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EXTENDING THE TRANSFER FUNCTION CALCULUS OF TIME-VARYING LINEAR SYSTEMS: A GENERALIZED UNDERSPREAD THEORY*

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ABSTRACT

We extend the approximate transfer function calculus of "underspread" linear time-varying (LTV) systems introduced by W. Kozek. Our extension is based on a new, generalized definition of underspread LTV systems that does not assume finite support of the systems' spreading function. We establish explicit bounds on various error quantities associated with the transfer function approximation. Our results yield a simple and convenient transfer function calculus for a significantly wider and practically more relevant class of LTV systems than that previously considered.

1 INTRODUCTION

Background. Linear time-varying (LTV) systems model a variety of phenomena as diverse as speech production and mobile radio channels. The input-output relation for an LTV system (linear operator [1]) $\mathbf{H}$ is given by[1]

$$(\mathbf{H}x)(t) = \int_{t'} h(t,t')\, x(t')\, dt', \qquad (1)$$

where $h(t,t')$ is the impulse response (kernel) of $\mathbf{H}$.

Unfortunately, general LTV systems are much more difficult to analyze and characterize than linear time-invariant (LTI) systems, i.e., systems with convolution-type impulse response of the form $h(t,t') = g(t-t')$. For an LTI system, the *transfer function* (frequency response) $G(f) = \int_\tau g(\tau)\, e^{-j2\pi f\tau}\, d\tau$ is an extremely simple and efficient system description. This is due to the following properties:

- The complex sinusoids $\{e^{j2\pi ft}\}$ are the eigenfunctions of any LTI system, with $G(f)$ the associated eigenvalue. Thus, the response of an LTI system to a complex sinusoid $e^{j2\pi f_0 t}$ equals $e^{j2\pi f_0 t}$ multiplied by $G(f_0)$.
- The Fourier transform of $(\mathbf{H}x)(t)$ equals the Fourier transform of $x(t)$ multiplied by $G(f)$.
- The transfer function of the series connection (composition) of two LTI systems $\mathbf{H}_1$ and $\mathbf{H}_2$ equals $G_1(f)G_2(f)$.
- The transfer function of the adjoint of an LTI system equals the complex conjugate of $G(f)$.
- The minimum and maximum system gain are reflected by the infimum and supremum, respectively, of $|G(f)|$.

A similarly simple characterization exists for "linear frequency-invariant" (LFI) systems which have an impulse response of the type $h(t,t') = m(t)\,\delta(t-t')$. Here, the factor $m(t)$ plays the role of a "temporal transfer function."

In contrast to LTI or LFI systems, general LTV systems do not allow a simple and efficient description via a universal "transfer function" with properties similar to those listed above.

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[1] All integrals go from $-\infty$ to ∞.

Outline of paper. This paper shows that the *generalized Weyl symbol* (GWS) introduced by W. Kozek [2] is an *approximate transfer function* for a practically important class of LTV systems. The GWS of an LTV system $\mathbf{H}$ is a (generally complex-valued) function of time t and frequency f defined as

$$L_{\mathbf{H}}^{(\alpha)}(t,f) \triangleq \int_\tau h^{(\alpha)}(t,\tau)\, e^{-j2\pi f\tau}\, d\tau, \qquad (2)$$

where

$$h^{(\alpha)}(t,\tau) = h\Big(t + \Big(\frac{1}{2}-\alpha\Big)\tau,\, t - \Big(\frac{1}{2}+\alpha\Big)\tau\Big) \qquad (3)$$

with α a real-valued parameter. For $\alpha = 0$, $1/2$, and $-1/2$, the GWS reduces to respectively the Weyl symbol [2]–[6], Zadeh's time-varying transfer function [7], and the Kohn-Nirenberg symbol [6, 8] (equivalently Bello's frequency-dependent modulation function [9]). For LTI and LFI systems, the GWS simplifies to the spectral and temporal transfer function, respectively.

Our results extend the pioneering work of W. Kozek who developed a GWS-based approximate transfer function calculus for a class of "underspread" LTV systems whose spreading function (see Section 2) has compact support of area $\ll 1$ [10, 11]. In Section 2 of this paper, we shall extend the concept of underspread systems using weighted integrals and moments of the spreading function. Subsequently, in Section 3 we will employ these integrals/moments to formulate explicit bounds on the errors incurred by the transfer function approximation, thereby extending the GWS-based transfer function calculus to a significantly wider and practically more relevant class of LTV systems than that considered in [10, 11]. We note that a different approach to related topics is taken in the theory of pseudo-differential operators [6, 8, 12].

