boldsymbol{S}^{-1} h_{k,m})[n]$ is also a frame (the "dual" frame), with frame bounds $A' = 1/B$ and $B' = 1/A$ (note that $B'/A' = B/A$) and frame operator $\boldsymbol{S}^{-1}$. This dual frame can be shown to be *again a UFBF* [BHF96a, BHF96b], i.e., it is obtained by uniformly time-shifting N functions $f_k[n]$,

$$f_{k,m}[n] = f_k[n - mM]\,.$$

The $f_k[n]$ are the synthesis filter impulse responses; they are derived from the analysis filter impulse responses $h_k[n]$ according to[4]

$$f_k[n] = (\boldsymbol{S}^{-1} \tilde{h}_k)[n] \qquad \text{with} \;\; \tilde{h}_k[n] = h_k^*[-n]\,. \qquad (9.2.5)$$

A frame is called *snug* if $B'/A' = B/A \approx 1$ and *tight* if $B'/A' = B/A = 1$. For a tight frame we have $\boldsymbol{S} = A\,\boldsymbol{I}$ and $\boldsymbol{S}^{-1} = A'\,\boldsymbol{I}$ (where $\boldsymbol{I}$ is the identity operator on $l^2(\mathbb{Z})$), and hence there is simply $f_k[n] = A'\,h_k^*[-n]$.

[4]There exists a basic difference between FB theory and frame theory: In FB theory one usually specifies the analysis FB $\{h_k[n]\}$ and computes the corresponding synthesis FB $\{f_k[n]\}$ for PR. In frame theory, however, the synthesis set $\{f_{k,m}[n]\}$ is often specified and then the corresponding analysis set $\{h_{k,m}[n]\}$ is calculated. Since this chapter is dealing with FBs, we shall here adopt the FB approach, i.e., assume knowledge of the analysis filters $h_k[n]$ and calculate the synthesis filters $f_k[n]$.

9.2.3 Frame operator and polyphase matrices

The connection between FBs and UFBFs is further expressed by the following representation of the UFBF frame operator in terms of the polyphase matrices of the corresponding FB. This result extends a similar result on continuous-time Weyl-Heisenberg frames [ZZ93b, ZZ95].

Theorem 9.2.1 *[BHF96a, BHF96b] Consider $u[n] = (\boldsymbol{S}x)[n]$ and $x[n] = (\boldsymbol{S}^{-1}u)[n]$, where $\boldsymbol{S}$ is the frame operator corresponding to a UFBF. Then, the polyphase components $U_n(z) = \sum_{m=-\infty}^{\infty} u[mM+n]\, z^{-m}$ of $U(z)$ and the polyphase components $X_{n'}(z) = \sum_{m=-\infty}^{\infty} x[mM+n']\, z^{-m}$ of $X(z)$ are related as*[5]

$$
\begin{aligned}
U_n(z) &= \sum_{n'=0}^{M-1} S_{n,n'}(z)\, X_{n'}(z) \qquad \text{with} \quad S_{n,n'}(z) = \sum_{k=0}^{N-1} \tilde{E}_{k,n}(z)\, E_{k,n'}(z) \\
X_{n'}(z) &= \sum_{n=0}^{M-1} S^{-1}_{n',n}(z)\, U_n(z) \qquad \text{with} \quad S^{-1}_{n',n}(z) = \sum_{k=0}^{N-1} R_{k,n'}(z)\, \tilde{R}_{k,n}(z)\,,
\end{aligned}
$$

or equivalently, the vectors $\mathbf{x}(z) = [X_0(z)\ X_1(z)\ \ldots\ X_{M-1}(z)]^T$ and $\mathbf{u}(z) = [U_0(z)\ U_1(z)\ \ldots\ U_{M-1}(z)]^T$ are related as[6]

$$
\begin{aligned}
\mathbf{u}(z) &= \mathbf{S}(z)\, \mathbf{x}(z) \qquad \text{with} \quad \mathbf{S}(z) = \tilde{\mathbf{E}}(z)\, \mathbf{E}(z)\,, \\
\mathbf{x}(z) &= \mathbf{S}^{-1}(z)\, \mathbf{u}(z) \qquad \text{with} \quad \mathbf{S}^{-1}(z) = \mathbf{R}(z)\, \tilde{\mathbf{R}}(z)\,.
\end{aligned}
$$

Thus, the frame operator $\boldsymbol{S}$ is expressed in the polyphase domain by the $M \times M$ *UFBF matrix* $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\, \mathbf{E}(z)$ defined in terms of the analysis polyphase matrix $\mathbf{E}(z)$. Similarly, the inverse frame operator $\boldsymbol{S}^{-1}$ is expressed by the $M \times M$ *inverse UFBF matrix* $\mathbf{S}^{-1}(z) = \mathbf{R}(z)\, \tilde{\mathbf{R}}(z)$ defined in terms of the synthesis polyphase matrix $\mathbf{R}(z)$.

Specializing to the unit circle ($z = e^{j2\pi\theta}$), it can be shown [BHF96a, BHF96b] that the positive definite $M \times M$ matrices

$$
\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\, \mathbf{E}(e^{j2\pi\theta}) \quad \text{and} \quad \mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\, \mathbf{R}^H(e^{j2\pi\theta})
$$

are *matrix representations* [NS82] of the frame operator $\boldsymbol{S}$ and the inverse frame operator $\boldsymbol{S}^{-1}$, respectively, with respect to the basis $\{b_{n,\theta}[n']\}$ of $l^2(\mathbb{Z})$ given by[7] $b_{n,\theta}[n'] = \sum_{m=-\infty}^{\infty} \delta[n'-n-mM]\, e^{j2\pi \frac{\theta}{M}(n'-n)}$ $(n =$

[5] $\tilde{R}_{k,n'}(z) = R^*_{k,n'}(1/z^*)$ denotes the paraconjugate of $R_{k,n'}(z)$ [Vai93].

[6] $\tilde{\mathbf{R}}(z) = \mathbf{R}^H(1/z^*)$ denotes the paraconjugate of the matrix $\mathbf{R}(z)$ [Vai93].

[7] Here, $\delta[n]$ denotes the unit sample ($\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$). The basis $\{b_{n,\theta}[n']\}$ induces the polyphase representation on the unit circle, $\langle x, b_{n,\theta}\rangle = X_n(e^{j2\pi\theta}) = \sum_{m=-\infty}^{\infty} x[mM+n]\, e^{-j2\pi\theta m}$. Equivalently, this is the *Zak transform* of $x[n]$ [Jan88, BHar]. Note that the functions $b_{n,\theta}[n']$ are not in $l^2(\mathbb{Z})$.

$0, 1, ..., M-1,\ 0 \leq \theta < 1$). A similar approach based on a matrix representation of the frame operator has been proposed in [RS95b] for the study of shift-invariant function systems. Furthermore, in Chapter 1 of this book Janssen presents an analysis of shift-invariant function systems based on different representations of the FB analysis and synthesis operators.

It is known [NS82] that the eigenvalues of an operator and those of its matrix representation are identical. Let $\lambda_n(\theta) > 0$ with $n = 0, 1, ..., M-1$ denote the eigenvalues of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$, defined by the eigenequation $\mathbf{S}(e^{j2\pi\theta})\,\mathbf{e}_n(\theta) = \lambda_n(\theta)\,\mathbf{e}_n(\theta)$ ($n = 0, 1, ..., M-1,\ 0 \leq \theta < 1$). Then, any eigenvalue $\lambda_n(\theta)$ is simultaneously an eigenvalue of $\boldsymbol{S}$. Conversely, any eigenvalue of $\boldsymbol{S}$ is simultaneously an eigenvalue of $\mathbf{S}(e^{j2\pi\theta})$. This means that the eigenanalysis of the frame operator $\boldsymbol{S}$ (a matrix of infinite size) is equivalent to that of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta})$ (an $M \times M$ matrix indexed by a real-valued parameter $\theta \in [0, 1)$). Similarly, the eigenvalues of the inverse frame operator $\boldsymbol{S}^{-1}$ are equal to those of the inverse UFBF matrix $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$, which will be denoted $\lambda'_n(\theta)$ in the following. Since $\mathbf{S}(e^{j2\pi\theta})$ and $\mathbf{S}^{-1}(e^{j2\pi\theta})$ are positive definite matrices, their eigenvalues are positive.

9.2.4 Perfect reconstruction property and design freedom

We shall now derive a PR condition for oversampled FBs and present a parameterization of all synthesis FBs providing PR for a given oversampled analysis FB. Transforming the FB input-output relation $y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]$ into the polyphase domain yields $\mathbf{y}(z) = \mathbf{R}(z)\,\mathbf{E}(z)\,\mathbf{x}(z)$. This gives the following result.