2 EXTENDED CONCEPT OF UNDERSPREAD SYSTEMS

In contrast to LTI or LFI systems (which cause only time *or* frequency shifts, respectively), general LTV systems shift the input signal with respect to both time and frequency. Indeed, the output signal in (1) can be written as [2, 6, 9, 10, 13]

$$(\mathbf{H}x)(t) = \int_\tau \int_\nu S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)\, x_{\tau,\nu}^{(\alpha)}(t)\, d\tau\, d\nu\,.$$

Here, $x_{\tau,\nu}^{(\alpha)}(t) = x(t-\tau)\, e^{j2\pi\nu t} e^{j2\pi(\alpha-1/2)\tau\nu}$ is the signal $x(t)$ shifted by τ in time and by ν in frequency, with the parameter $\alpha \in \mathbb{R}$ expressing a freedom in defining joint time-frequency (TF) shifts, and $S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)$ is the *generalized spreading function* (GSF) of $\mathbf{H}$, defined as [2, 6, 9, 10, 13]

$$S_{\mathbf{H}}^{(\alpha)}(\tau,\nu) \triangleq \int_t h^{(\alpha)}(t,\tau)\, e^{-j2\pi\nu t}\, dt\,,$$

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with $h^{(\alpha)}(t,\tau)$ as in (3). The GSF is the 2-D Fourier transform of the GWS in (2). It can be shown that $S_{\mathbf{H}}^{(\alpha_2)}(\tau,\nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)\, e^{j2\pi\tau\nu(\alpha_1-\alpha_2)}$ so that $\left|S_{\mathbf{H}}^{(\alpha_2)}(\tau,\nu)\right| = \left|S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)\right|$. Hence, we will write $|S_{\mathbf{H}}(\tau,\nu)|$ instead of $|S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)|$.

Conceptually, an LTV system is *underspread* if its GSF is concentrated in a small region about the origin of the (τ,ν)-plane, which indicates that the system introduces only small TF shifts τ, ν. In [10, 11], the GSF of an underspread system was required to be *exactly* zero outside a small support region about the origin. In practice, however, this condition is often not satisfied exactly but only *effectively*. This poses the problem of how to choose the effective support region and how the resulting modeling error affects the validity of the results based on the finite support model.

To circumvent these problems, we here propose to characterize an underspread system by means of the following (α-independent) normalized[2] *weighted GSF integrals*:

$$m_{\mathbf{H}}^{(\phi)} \triangleq \frac{1}{\|S_{\mathbf{H}}\|_1} \int_\tau \int_\nu \phi(\tau,\nu)\, |S_{\mathbf{H}}(\tau,\nu)|\, d\tau\, d\nu$$

$$M_{\mathbf{H}}^{(\phi)} \triangleq \frac{1}{\|\mathbf{H}\|_2} \left[\int_\tau \int_\nu \phi^2(\tau,\nu)\, |S_{\mathbf{H}}(\tau,\nu)|^2\, d\tau\, d\nu\right]^{1/2}.$$

Here, $\phi(\tau,\nu)$ with $\phi(\tau,\nu) \geq \phi(0,0) = 0$ is a weighting function which penalizes GSF contributions that are far away from the origin. Fig. 1 shows some weighting functions to be used in Section 3. We also define the *GSF moments* $m_{\mathbf{H}}^{(k,l)}$ and $M_{\mathbf{H}}^{(k,l)}$ ($k, l \in \mathbb{N}_0$) as special cases of $m_{\mathbf{H}}^{(\phi)}$ and $M_{\mathbf{H}}^{(\phi)}$ using the weighting functions $\phi(\tau,\nu) = |\tau|^k |\nu|^l$. Moments with $k=0$ or $l=0$ penalize mainly GSF contributions located away from the τ axis or away from the ν axis, respectively, whereas moments with $k=l$ penalize mainly GSF contributions located away from the τ *and* ν axes, i.e., lying in oblique directions in the (τ,ν)-plane (cf. Fig. 1).

The GSF integrals and moments measure the spread of $|S_{\mathbf{H}}(\tau,\nu)|$ about the origin of the (τ,ν)-plane. Hence, without being forced to assume that the GSF has finite support, we can consider a system $\mathbf{H}$ to be *underspread* if $m_{\mathbf{H}}^{(\phi)}$ and $M_{\mathbf{H}}^{(\phi)}$ are "small." (Note that this concept is much less restrictive than the concept of "slow time-variation" which requires $|S_{\mathbf{H}}(\tau,\nu)|$ to be narrow with respect to ν.) Since the GSF and the GWS are a 2-D Fourier transform pair, the GWS of an underspread system is a *smooth* function.

3 TRANSFER FUNCTION CALCULUS

In this section, we show that for systems that are underspread in the extended sense of Section 2, the GWS $L_{\mathbf{H}}^{(\alpha)}(t,f)$ defined in (2) is an approximate "TF transfer function" that generalizes the spectral (temporal) transfer function of LTI (LFI) systems. More specifically, we establish explicit upper bounds on the errors associated with the transfer function approximation. These bounds are partly similar to bounds derived in [10, 11], but they use the GSF integrals/moments defined in Section 2 and do not assume the GSF to have finite support. Hence, our subsequent results show that a GWS-based transfer function calculus is valid for a significantly wider class of underspread systems than that considered in [10, 11].

3.1 Approximate Uniqueness of the GWS

Whereas the transfer functions of LTI and LFI systems are uniquely defined, the GWS of an LTV system depends on the parameter α. However, this dependence is bounded according to the following theorem.[3]

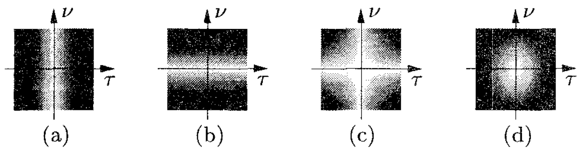

Fig. 1. *Gray-scale plots of some weighting functions: (a) $\phi(\tau,\nu) = |\tau|^k$, (b) $\phi(\tau,\nu) = |\nu|^k$, (c) $\phi(\tau,\nu) = |\tau\nu|^k$, (d) $\phi(\tau,\nu) = |1 - A_s^{(0)}(\tau,\nu)|$ with $A_s^{(0)}(\tau,\nu)$ the ambiguity function of a normalized Gaussian function (cf. Subsection 3.4). We note that darker shades correspond to larger values.*

Theorem 3.1 *For any LTV system* $\mathbf{H}$, *the difference* $\Delta_1(t,f) = L_{\mathbf{H}}^{(\alpha_1)}(t,f) - L_{\mathbf{H}}^{(\alpha_2)}(t,f)$ *between two GWSs with parameters* α_1 *and* α_2 *is bounded as*[4]

$$\frac{|\Delta_1(t,f)|}{\|S_{\mathbf{H}}\|_1} \leq 2\pi|\alpha_1-\alpha_2|\, m_{\mathbf{H}}^{(1,1)}, \qquad \frac{\|\Delta_1\|_2}{\|\mathbf{H}\|_2} \leq 2\pi|\alpha_1-\alpha_2|\, M_{\mathbf{H}}^{(1,1)}.$$

For an *underspread* system whose moments $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ will be small, these bounds show that the GWS is approximately independent of α, i.e.,

$$L_{\mathbf{H}}^{(\alpha_1)}(t,f) \approx L_{\mathbf{H}}^{(\alpha_2)}(t,f).$$

Hence, the TF transfer function of an underspread system is approximately unique. We note that small $m_{\mathbf{H}}^{(1,1)}$ or $M_{\mathbf{H}}^{(1,1)}$ requires the GSF to be concentrated along the τ and ν axes (i.e., not oriented in oblique directions).

3.2 Adjoint Systems

The transfer function of the adjoint [1] of an LTI system is $G^*(f)$, and similarly for an LFI system. In contrast, the GWS of the adjoint $\mathbf{H}^+$ of an LTV system $\mathbf{H}$ is $L_{\mathbf{H}^+}^{(\alpha)}(t,f) = L_{\mathbf{H}}^{(-\alpha)*}(t,f)$, which does *not* equal $L_{\mathbf{H}}^{(\alpha)*}(t,f)$ unless $\alpha = 0$. However, Theorem 3.1 leads to the following bounds.