Theorem 9.2.2 *[BHF96a, BHF96b]* *A FB satisfies the PR property $y[n] = x[n]$ if and only if*

$$\mathbf{R}(z)\,\mathbf{E}(z) = \mathbf{I}_M, \tag{9.2.6}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. In the critically sampled case ($N = M$), $\mathbf{R}(z)$ is uniquely defined by (9.2.6) as [Vet87, Vai87, Vai93, VK95]

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z)\,,$$

where we assumed $\operatorname{rank}\{\mathbf{E}(z)\} = M$ almost everywhere so that $\mathbf{E}^{-1}(z)$ exists. In the oversampled case ($N > M$), $\mathbf{R}(z)$ is not uniquely determined: any solution of (9.2.6) can be written as [Kai80] (still assuming $\operatorname{rank}\{\mathbf{E}(z)\} = M$ almost everywhere)

$$\mathbf{R}(z) \;=\; \mathbf{R}^{(m)}(z) \;+\; \mathbf{P}(z)\left[\mathbf{I}_N - \mathbf{E}(z)\,\mathbf{R}^{(m)}(z)\right], \tag{9.2.7}$$

where $\mathbf{R}^{(m)}(z)$ *is the parapseudo-inverse of* $\mathbf{E}(z)$*, which is a particular solution of (9.2.6) defined as*

$$\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1} \tilde{\mathbf{E}}(z)\,, \tag{9.2.8}$$

and $\mathbf{P}(z)$ *is an* $M \times N$ *matrix with arbitrary elements* $[\mathbf{P}(z)]_{n,k}$ *(*$n = 0, 1, ..., M-1$*,* $k = 0, 1, ..., N-1$*) satisfying* $|[\mathbf{P}(e^{j2\pi\theta})]_{n,k}| < \infty$*.*

We shall now discuss the above theorem. For critical sampling ($N = M$), $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are square ($M \times M$) matrices and thus (9.2.6) has the unique solution $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$ [Vet87, Vai87, Vai93, VK95]. In the oversampled case ($N > M$), the matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are rectangular ($N \times M$ and $M \times N$, respectively) and thus the solution of (9.2.6) is not uniquely determined; in fact, any *left-inverse* of $\mathbf{E}(z)$ is a valid solution. Expression (9.2.7) is a parameterization of all left-inverses $\mathbf{R}(z)$ in terms of the MN entries $[\mathbf{P}(z)]_{n,k}$ that can be chosen arbitrarily [BHF96b, BH97b]. The nonuniqueness of the synthesis FB for given analysis FB in the oversampled case entails a (desirable) freedom of design that does not exist in the case of critical sampling. In either case, PR requires that $\mathbf{E}(z)$ has full rank.

The particular synthesis polyphase matrix given by the parapseudo-inverse $\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$ corresponds to the synthesis filter impulse responses $f_k^{(m)}[n]$ provided by frame theory via (9.2.5), i.e., $f_k[n] = f_k^{(m)}[n] = (\boldsymbol{S}^{-1}\tilde{h}_k)[n]$ with $\tilde{h}_k[n] = h_k^*[-n]$, or in other words, $\{f_{k,m}[n]\}$ is the UFBF that is dual to $\{h_{k,m}[n]\}$. This frame-theoretic solution minimizes $\sum_{k=0}^{N-1} \|f_k\|^2$ among all left-inverses or, equivalently, all PR synthesis FBs (hence the superscript $^{(m)}$). We note that the relation between pseudo-inverses and frames has been discussed in a more general context in [Chr95a].

The parameterization (9.2.7) can be reformulated in the time domain as

$$f_k[n] \;=\; f_k^{(m)}[n] \,+\, p_k[n] \,-\, \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle f_k^{(m)}, h_{l,m}\rangle \, p_{l,m}[n]\,, \tag{9.2.9}$$

where $p_k[n]$ is the impulse response of the filter with polyphase components $[\mathbf{P}(z)]_{n,k}$, i.e., $P_k(z) = \sum_{n=0}^{M-1} z^{-n}\,[\mathbf{P}(z^M)]_{n,k}$, and $p_{k,m}[n] = p_k[n - mM]$. In the z-transform domain, (9.2.7) can be reformulated as

$$F_k(z) \;=\; F_k^{(m)}(z) \,+\, P_k(z) \,-\, \frac{1}{M} \sum_{i=0}^{M-1} F_k^{(m)}(z\,W_M^i) \left[\sum_{l=0}^{N-1} H_l(z\,W_M^i)\,P_l(z)\right],$$

where $W_M = e^{-j2\pi/M}$. Thus, all PR synthesis filters are parameterized in terms of the N filters $p_k[n] \leftrightarrow P_k(z)$ that can be chosen arbitrarily.

Note that the frame-theoretic, minimum norm solution $f_k^{(m)}[n] \leftrightarrow F_k^{(m)}(z)$ (corresponding to the particular synthesis polyphase matrix $\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\,\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$) is reobtained for $p_k[n] \equiv 0$ or equivalently $P_k(z) \equiv 0$. In the following we shall mainly use this minimum norm synthesis FB, which will hereafter be denoted simply by $\{f_k[n]\}$ or $\mathbf{R}(z)$.

9.2.5 Frame property

Any synthesis FB of the form (9.2.7) satisfies PR, but it need not correspond to a frame (i.e., UFBF). The frame property is desirable as it guarantees a certain degree of numerical stability (see Subsection 9.2.6). We shall now provide conditions under which a FB corresponds to a frame (UFBF).

Theorem 9.2.3 *[BHF96a, BHF96b] An oversampled or critically sampled FB with BIBO stable*[8] *analysis filters $h_k[n]$ corresponds to a UFBF for $l^2(\mathbb{Z})$, i.e., the analysis set $\{h_{k,m}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, if and only if the analysis polyphase matrix $\mathbf{E}(z)$ has full rank on the unit circle, i.e.,*

$$\operatorname{rank}\{\mathbf{E}(e^{j2\pi\theta})\} = M \qquad \textit{for} \quad 0 \le \theta < 1.$$

Since FIR (i.e., finite-length) filters are inherently BIBO stable, an oversampled or critically sampled FB with FIR analysis filters corresponds to a UFBF for $l^2(\mathbb{Z})$ if and only if the analysis polyphase matrix $\mathbf{E}(z)$ has full rank on the unit circle. This condition for the special case of FIR filters has been found previously in [CV].

Alternatively, it can be shown that a FB corresponds to a UFBF for $l^2(\mathbb{Z})$ if $\mathbf{E}(e^{j2\pi\theta})$ has full rank for $0 \le \theta < 1$ and the $E_{k,n}(e^{j2\pi\theta})$ are continuous and bounded functions of θ [BHF96a, BHF96b]. Yet another condition is phrased in terms of the eigenvalues of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta})$: It can be shown [BHF96a, BHF96b] that an oversampled or critically sampled FB corresponds to a UFBF for $l^2(\mathbb{Z})$ if and only if the eigenvalues $\lambda_n(\theta)$ of $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$ satisfy[9]

$$\operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda_n(\theta) > 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda_n(\theta) < \infty.$$

If this condition is satisfied, then the (tightest possible) frame bounds are given by

$$A = \operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda_n(\theta)\,, \qquad B = \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda_n(\theta)\,.$$

[8]BIBO (bounded input bounded output) stability means that $h_k[n] \in l^1(\mathbb{Z})$, i.e., $\sum_{n=-\infty}^{\infty} |h_k[n]| < \infty$, for $k = 0, 1, \dots, N-1$.

[9]ess inf and ess sup denote the essential infimum and essential supremum, respectively.

Similarly, we have

$$A' = \operatorname*{ess\ inf}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda'_n(\theta)\,, \qquad B' = \operatorname*{ess\ sup}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \lambda'_n(\theta)\,,$$

where $\lambda'_n(\theta)$ are the eigenvalues of $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$. Note that in practice the frame bounds have to be estimated by sampling $\mathbf{S}(e^{j2\pi\theta})$ on the unit circle and performing an eigenanalysis of $\mathbf{S}(e^{j2\pi\frac{l}{L}})$ for $l = 0, 1, ..., L-1$. Some comments on the quality of this approximate calculation of frame bounds can be found for the FIR case in [Vel93].

The analysis UFBF $\{h_{k,m}[n]\}$ is *tight* if $A = B$ or equivalently $A' = B'$. In this case, $\boldsymbol{S} = A\boldsymbol{I}$ and $\boldsymbol{S}^{-1} = \frac{1}{A}\boldsymbol{I}$ [Dau92]. With (9.2.5), this implies that the frame-theoretic (minimum norm) PR synthesis FB is $f_k[n] = \frac{1}{A}\, h_k^*[-n]$ or $\mathbf{R}(z) = \frac{1}{A}\,\tilde{\mathbf{E}}(z)$. This is precisely the relation between the synthesis and analysis filters in a paraunitary[10] FB [Vai93]. In fact, a FB (oversampled or critically sampled) corresponds to a *tight* UFBF for $l^2(\mathbb{Z})$ if and only if it is *paraunitary*, i.e., $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z) \equiv A\,\mathbf{I}_M$; the frame bound is here $A = [\mathbf{S}(z)]_{n,n} = \sum_{k=0}^{N-1} \tilde{E}_{k,n}(z)\, E_{k,n}(z)$ [BHF96a, BHF96b, CV]. The equivalence of tight Weyl-Heisenberg frames (an important subclass of UFBFs) and paraunitary DFT FBs (cf. Section 9.3) has been noted in [BHF95, Cve95b]. For the special case of FIR oversampled FBs, this equivalence has been stated previously in [CV].

The following theorem describes a method for the derivation of a paraunitary FB from a given nonparaunitary FB. This is an adaptation of a method for the derivation of tight frames from nontight frames [Dau92].

Theorem 9.2.4 *[BHF96a, BHF96b] Let $\mathbf{E}(z)$ and $\mathbf{R}(z)$ be the polyphase matrices of a FB corresponding to a UFBF, and let the $M \times M$ matrix $\mathbf{U}(z)$ be an invertible matrix defined by $\mathbf{U}^2(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$ with $\tilde{\mathbf{U}}(z) = \mathbf{U}(z)$. Then the FB with analysis polyphase matrix*

$$\mathbf{E}^{(p)}(z) = \mathbf{E}(z)\,\mathbf{U}^{-1}(z)$$

is paraunitary with frame bound $A = 1$, i.e., $\mathbf{S}^{(p)}(z) = \tilde{\mathbf{E}}^{(p)}(z)\,\mathbf{E}^{(p)}(z) \equiv \mathbf{I}_M$. The corresponding synthesis polyphase matrix is given by $\mathbf{R}^{(p)}(z) = \tilde{\mathbf{E}}^{(p)}(z)$.

9.2.6 Frame bounds and noise sensitivity

The frame bounds A and B or, equivalently, $A' = 1/B$ and $B' = 1/A$ determine important numerical properties of the UFBF $\{h_{k,m}[n]\}$, and thus

[10] A FB is *paraunitary* [Vai93] if it satisfies PR and the analysis and synthesis filters satisfy $f_k[n] \propto h_k^*[-n]$; this implies $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z) \propto \mathbf{I}_M$.

also of the associated FB [Dau92]. Due to (9.2.3), the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$ of a FB corresponding to a UFBF satisfy

$$A\|x\|^2 \;\leq\; \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 \;\leq\; B\|x\|^2 \qquad \forall\, x[n] \in l^2(\mathbb{Z}) \qquad (9.2.10)$$

with $0 < A \leq B < \infty$. This double inequality generalizes the energy conservation equation $\sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 = \|x\|^2$ in orthogonal FBs [CV94], which is reobtained for $A = B = 1$. It also shows that the subband signals $v_k[m]$ are in $l^2(\mathbb{Z})$ if the input signal $x[n]$ is in $l^2(\mathbb{Z})$.

The frame bounds are related to the oversampling factor. It has been shown for the FIR case in [Vel93] and for the general case in [BHF96a, BHF96b] that the analysis filters of a FB corresponding to a UFBF satisfy

$$A \;\leq\; \frac{1}{M} \sum_{k=0}^{N-1} \|h_k\|^2 \;\leq\; B\,.$$

Assuming normalized analysis filters, $\|h_k\|^2 = 1$ for $k = 0, 1, ..., N-1$, this yields the following important relation between the frame bounds and the oversampling factor N/M,

$$A \;\leq\; \frac{N}{M} \;\leq\; B\,.$$

For a tight frame (paraunitary FB) where $A = B$, we obtain

$$A = B = \frac{N}{M} \qquad \text{for } \; \|h_k\|^2 = 1\,. \qquad (9.2.11)$$

Hence, the frame bounds of a tight frame (paraunitary FB) with normalized analysis filters equal the oversampling factor N/M.

To show that the frame bounds characterize important numerical properties of a FB, we consider the subband signals $v_k[m]$ corresponding to input signal $x[n]$ and reconstructed signal $y[n]$, and perturbed subband signals, $v_k'[m] = v_k[m] + \Delta v_k[m]$, corresponding to input signal $x'[n] = x[n] + \Delta x[n]$ and reconstructed signal $y'[n] = y[n] + \Delta y[n]$. (In practice, the perturbations $\Delta v_k[m]$ are usually caused by a quantization of the subband signals.) Using the PR property, $y[n] = x[n]$ and $y'[n] = x'[n]$, and the linearity of FB analysis and synthesis, it follows from (9.2.10) that the energy of the reconstruction error $\Delta y[n] = y'[n] - y[n] = x'[n] - x[n]$ is related to the frame bounds A, B and the total energy $\|\Delta v\|^2 = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\Delta v_k[m]|^2$ of the subband signal perturbations $\Delta v_k[m]$ as

$$A\,\|\Delta y\|^2 \;\leq\; \|\Delta v\|^2 \;\leq\; B\,\|\Delta y\|^2\,.$$

With $A' = 1/B$ and $B' = 1/A$, this implies

$$A' \leq \frac{\|\Delta y\|^2}{\|\Delta v\|^2} \leq B' .$$

Hence, for given subband perturbation energy $\|\Delta v\|^2$, the frame bounds A' and B' provide lower and upper bounds on the resulting reconstruction error energy $\|\Delta y\|^2$. The reconstruction error energy is minimized by making A' as small as possible and B' as close to A' as possible. Thus it is desirable to have $A' \approx B'$ or equivalently $A \approx B$, i.e., a snug frame.

For a tight frame (paraunitary FB), we obtain with (9.2.11)

$$\frac{\|\Delta y\|^2}{\|\Delta v\|^2} = \frac{1}{N/M} ,$$

i.e., *the reconstruction error energy is inversely proportional to the oversampling factor.* We note that a stochastic approach (assuming white uncorrelated noise added to the subband signals) leads to an analogous result [BH97b]. Thus, oversampled FBs feature better noise immunity than critically sampled FBs which, in turn, allows a coarser quantization of the subband signals. A similar result exists for oversampled A/D conversions, where the mean squared error is inversely proportional to the oversampling factor [CMH80, TV94]. The use of noise shaping techniques in oversampled FBs can achieve a further reduction of the reconstruction error [BH97b].

9.3 Oversampled DFT filter banks

DFT FBs (also known as *complex modulated FBs*/) [CR83] are an important class of uniform FBs that can be implemented very efficiently using FFT-based methods [CR83]. In this section, we specialize the results of Section 2 to DFT FBs. We apply the theory of *Weyl-Heisenberg frames* [DGM86, Dau92, BW94] to FIR and IIR, oversampled DFT FBs [Var79, SI87, CR83, Cve95b, BHF96c]. Although the connection between DFT FBs and signal expansions (short time Fourier transforms [Por80, NQ88] or Gabor expansions [Bas80b, Jan81, WR90, Jan95b, DLL95]) is well established [CR83, Vai93, PRV93], a frame-theoretic approach to DFT FBs has been proposed only recently [Cve95b, BHF95, BHF96c].

9.3.1 DFT filter banks and Weyl-Heisenberg sets

In the following we restrict our attention to *even-stacked* DFT FBs [CR83] (*odd-stacked* DFT FBs will be briefly considered in Subsection 9.3.5). The analysis and synthesis filters of an even-stacked DFT FB with N channels and decimation factor M are derived from a single analysis prototype filter $h[n] \leftrightarrow H(z)$ and a single synthesis prototype filter $f[n] \leftrightarrow F(z)$, respectively, as

$$h_k[n] = h[n]\, W_N^{-kn}\,, \qquad f_k[n] = f[n]\, W_N^{-kn}\,, \qquad k = 0, 1, ..., N-1$$

with $W_N = e^{-j2\pi/N}$, or equivalently as

$$H_k(z) = H(zW_N^k)\,, \qquad F_k(z) = F(zW_N^k)\,, \qquad k = 0, 1, ..., N-1\,.$$

The polyphase decomposition of the analysis prototype is given by

$$H(z) = \sum_{n=0}^{M-1} z^n\, E_n(z^M) \qquad \text{with} \qquad E_n(z) = \sum_{m=-\infty}^{\infty} h[mM-n]\, z^{-m}.$$

Note that furthermore $E_{k,n}(z) = W_N^{kn}\, E_n(z\, W_N^{Mk})$, so that the analysis polyphase matrix $\mathbf{E}(z)$ is fully determined by $E_n(z)$ ($n = 0, 1, ..., M-1$). Similarly, the polyphase decomposition of the synthesis prototype reads

$$F(z) = \sum_{n=0}^{M-1} z^{-n} R_n(z^M) \qquad \text{with} \qquad R_n(z) = \sum_{m=-\infty}^{\infty} f[mM+n]\, z^{-m},$$

and there is $R_{k,n}(z) = W_N^{-kn}\, R_n(z\, W_N^{Mk})$.