Corollary 3.2 *For any LTV system* $\mathbf{H}$, *the difference* $\Delta_2(t,f) = L_{\mathbf{H}^+}^{(\alpha)}(t,f) - L_{\mathbf{H}}^{(\alpha)*}(t,f)$ *is bounded as*

$$\frac{|\Delta_2(t,f)|}{\|S_{\mathbf{H}}\|_1} \leq 4\pi|\alpha|\, m_{\mathbf{H}}^{(1,1)}, \qquad \frac{\|\Delta_2\|_2}{\|\mathbf{H}\|_2} \leq 4\pi|\alpha|\, M_{\mathbf{H}}^{(1,1)}.$$

For an underspread system, these bounds show that

$$L_{\mathbf{H}^+}^{(\alpha)}(t,f) \approx L_{\mathbf{H}}^{(\alpha)*}(t,f).$$

Unlike in the LTI or LFI case, the GWS of a *self-adjoint (Hermitian)* system (i.e., a system satisfying $\mathbf{H}^+ = \mathbf{H}$) is not real-valued for $\alpha \neq 0$. Corollary 3.2 implies that the imaginary part of the GWS of a self-adjoint system, $\mathcal{I}(t,f) =$

[2] The weighted integrals are normalized by the L_1 and L_2 norm of $S_{\mathbf{H}}^{(\alpha)}(\tau,\nu)$, $\|S_{\mathbf{H}}\|_1 = \int_\tau \int_\nu |S_{\mathbf{H}}(\tau,\nu)|\, d\tau\, d\nu$ and $\|S_{\mathbf{H}}\|_2^2 = \int_\tau \int_\nu |S_{\mathbf{H}}(\tau,\nu)|^2\, d\tau\, d\nu = \int_{t_1} \int_{t_2} |H(t_1,t_2)|^2\, dt_1\, dt_2 = \|\mathbf{H}\|_2^2$.

[3] The proof of this theorem is outlined in the Appendix. The proofs of subsequent theorems are partly similar; they can be found in [14] and will here be suppressed due to space limitations. For all bounds, the respective GSF integrals/moments are assumed to exist.

[4] Here, $\|\Delta_1\|_2$ denotes the L_2 norm of $\Delta_1(t,f)$ defined as $\|\Delta_1\|_2^2 = \int_t \int_f |\Delta_1(t,f)|^2\, dt\, df$. We shall consider bounds on the *normalized* error magnitude and L_2 norm. The magnitude is normalized by $\|S_{\mathbf{H}}\|_1$ since $|L_{\mathbf{H}}^{(\alpha)}(t,f)| \leq \|S_{\mathbf{H}}\|_1$, and the L_2 norm is normalized by $\|\mathbf{H}\|_2$ since $\|L_{\mathbf{H}}^{(\alpha)}\|_2 = \|\mathbf{H}\|_2$.

$\frac{1}{2j}\big[L_{\mathbf{H}}^{(\alpha)}(t,f) - L_{\mathbf{H}}^{(\alpha)*}(t,f)\big] = \frac{1}{2j}\big[L_{\mathbf{H}^+}^{(\alpha)}(t,f) - L_{\mathbf{H}}^{(\alpha)*}(t,f)\big] = \frac{1}{2j}\Delta_2(t,f)$, is bounded as

$$\frac{|\mathcal{I}(t,f)|}{\|S_{\mathbf{H}}\|_1} \le 2\pi|\alpha|\, m_{\mathbf{H}}^{(1,1)}, \qquad \frac{\|\mathcal{I}\|_2}{\|\mathbf{H}\|_2} \le 2\pi|\alpha|\, M_{\mathbf{H}}^{(1,1)}.$$

Hence, the GWS of an underspread, self-adjoint system is approximately real-valued even if $\alpha \neq 0$.

3.3 Composition of Systems

The transfer function of the series connection (composition) of two LTI systems $\mathbf{H}_1$ and $\mathbf{H}_2$ equals the product of the individual transfer functions, $G_1(f)\,G_2(f)$, and similarly for LFI systems. In contrast, the GWS of the composition $\mathbf{H}_2\mathbf{H}_1$ of two general LTV systems is *not* equal to the product of the individual GWSs of $\mathbf{H}_1$ and $\mathbf{H}_2$.