The input-output relation of the DFT FB is

$$y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]\,, \tag{9.3.1}$$

where the analysis and synthesis functions are the *Weyl-Heisenberg (WH) sets* [Dau92] generated by $h^*[-n]$ and $f[n]$, respectively,

$$h_{k,m}[n] = h^*[mM-n]\, W_N^{-k(n-mM)}\,, \qquad f_{k,m}[n] = f[n-mM]\, W_N^{-k(n-mM)}$$

with $k = 0, 1, ..., N-1$, $-\infty < m < \infty$.

9.3.2 Perfect reconstruction property and design freedom

If the PR property $y[n] = x[n]$ is satisfied, then (9.3.1) becomes

$$x[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]\,.$$

Hence, a DFT FB with PR provides an expansion of the input signal $x[n]$ into the WH set $f_{k,m}[n] = f[n - mM]\, W_N^{-k(n-mM)}$. This expansion is known as the (discrete-time) *Gabor expansion* [WR90, BHar, Jan94a]. Thus, PR DFT FBs and Gabor expansions are mathematically equivalent [PRV93, BHF95].

A DFT FB (oversampled or critically sampled) with analysis prototype $h[n]$ and synthesis prototype $f[n]$ yields PR if and only if [CR83, WR90, Jan94a]

$$N \sum_{m=-\infty}^{\infty} f[n - mM]\; h[mM - n + lN] \;=\; \delta[l]\,.$$

Equivalently, in the frequency domain the PR condition reads

$$\frac{1}{M} \sum_{m=0}^{N-1} F\Big(e^{j2\pi(\theta - \frac{m}{N})}\Big)\, H\Big(e^{j2\pi(\theta - \frac{m}{N} - \frac{l}{M})}\Big) \;=\; \delta[l]\,.$$

Finally, setting $\frac{N}{M} = \frac{P}{Q}$ where P, Q are relatively prime (i.e., $\gcd(P,Q)=1$ with $\gcd(P,Q)$ denoting the greatest common divisor of P and Q), the PR condition in the polyphase domain is [ZZ93b, Jan94a, Bas95]

$$\frac{M}{Q}\, \mathbf{R}_n(z)\, \mathbf{E}_n(z) = \mathbf{I}_Q\,, \qquad n = 0, 1, ..., M-1\,,$$

where the $Q \times P$ matrices $\mathbf{R}_n(z)$ and the $P \times Q$ matrices $\mathbf{E}_n(z)$ are defined as[11] $[\mathbf{R}_n(z)]_{k,l} = R_{n-kN}(zW_P^l)$ $(k = 0, 1, ..., Q-1,\; l = 0, 1, ..., P-1)$ and $[\mathbf{E}_n(z)]_{k,l} = E_{n-lN}(zW_P^k)$ $(k = 0, 1, ..., P-1,\; l = 0, 1, ..., Q-1)$.

From Subsection 9.2.4 we know that the synthesis FB yielding PR for a given oversampled analysis FB is not uniquely determined. In (9.2.9), all PR synthesis FBs were parameterized in terms of N filters $p_k[n] \leftrightarrow P_k(z)$. In the special case of a DFT analysis FB, the *minimum norm* synthesis FB $\{f_k^{(m)}[n]\}$ is always a DFT FB [BHF96c]; this follows immediately from the fact that the dual frame of a WH frame is again a WH frame [Dau92] (cf. Subsection 9.3.3). Thus we conclude that

$$f_k^{(m)}[n] = f^{(m)}[n]\, W_N^{-kn}\,,$$

where $f^{(m)}[n] = (\boldsymbol{S}^{-1}\tilde{h})[n]$ with $\tilde{h}[n] = h^*[-n]$ is the minimum norm synthesis prototype (cf. Subsection 9.3.3). In general, however, a PR synthesis FB for a given oversampled analysis DFT FB need not have DFT structure, i.e., the $f_k[n]$ need not be modulated versions of a single prototype

[11] Usually, the polyphase components are defined only for $n = 0, 1, ..., M-1$. However, using $X_{n+lM}(z) = z^l\, X_n(z)$, where $X_n(z) = \sum_{m=-\infty}^{\infty} x[mM + n]\, z^{-m}$, this definition can be extended to arbitrary $n \in \mathbb{Z}$.

$f[n]$. Li and Healy have shown in the context of WH frames [LH96] that

$$p_k[n] = p[n]\, W_N^{-kn}$$

(with arbitrary $p[n]$) is a sufficient condition for the synthesis FB to have DFT structure. With $p_k[n] = p[n]\, W_N^{-kn}$, (9.2.9) yields

$$\begin{aligned} f[n] &= f^{(m)}[n] + p[n] \\ &\quad - N \sum_{l=-\infty}^{\infty} f^{(m)}[n-lN] \left[\sum_{m=-\infty}^{\infty} h[mM-n+lN]\, p[n-mM] \right], \end{aligned} \tag{9.3.2}$$

which is a parameterization of the synthesis prototype $f[n]$ in terms of the single filter $p[n]$ that may be chosen arbitrarily. Note that the frame-theoretic, minimum norm prototype $f^{(m)}[n]$ is reobtained for $p[n] \equiv 0$.

9.3.3 Frame-theoretic properties

A WH set $\{h_{k,m}[n]\}$ that is a frame (cf. (9.2.3)) is called a *WH frame*. The dual frame can be shown to be again a WH frame [Dau92],

$$f_{k,m}[n] = f[n-mM]\, W_N^{-k(n-mM)},$$

with synthesis prototype $f[n]$ given by

$$f[n] = f^{(m)}[n] = (\boldsymbol{S}^{-1}\tilde{h})[n] \qquad \text{with } \tilde{h}[n] = h^*[-n]\,. \tag{9.3.3}$$

Here, $\boldsymbol{S}^{-1}$ is again the inverse frame operator (cf. (9.2.4)). Among all synthesis prototypes satisfying PR, (9.3.3) defines the synthesis prototype with minimum energy (norm) [Jan94a].

We next provide time, frequency, and polyphase domain expressions for the WH frame operator, and we formulate paraunitarity conditions in the various domains. The *Walnut representation* [Wal92] of the WH frame operator reads

$$(\boldsymbol{S}x)[n] = \sum_{l=-\infty}^{\infty} x[n-lN] \left[N \sum_{m=-\infty}^{\infty} h^*[-n+mM]\, h[-n+mM+lN] \right].$$

(The inverse frame operator can be represented in a similar manner by replacing $h[n]$ with $f^*[-n]$.) Using this representation, it is seen that a DFT FB is paraunitary, i.e., $\{h_{k,m}[n]\}$ is a tight frame for $l^2(\mathbb{Z})$ with frame bound A, or equivalently $\boldsymbol{S} = A\boldsymbol{I}$, if and only if

$$N \sum_{m=-\infty}^{\infty} h^*[-n+mM]\, h[-n+mM+lN] = A\,\delta[l]\,.$$

In the frequency domain, the WH frame operator can be expressed as

$$(\hat{\boldsymbol{S}}X)(e^{j2\pi\theta}) = \sum_{l=0}^{M-1} X\left(e^{j2\pi(\theta-\frac{l}{M})}\right)\left[\frac{1}{M}\sum_{m=0}^{N-1} H^*\left(e^{j2\pi(\theta-\frac{m}{N})}\right) H\left(e^{j2\pi(\theta-\frac{m}{N}-\frac{l}{M})}\right)\right],$$

where $\hat{\boldsymbol{S}} = \boldsymbol{\mathcal{F}}\boldsymbol{S}\boldsymbol{\mathcal{F}}^{-1}$ (with $\boldsymbol{\mathcal{F}}$ denoting the Fourier transform operator) is the frequency domain representation of $\boldsymbol{S}$. (The inverse frame operator can be represented in a similar manner by replacing $H(e^{j2\pi\theta})$ with $F^*(e^{j2\pi\theta})$.) Hence, the FB is paraunitary with frame bound A if and only if

$$\frac{1}{M}\sum_{m=0}^{N-1} H^*\left(e^{j2\pi(\theta-\frac{m}{N})}\right) H\left(e^{j2\pi(\theta-\frac{m}{N}-\frac{l}{M})}\right) = A\,\delta[l]\,.$$

In Subsection 9.2.3, it has been shown that the frame operator $\boldsymbol{S}$ of a general UFBF can be represented in the polyphase domain by the $M \times M$ UFBF matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$. For DFT FBs, the frame operator can alternatively be represented in terms of M (often smaller) matrices of size $Q \times Q$, where again $\frac{N}{M} = \frac{P}{Q}$ with $\gcd(P,Q) = 1$. As in Theorem 9.2.1, let $u[n] = (\boldsymbol{S}x)[n]$ and $x[n] = (\boldsymbol{S}^{-1}u)[n]$, and define the polyphase components $U_n(z) = \sum_{m=-\infty}^{\infty} u[mM+n]\,z^{-m}$ and $X_n(z) = \sum_{m=-\infty}^{\infty} x[mM+n]\,z^{-m}$. Then, the polyphase vectors $\mathbf{u}_n(z) = [U_n(z)\; U_{n-N}(z)\; \ldots\; U_{n-(Q-1)N}(z)]^T$ and $\mathbf{x}_n(z) = [X_n(z)\; X_{n-N}(z)\; \ldots\; X_{n-(Q-1)N}(z)]^T$ can be shown [ZZ93b, Bön] to be related as

$$\mathbf{u}_n(z) = \mathbf{S}_n(z)\,\mathbf{x}_n(z) \qquad \text{with} \quad \mathbf{S}_n(z) = \frac{M}{Q}\,\tilde{\mathbf{E}}_n(z)\,\mathbf{E}_n(z)\,,$$

$$\mathbf{x}_n(z) = \mathbf{S}_n^{-1}(z)\,\mathbf{u}_n(z) \qquad \text{with} \quad \mathbf{S}_n^{-1}(z) = \frac{M}{Q}\,\mathbf{R}_n(z)\,\tilde{\mathbf{R}}_n(z)$$

for $n = 0, 1, \ldots, M-1$. This representation of $\boldsymbol{S}$ in terms of M matrices $\mathbf{S}_n(z)$ of size $Q \times Q$ is known in WH frame theory as the *Zibulski-Zeevi representation* of the WH frame operator [ZZ93b, BHar]. In particular, the inversion of the frame operator—which, in the general UFBF case, requires the inversion of the $M \times M$ UFBF matrix $\mathbf{S}(z)$—here reduces to the inversion of M matrices of size $Q \times Q$. It can easily be seen that the DFT FB is paraunitary with frame bound A if and only if

$$\mathbf{S}_n(z) = A\,\mathbf{I}_Q \qquad \text{for} \quad n = 0, 1, \ldots, M-1.$$

9.3.4 Integer oversampling

For integer oversampled DFT FBs ($N = PM$ with $P \in \mathbb{N}$), it can be shown [BHF96c] that the UFBF matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$ is diagonal with diagonal elements $[\mathbf{S}(z)]_{nn} = \Lambda_n(z)$, where

$$\Lambda_n(z) \stackrel{\triangle}{=} M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r)\, E_n(zW_P^r)\,, \qquad n = 0, 1, .., M-1.$$

In this case the $Q \times Q$ matrices $\mathbf{S}_n(z)$ reduce to scalars since $Q = 1$ and furthermore $\mathbf{S}(z) = M\, \mathrm{diag}\{\mathbf{S}_n(z)\}_{n=0}^{M-1}$ or equivalently $[\mathbf{S}(z)]_{nn} = M\, \mathbf{S}_n(z)$.

The eigenvalues of $\mathbf{S}(e^{j2\pi\theta})$ follow from the frequency responses $E_n(e^{j2\pi\theta})$ of the analysis prototype's polyphase components according to

$$\lambda_n(\theta) = \Lambda_n(e^{j2\pi\theta}) = M \sum_{r=0}^{P-1} \left| E_n\left(e^{j2\pi(\theta - \frac{r}{P})}\right)\right|^2.$$

Hence, it follows from Subsection 9.2.5 that an integer oversampled DFT FB corresponds to a WH frame if and only if

$$\operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \Lambda_n(e^{j2\pi\theta}) > 0, \qquad \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \Lambda_n(e^{j2\pi\theta}) < \infty,$$

and that the (tightest possible) frame bounds are given by

$$A = \operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \Lambda_n(e^{j2\pi\theta})\,, \quad B = \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\dots,M-1} \Lambda_n(e^{j2\pi\theta})\,.$$

Note that the frequency responses $E_n(e^{j2\pi\theta})$ of the analysis prototype's polyphase components determine the frame bounds (important numerical properties) of the FB. An integer oversampled DFT FB is paraunitary with frame bound A if and only if

$$\Lambda_n(z) \equiv A \qquad \text{for } n = 0, 1, ..., M-1. \tag{9.3.4}$$

With (9.2.8) it follows that the polyphase components of the minimum norm synthesis prototype are given by

$$R_n(z) = \frac{\tilde{E}_n(z)}{\Lambda_n(z)}\,.$$

Thus, in the case of integer oversampling the synthesis prototype can be calculated in the polyphase domain by simple divisions and the matrix inversion in (9.2.8) is avoided.

According to Theorem 9.2.4, a paraunitary FB can be constructed by factoring the matrix $\mathbf{S}(z)$ of an arbitrary FB corresponding to a frame. For integer oversampled DFT FBs, this reduces to a factorization of polynomials (FIR case) or rational functions (IIR case) in z^{-1}. Let $E_n(z)$ be the analysis polyphase components of an integer oversampled DFT FB corresponding to a WH frame in $l^2(\mathbb{Z})$. Furthermore, let $U_n(z)$ be such that

$$U_n^2(z) = M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r)\, E_n(zW_P^r) \qquad \text{and} \qquad \tilde{U}_n(z) = U_n(z)\,. \tag{9.3.5}$$

Then, the DFT FB with analysis polyphase components

$$E_n^{(p)}(z) = \frac{E_n(z)}{U_n(z)}$$

is paraunitary with frame bound $A = 1$, i.e., $\tilde{\mathbf{E}}^{(p)}(z)\, \mathbf{E}^{(p)}(z) = \mathbf{I}_M$.

In the case of critical sampling ($P = 1$), we have

$$\Lambda_n(z) = M\, \tilde{E}_n(z)\, E_n(z)\,, \qquad \lambda_n(\theta) = \Lambda_n(e^{j2\pi\theta}) = M\, |E_n(e^{j2\pi\theta})|^2\,,$$

and the above relations simplify accordingly. In particular, (9.3.4) becomes

$$\tilde{E}_n(z)\, E_n(z) = \frac{A}{M} \qquad \text{for } n = 0, 1, ..., M-1\,,$$

or $|E_n(e^{j2\pi\theta})|^2 \equiv A/M$, which means that the polyphase filters $E_n(z)$ are allpass filters. Thus, the design of a critically sampled paraunitary DFT FB reduces to finding an arbitrary set of M allpass filters. Furthermore, (9.3.5) simplifies to $U_n^2(z) = M\, \tilde{E}_n(z)\, E_n(z)$.

For $P = 2$, a paraunitary DFT FB can be constructed by choosing polyphase filters satisfying the power symmetry conditions [Vai93]

$$\tilde{E}_n(z)\, E_n(z) + \tilde{E}_n(-z)\, E_n(-z) \;=\; \frac{A}{M} \qquad \text{for } n = 0, 1, ..., M-1.$$

In [SV86, Vet87] it has been shown that for critical sampling, a DFT FB with FIR filters in both the analysis and the synthesis section is possible only if all the polyphase filters are pure delays. In the oversampled case this restriction is relaxed. A necessary and sufficient condition for a FB to have FIR analysis and FIR minimum norm synthesis filters is $\det[\tilde{\mathbf{E}}(z)\mathbf{E}(z)] = C$ with $C \neq 0$ [CV]. For an integer oversampled DFT FB, $\tilde{\mathbf{E}}(z)\mathbf{E}(z)$ is a diagonal matrix and thus the condition reads

$$\det[\tilde{\mathbf{E}}(z)\mathbf{E}(z)] = \prod_{n=0}^{M-1} [\tilde{\mathbf{E}}(z)\mathbf{E}(z)]_{n,n} = \prod_{n=0}^{M-1} \left[M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r) E_n(zW_P^r) \right] = C.$$

For example, for $P = 2$ the above condition can be satisfied by choosing the polyphase filters $E_0(z)$ and $E_1(z)$ such that the power symmetry conditions

$$\tilde{E}_n(z)\, E_n(z) + \tilde{E}_n(-z)\, E_n(-z) \;=\; C_n, \qquad n = 0, 1$$

hold with some C_n. These polyphase filters are not necessarily pure delays.

In the general case (i.e., general oversampling) the UFBF matrix $\mathbf{S}(z)$ is not diagonal. However, by imposing restrictions on the length or bandwidth of $h[n]$, it is nevertheless possible to obtain simple expressions for the frame bounds and for the synthesis prototype (see [BHF96c]).

9.3.5 *Odd-stacked DFT filter banks*

The distinction between *even-stacked* and *odd-stacked* DFT FBs has been introduced in [NP78]. For even-stacked DFT FBs (considered so far) the subbands are centered about frequencies $\theta_k = \frac{k}{N}$ $(k = 0, 1, ..., N-1)$; in particular, the subband corresponding to frequency index $k = 0$ is centered about $\theta_0 = 0$. For odd-stacked DFT FBs the subbands are centered about frequencies $\theta_k = \frac{k+1/2}{N}$ $(k = 0, 1, ..., N-1)$; in particular, the subband corresponding to frequency index $k = 0$ is centered about $\theta_0 = \frac{1}{2N}$.

The impulse responses and transfer functions of the analysis and synthesis filters in an odd-stacked DFT FB with N channels and decimation factor M are given by

$$h_k[n] = h[n]\, W_N^{-(k+1/2)n}, \qquad f_k[n] = f[n]\, W_N^{-(k+1/2)n},$$
$$H_k(z) = H(zW_N^{k+1/2}), \qquad F_k(z) = F(zW_N^{k+1/2})$$

$(k = 0, 1, ..., N-1)$, and the corresponding polyphase components are

$$\begin{aligned} E_{k,n}(z) &= W_N^{(k+1/2)n} E_n\left(zW_N^{M(k+1/2)}\right) \\ R_{k,n}(z) &= W_N^{-(k+1/2)n} R_n\left(zW_N^{M(k+1/2)}\right). \end{aligned}$$

The FB's input-output relation is (9.3.1) with analysis functions $h_{k,m}[n]$ and synthesis functions $f_{k,m}[n]$ given by

$$\begin{aligned} h_{k,m}[n] &= h^*[mM - n]\, W_N^{-(k+1/2)(n-mM)} \\ f_{k,m}[n] &= f[n - mM]\, W_N^{-(k+1/2)(n-mM)}. \end{aligned}$$

It can be shown that an odd-stacked DFT FB with prototypes $h[n]$ and $f[n]$ is PR or paraunitary if and only if the associated even-stacked DFT FB with the same prototypes is PR or paraunitary, respectively. Thus, the PR and paraunitarity conditions provided for even-stacked DFT FBs in Subsections 9.3.2–9.3.4 are also applicable to odd-stacked DFT FBs.

9.3.6 *Simulation results*

Fig. 9.3.1(a) shows an analysis prototype in a DFT FB with $N = 64$ and $M = 8$ (oversampling factor $P = 8$). The corresponding minimum norm

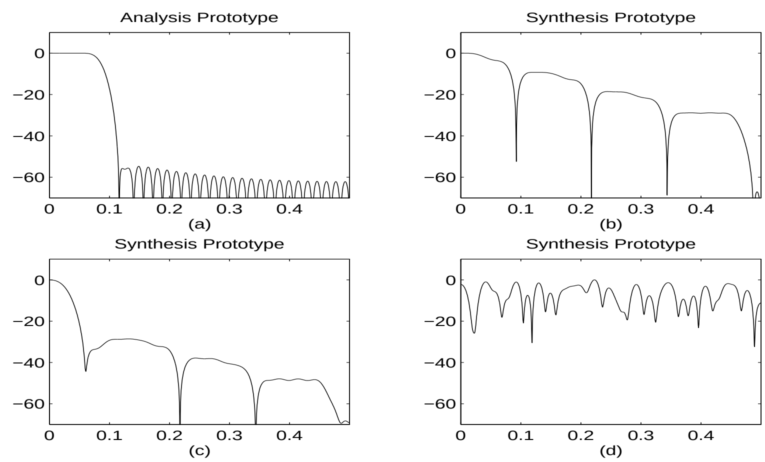

FIGURE 9.3.1. 64-channel DFT FB with oversampling factor $P = 8$: Transfer function magnitude of (a) analysis prototype, (b) minimum norm PR synthesis prototype, (c) PR synthesis prototype with improved frequency selectivity, and (d) “random” PR synthesis prototype.

PR synthesis prototype, depicted in Fig. 9.3.1(b), is seen to have poor frequency selectivity[12]. In Fig. 9.3.1(c) a PR synthesis prototype with improved frequency selectivity is shown. Finally, Fig. 9.3.1(d) shows a PR synthesis prototype which was obtained by a random choice of the parameter function $p[n]$ in (9.3.2). This variety of quite different synthesis prototypes—all satisfying PR—demonstrates the extent of design freedom existing for oversampling factors as high as 8.

9.4 Oversampled cosine modulated filter banks

Cosine modulated FBs (CMFBs) are often preferred over DFT FBs since their subband signals are real-valued if the input signal and the analysis prototype are real-valued. It seems that so far only critically sampled CMFBs have been considered in the literature [Mal92, Rot83, Chu85, KV92, RT91, GB95, NK96, Vai93, Gop96, VK95, LV95]. This section introduces and studies *oversampled* CMFBs. We note that CMFBs can be efficiently implemented using the DCT and DST [Mal92, Vai93, VK95,

[12]Frequency selectivity of the synthesis filters is important in image coding applications where high frequency components are often coarsely quantized. It is here important that the resulting quantization error does not affect low frequency components, which would cause perceptually annoying artifacts in the reconstructed image.

Gop96, BH96b].

9.4.1 Odd-stacked cosine modulated filter banks

We first extend the conventional type of CMFBs [Mal92, Rot83, Chu85, KV92, RT91, GB95, NK96, Vai93, VK95] to the oversampled case [BH97a, BH96c]. For critical sampling, these CMFBs have been termed "class B CMFBs" in [Gop96]; however, we shall here call them "odd-stacked" due to their close relation to odd-stacked DFT FBs (see Subsection 9.4.3).

In the general case with N channels and decimation factor M (note that the CMFB is oversampled for $N > M$), the analysis and synthesis filters of an odd-stacked CMFB are derived from an analysis prototype $h[n]$ and a synthesis prototype $f[n]$, respectively, as[13]

$$h_k^{\text{C-o}}[n] = \sqrt{2}\, h[n] \cos\left(\frac{(k+1/2)\pi}{N} n + \phi_k^{\text{o}}\right), \tag{9.4.1}$$

$$f_k^{\text{C-o}}[n] = \sqrt{2}\, f[n] \cos\left(\frac{(k+1/2)\pi}{N} n - \phi_k^{\text{o}}\right) \tag{9.4.2}$$

for $k = 0, 1, ..., N-1$. Extending the phase definition given for critical sampling ($N = M$) by Gopinath and Burrus [GB95] to the oversampled case, we define the phases ϕ_k^{o} as

$$\phi_k^{\text{o}} = -\alpha \frac{\pi}{2N}\left(k + \frac{1}{2}\right) + r\frac{\pi}{2} \qquad \text{with } \alpha \in \mathbb{Z},\ r \in \{0, 1\}.$$

The choice $r = 1$ corresponds to replacing the cos in (9.4.1) and (9.4.2) by $-$sin and sin, respectively. The above phase expression contains the phases proposed in [Chu85, Rot83, Mal90a, RT91, KV92] as special cases.

The transfer functions of the analysis filters are

$$H_k^{\text{C-o}}(z) = \frac{1}{\sqrt{2}}\left[H\left(zW_{2N}^{k+1/2}\right) e^{j\phi_k^{\text{o}}} + H\left(zW_{2N}^{-(k+1/2)}\right) e^{-j\phi_k^{\text{o}}}\right]$$

for $k = 0, 1, ..., N-1$. A similar expression exists for the transfer functions of the synthesis filters. Note that the channel frequencies in an odd-stacked CMFB are $\theta_k = \frac{k+1/2}{2N}$, as depicted in Fig. 9.4.1(a). In particular, the channel with index $k = 0$ is centered at frequency $\theta_0 = \frac{1}{4N}$.

An important disadvantage of odd-stacked CMFBs is that the channel filters do not have linear phase even if the prototypes have linear phase [Gop96]. (Linear phase filters are especially important in image coding applications [MS74].)

[13] The superscripts C-o and C-e indicate that the respective quantity belongs to an odd- and even-stacked CMFB, respectively.

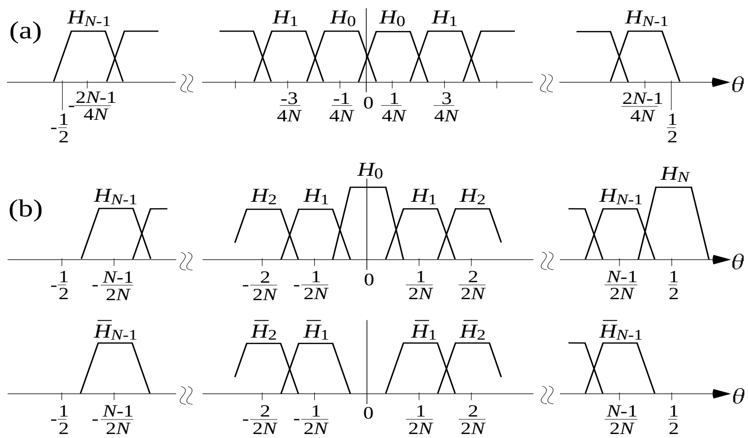

FIGURE 9.4.1. Transfer functions of the channel filters in (a) an N-channel odd-stacked CMFB and (b) a $2N$-channel even-stacked CMFB.

9.4.2 Even-stacked cosine modulated filter banks

We next generalize the "class A" CMFBs recently proposed for critical sampling by Gopinath [Gop96] to the oversampled case [BH96c, BH96b, BH97a]. We call this CMFB type "even-stacked" due to its close relation to even-stacked DFT FBs (see Subsection 9.4.3). The CMFBs recently introduced (for critical sampling) by Lin and Vaidyanathan [LV95] and the recently proposed Wilson FBs [BH96c, BH97a] (corresponding to the discrete-time Wilson expansion [BFGH96]) are special even-stacked CMFBs.

The analysis FB in an even-stacked CMFB with $2N$ channels and decimation factor $2M$ (the CMFB is oversampled for $N > M$) consists of two partial FBs $\{h_k^{\text{C-e}}[n]\}_{k=0,..,N}$ and $\{h_k^{\text{C-e}'}[n]\}_{k=1,..,N-1}$ derived from an analysis prototype $h[n]$ as [BH96c, BH96b, BH97a]

$$h_k^{\text{C-e}}[n] = \begin{cases} h[n-rM], & k=0 \\ \sqrt{2}\,h[n]\,\cos\!\left(\frac{k\pi}{N}n+\phi_k^{\text{e}}\right), & k=1,..,N-1 \\ h[n-sM]\,(-1)^{n-sM}, & k=N \end{cases}$$

$$h_k^{\text{C-e}'}[n] = \sqrt{2}\,h[n-M]\,\sin\!\left(\frac{k\pi}{N}(n-M)+\phi_k^{\text{e}}\right), \quad k=1,..,N-1\,.$$

Similarly, the synthesis FB consists of two partial FBs $\{f_k^{\text{C-e}}[n]\}_{k=0,..,N}$

and $\{f_k^{\text{C-e}'}[n]\}_{k=1,..,N-1}$ defined in terms of a synthesis prototype $f[n]$ as

$$f_k^{\text{C-e}}[n] = \begin{cases} f[n+rM], & k=0 \\ \sqrt{2}\, f[n]\, \cos\!\left(\frac{k\pi}{N}n - \phi_k^{\text{e}}\right), & k=1,..,N-1 \\ f[n+sM]\,(-1)^{n+sM}, & k=N \end{cases}$$

$$f_k^{\text{C-e}'}[n] = -\sqrt{2}\, f[n+M]\, \sin\!\left(\frac{k\pi}{N}(n+M) - \phi_k^{\text{e}}\right), \quad k=1,..,N-1\,.$$

Here, extending the phase definition given for critical sampling in [Gop96], we define the phases as

$$\phi_k^{\text{e}} = -\alpha\frac{\pi}{2N}k + r\frac{\pi}{2} \qquad \text{with } \alpha \in \mathbb{Z}\,,\ r \in \{0,1\}\,;$$

furthermore, $s \in \{0,1\}$ with $s = r$ for α even and $s = 1-r$ for α odd.

The transfer functions of the analysis filters are

$$H_k^{\text{C-e}}(z) = \begin{cases} z^{-rM}\, H(z), & k=0 \\ \frac{1}{\sqrt{2}}\left[H(zW_{2N}^k)\, e^{j\phi_k^{\text{e}}} + H(zW_{2N}^{-k})\, e^{-j\phi_k^{\text{e}}}\right], & k=1,..,N-1 \\ z^{-sM} H(-z), & k=N \end{cases}$$

$$H_k^{\text{C-e}'}(z) = \frac{z^{-M}}{j\sqrt{2}}\left[H(zW_{2N}^k)\, e^{j\phi_k^{\text{e}}} - H(zW_{2N}^{-k})\, e^{-j\phi_k^{\text{e}}}\right], \quad k=1,..,N-1.$$

Similar expressions exist for the transfer functions of the synthesis filters. Note that an even-stacked CMFB has $2N$ channels but there are only $N+1$ different channel frequencies $\theta_k = \frac{k}{2N}$ $(k = 0,...,N)$, as depicted in Fig. 9.4.1(b). In particular, the $k=0$ channel is centered at frequency $\theta_0 = 0$.

For *any* choice of the parameters $\alpha \in \mathbb{Z}$ and $r \in \{0,1\}$, all analysis filters have linear phase if the analysis prototype $h[n]$ satisfies the linear phase (symmetry) property $h[\alpha + (2l-1)N - n] = h[n]$ for some $l \in \mathbb{Z}$. Similarly, all synthesis filters have linear phase if the synthesis prototype satisfies $f[-\alpha - (2l-1)N - n] = f[n]$ for some $l \in \mathbb{Z}$. This linear phase property of even-stacked CMFBs is an important advantage over odd-stacked CMFBs. For the special case of critical sampling the linear phase property of even-stacked (class A) CMFBs has first been recognized by Gopinath [Gop96].

9.4.3 Perfect reconstruction property

Our discussion of the PR property for CMFBs will be based on an important relation between any CMFB and a corresponding DFT FB of the same stacking type but with twice the CMFB's oversampling factor. This relation is epitomized by the following fundamental decomposition of the reconstructed signal $y[n]$ in a CMFB.

Theorem 9.4.1 *[BH96b, BH96c, BH97a] The reconstructed signal in a CMFB (odd- or even-stacked, oversampled or critically sampled) can be decomposed as*[14]

$$y[n] = \frac{1}{2}\left[\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n] + \left(\boldsymbol{T}_{\mathrm{D}}^{(h,f)}x\right)[n]\right]. \tag{9.4.3}$$

Here, $\boldsymbol{S}_{\mathrm{D}}^{(h,f)}$ *is the input-output operator of a DFT FB with* $2N$ *channels and decimation factor* M*,*

$$\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty}\left\langle x, h_{k,m}^{\mathrm{D}}\right\rangle f_{k,m}^{\mathrm{D}}[n],$$

and $\boldsymbol{T}_{\mathrm{D}}^{(h,f)}$ *is given by*

$$\left(\boldsymbol{T}_{\mathrm{D}}^{(h,f)}x\right)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \left\langle x, h_{k,m}^{\mathrm{D}}\right\rangle f_{k,m}^{\mathrm{D}\dagger}[n],$$

where for an odd-stacked CMFB $h_{k,m}^{\mathrm{D}}[n] = h^*[mM-n]\,W_{2N}^{-(k+1/2)(n-mM)}$, $f_{k,m}^{\mathrm{D}}[n] = f[n-mM]\,W_{2N}^{-(k+1/2)(n-mM)}$, $f_{k,m}^{\mathrm{D}\dagger}[n] = f_{2N-k-1,m}^{\mathrm{D}}[n]$, $\phi_k = \phi_k^{\mathrm{o}}$*, and* $c_m = 1$*, and for an even-stacked CMFB* $h_{k,m}^{\mathrm{D}}[n] = h^*[mM-n]\,W_{2N}^{-k(n-mM)}$, $f_{k,m}^{\mathrm{D}}[n] = f[n-mM]\,W_{2N}^{-k(n-mM)}$, $f_{k,m}^{\mathrm{D}\dagger}[n] = f_{2N-k,m}^{\mathrm{D}}[n]$, $\phi_k = \phi_k^{\mathrm{e}}$*, and* $c_m = (-1)^m$*.*

We emphasize that the first component, $\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n]$, is the output signal of a DFT FB of the same stacking type as the CMFB but with $2N$ channels and decimation factor M, i.e., with twice the oversampling factor (cf. (9.3.1) with N replaced by $2N$). Time, frequency, and polyphase domain expressions of the operators $\boldsymbol{S}_{\mathrm{D}}^{(h,f)}$ and $\boldsymbol{T}_{\mathrm{D}}^{(h,f)}$ are provided in [BH96b].

The above decomposition is the basis for the following fundamental PR condition.

Theorem 9.4.2 *[BH96b, BH96c, BH97a] A CMFB (odd- or even-stacked, oversampled or critically sampled) satisfies the PR property* $y[n] = x[n]$ *if and only if*

$$\boldsymbol{S}_{\mathrm{D}}^{(h,f)} = 2\boldsymbol{I} \qquad \textit{and} \qquad \boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O},$$

where $\boldsymbol{I}$ *and* $\boldsymbol{O}$ *denote the identity and zero operator, respectively, on* $l^2(\mathbb{Z})$*.*

If the second PR condition, $\boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O}$, is satisfied, the CMFB's input-output relation (9.4.3) reduces to $y[n] = \frac{1}{2}\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n]$, which is (up to

[14]The subscript or superscript D indicates that the respective quantity belongs to a DFT FB.

the constant factor 1/2) the input-output relation of a DFT FB with $2N$ channels and decimation factor M. This DFT FB is odd-stacked (even-stacked) for an odd-stacked (even-stacked) CMFB. Thus, we conclude that *any CMFB with PR corresponds to a PR DFT FB of the* same *stacking type and with twice the oversampling factor.* In view of this correspondence, it is not surprising that the first PR condition, $\boldsymbol{S}_{\mathrm{D}}^{(h,f)} = 2\boldsymbol{I}$ is (up to the constant factor 2) the PR condition for a DFT FB with $2N$ channels and decimation factor M. This PR condition is *the same* for odd-stacked and even-stacked CMFBs (cf. the explicit time, frequency, and polyphase domain formulations in Subsection 9.3.2). Furthermore, explicit time, frequency, and polyphase domain formulations of the second PR condition, $\boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O}$, are provided in [BH96b]. We note that the idea of constructing CMFBs from DFT FBs has been previously used in the case of critical sampling and near-PR (see for example [Vai93]).

9.4.4 Frame-theoretic properties

The frame operator $\boldsymbol{S}_{\mathrm{C}}$ of a CMFB is given by

$$(\boldsymbol{S}_{\mathrm{C}}\, x)[n] = \sum_{k=0}^{N'} \sum_{m=-\infty}^{\infty} \left\langle x, h_{k,m}^{\mathrm{C}} \right\rangle h_{k,m}^{\mathrm{C}}[n]\,,$$

where in the odd-stacked case $N' = N-1$ and

$$h_{k,m}^{\mathrm{C}}[n] = h_{k}^{\mathrm{C\text{-}o}^*}[mM-n]\,,$$

and in the even-stacked case $N' = N$ and[15]

$$h_{k,m}^{\mathrm{C}}[n] \;=\; \begin{cases} h_{k}^{\mathrm{C\text{-}e}^*}[2\mu M-n]\,, & m=2\mu,\;\; k=0,1,...,N \\ h_{k}^{\mathrm{C\text{-}e}'^*}[2\mu M-n]\,, & m=2\mu-1,\;\; k=1,2,...,N-1. \end{cases}$$

Our frame-theoretic analysis of CMFBs will be based on the following fundamental decomposition of the CMFB frame operator.

Theorem 9.4.3 *[BH96b, BH97a] The frame operator of a CMFB (odd- or even-stacked, oversampled or critically sampled) can be decomposed as*

$$\boldsymbol{S}_{\mathrm{C}} \;=\; \frac{1}{2}\left(\boldsymbol{S}_{\mathrm{D}} + \boldsymbol{T}_{\mathrm{D}}\right). \tag{9.4.4}$$

[15]Note that in the even-stacked case no analysis functions $h_{k,m}^{\mathrm{C}}[n]$ exist for $k=0$, $m=2\mu-1$ and $k=N$, $m=2\mu-1$. Furthermore note that for the sake of simplicity we here choose an indexing of the analysis functions $h_{k,m}^{\mathrm{C}}[n]$ of even-stacked CMFBs that is not strictly consistent with the UFBF format in Subsection 9.2.2.

Here, $\boldsymbol{S}_{\rm D}$ is the frame operator of a DFT FB with $2N$ channels and decimation factor M,

$$(\boldsymbol{S}_{\rm D}\, x)[n] \;=\; \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h^{\rm D}_{k,m}\rangle\, h^{\rm D}_{k,m}[n]\,,$$

and $\boldsymbol{T}_{\rm D}$ is given by

$$(\boldsymbol{T}_{\rm D}\, x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m\, \langle x, h^{\rm D}_{k,m}\rangle\, h^{{\rm D}\dagger}_{k,m}[n]\,,$$

with $h^{\rm D}_{k,m}[n]$, $h^{{\rm D}\dagger}_{k,m}[n]$, ϕ_k, and c_m as defined in Theorem 9.4.1.

We emphasize that $\boldsymbol{S}_{\rm D}$ is the frame operator of a DFT FB of the same stacking type as the CMFB but with $2N$ channels and decimation factor M, i.e., with twice the oversampling factor of the CMFB. Furthermore, $\boldsymbol{S}_{\rm D} = \boldsymbol{S}^{(h,\tilde{h})}_{\rm D}$ and $\boldsymbol{T}_{\rm D} = \boldsymbol{T}^{(h,\tilde{h})}_{\rm D}$ with $\tilde{h}[n] = h^*[-n]$ (cf. Theorem 9.4.1).

Based on the above decomposition it can be shown [BH96b, BH97a] that, under the condition $\boldsymbol{T}_{\rm D} = \boldsymbol{O}$, the CMFB inherits the frame-theoretic properties of the corresponding DFT FB.

Theorem 9.4.4 *[BH96b, BH97a] Let $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, in an odd-stacked CMFB with N channels and decimation factor M, or in an even-stacked CMFB with $2N$ channels and decimation factor $2M$. Let $h[n]$ be such that[16] $\{h^{\rm D}_{k,m}[n]\}$ is a frame for $l^2(\mathbb{Z})$, i.e.,*

$$A_{\rm D}\|x\|^2 \le \langle \boldsymbol{S}_{\rm D}\, x, x\rangle \le B_{\rm D}\|x\|^2 \qquad \forall\; x[n] \in l^2(\mathbb{Z})\,.$$

Furthermore, let $h[n]$ be such that $\boldsymbol{T}_{\rm D} = \boldsymbol{O}$. Then, the following holds:

(i) The CMFB analysis functions $\{h^{\rm C}_{k,m}[n]\}$ are a frame for $l^2(\mathbb{Z})$ with frame bounds $A_{\rm C} = A_{\rm D}/2$ and $B_{\rm C} = B_{\rm D}/2$, i.e.,

$$\frac{A_{\rm D}}{2}\|x\|^2 \le \langle \mathbf{S}_{\rm C}\, x, x\rangle \le \frac{B_{\rm D}}{2}\|x\|^2 \qquad \forall\; x[n] \in l^2(\mathbb{Z})\,.$$

(ii) For $f[n] = 2\left(\mathbf{S}_{\rm D}^{-1}\tilde{h}\right)[n]$ with $\tilde{h}[n] = h^*[-n]$, the synthesis CMFB $\{f^{\rm C}_k[n]\}$ constructed from $f[n]$ is the PR synthesis CMFB with minimum norm filters.

The following interpretations and conclusions apply for $\boldsymbol{T}_{\rm D} = \boldsymbol{O}$.

[16]Note that the $h^{\rm D}_{k,m}[n]$ are differently defined for odd-stacked and even-stacked CMFBs (see Theorem 9.4.1).

- Eq. (9.4.4) implies $\boldsymbol{S}_\mathrm{C} = \frac{1}{2}\boldsymbol{S}_\mathrm{D}$, which means that the CMFB frame operator reduces to the frame operator of the corresponding DFT FB. Since $\left(\boldsymbol{S}_\mathrm{D}^{-1}\tilde{h}\right)[n]$ is the minimum-norm synthesis prototype of the corresponding DFT FB [BHF96c], the minimum norm PR synthesis prototype in the CMFB, $f[n] = 2\left(\boldsymbol{S}_\mathrm{D}^{-1}\tilde{h}\right)[n]$, is equal (up to a constant factor) to the minimum norm PR synthesis prototype in the corresponding DFT FB. Thus, for $\boldsymbol{T}_\mathrm{D} = \boldsymbol{O}$ the design of a CMFB reduces to that of a DFT FB of the same stacking type and with twice the oversampling factor.

- The CMFB frame bounds $A_\mathrm{C} = A_\mathrm{D}/2$ and $B_\mathrm{C} = B_\mathrm{D}/2$ are trivially related to the frame bounds A_D and B_D of the corresponding DFT FB. Since $B_\mathrm{C}/A_\mathrm{C} = B_\mathrm{D}/A_\mathrm{D}$, the CMFB inherits important numerical properties (noise sensitivity [BH97b]) of the corresponding DFT FB even though it has just half the oversampling factor of the DFT FB. This is remarkable, since usually a decrease of redundancy leads to a deterioration of the numerical properties of a frame.

- In particular, if the DFT FB is paraunitary ($A_\mathrm{D} = B_\mathrm{D}$), then the corresponding CMFB is paraunitary as well ($A_\mathrm{C} = B_\mathrm{C}$).

All of these results hinge on the condition $\boldsymbol{T}_\mathrm{D} = \boldsymbol{O}$. Time, frequency, and polyphase domain versions of this condition are provided in [BH96b]. Furthermore, for odd-stacked CMFBs with arbitrary integer oversampling factor P, and for even-stacked CMFBs with odd P, the symmetry property

$$h^*[\alpha + (2l+1)PM - n] = h[n] \qquad \text{(with some } l \in \mathbb{Z}) \tag{9.4.5}$$

can be shown to be a sufficient condition for $\boldsymbol{T}_\mathrm{D} = \boldsymbol{O}$ [BH96b, BH96c, BH97a]. This condition implies that $h[n]$ has linear phase. Thus, PR (with linear phase filters in the case of an even-stacked CMFB) is achieved by choosing $h[n]$ according to (9.4.5) and $f[n] = 2\left(\boldsymbol{S}_\mathrm{D}^{-1}\tilde{h}\right)[n]$. In particular, the CMFB will be *paraunitary* with frame bound $A = 1$ if $\boldsymbol{S}_\mathrm{D} = 2\boldsymbol{I}$.

9.5 Conclusion

Oversampled filter banks (FBs) have several attractive properties such as increased design freedom and numerical stability. In this chapter, we studied oversampled uniform FBs using a frame-theoretic approach. Special attention has been given to DFT FBs and cosine modulated FBs (CMFBs), which are practically important due to the existence of efficient implementations. Our analysis has emphasized perfect reconstruction and frame-theoretic properties. Among other results, we showed that oversampled even-stacked CMFBs allow perfect reconstruction and paraunitarity as well as linear phase filters in all the channels, and that CMFBs can be derived from DFT FBs with twice the oversampling factor.

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