Theorem 3.3 *For any two LTV systems $\mathbf{H}_1$, $\mathbf{H}_2$, the error $\Delta_3(t,f) = L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t,f) - L_{\mathbf{H}_1}^{(\alpha)}(t,f)\,L_{\mathbf{H}_2}^{(\alpha)}(t,f)$ is bounded as*

$$\frac{|\Delta_3(t,f)|}{\|S_{\mathbf{H}_1}\|_1 \|S_{\mathbf{H}_2}\|_1} \le 2\pi\, B_{\mathbf{H}_1,\mathbf{H}_2}^{(\alpha)}, \qquad (4)$$

with

$$B_{\mathbf{H}_1,\mathbf{H}_2}^{(\alpha)} \triangleq \left|\alpha + \frac{1}{2}\right| m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)} + \left|\alpha - \frac{1}{2}\right| m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$$

The theorem shows that if $\mathbf{H}_1$ and $\mathbf{H}_2$ are such that $m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$ and $m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$ are both small, we have

$$L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t,f) \approx L_{\mathbf{H}_1}^{(\alpha)}(t,f)\, L_{\mathbf{H}_2}^{(\alpha)}(t,f).$$

Small $m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$ and $m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$ requires that the GSFs of $\mathbf{H}_1$ and $\mathbf{H}_2$ are both concentrated about the origin of the (τ,ν)-plane, with *similar* orientation parallel to the τ or ν axis. That is, $|S_{\mathbf{H}_1}(\tau,\nu)|$ and $|S_{\mathbf{H}_2}(\tau,\nu)|$ may not be oriented in oblique directions or in significantly different directions. For $\alpha = 0$, the bound in (4) simplifies since $B_{\mathbf{H}_1,\mathbf{H}_2}^{(0)} = \frac{1}{2}\big(m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)} + m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}\big)$. It can here be shown that (4) remains valid if $B_{\mathbf{H}_1,\mathbf{H}_2}^{(0)}$ is replaced by $B_{\widetilde{\mathbf{H}}_1,\widetilde{\mathbf{H}}_2}^{(0)}$ with $\widetilde{\mathbf{H}}_1 = \mathbf{U}\mathbf{H}_1\mathbf{U}^+$ and $\widetilde{\mathbf{H}}_2 = \mathbf{U}\mathbf{H}_2\mathbf{U}^+$, where $\mathbf{U}$ is any unitary operator corresponding to a rotation or some other symplectic coordinate transform of the (τ,ν)-plane [6]. Thus, for $\alpha = 0$, $\Delta_3(t,f)$ may be small even if the GSFs of $\mathbf{H}_1$ and $\mathbf{H}_2$ are oriented in (similar) *oblique* directions. For $\alpha = 1/2$, we obtain $B_{\mathbf{H}_1,\mathbf{H}_2}^{(1/2)} = m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$, which may be small even if $|S_{\mathbf{H}_1}(\tau,\nu)|$ is located along the τ axis and $|S_{\mathbf{H}_2}(\tau,\nu)|$ is located along the ν axis. Similarly, for $\alpha = -1/2$ we have $B_{\mathbf{H}_1,\mathbf{H}_2}^{(-1/2)} = m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$, which may be small even if $|S_{\mathbf{H}_1}(\tau,\nu)|$ is located along the ν axis and $|S_{\mathbf{H}_2}(\tau,\nu)|$ is located along the τ axis.

Of particular importance (cf. Sections 3.4 and 3.6) is the composition of $\mathbf{H}$ with $\mathbf{H}^+$, i.e., $\mathbf{H}_1 = \mathbf{H}$ and $\mathbf{H}_2 = \mathbf{H}^+$. Using $\|S_{\mathbf{H}^+}\|_1 = \|S_{\mathbf{H}}\|_1$ and $m_{\mathbf{H}^+}^{(k,l)} = m_{\mathbf{H}}^{(k,l)}$ in Theorem 3.3, and combining with Corollary 3.2, it can be shown that the difference $\Delta_4(t,f) = L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t,f) - |L_{\mathbf{H}}^{(\alpha)}(t,f)|^2$ is bounded as (the same bound holds for $\mathbf{H}\mathbf{H}^+$)

$$\frac{|\Delta_4(t,f)|}{\|S_{\mathbf{H}}\|_1^2} \le C_{\mathbf{H}}^{(\alpha)} \triangleq 2\pi\big(c_\alpha m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + 2|\alpha|\, m_{\mathbf{H}}^{(1,1)}\big), \quad (5)$$

with $c_\alpha = |\alpha + 1/2| + |\alpha - 1/2|$. Hence, for an underspread system we have

$$L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t,f) \approx \big|L_{\mathbf{H}}^{(\alpha)}(t,f)\big|^2.$$

The bound $C_{\mathbf{H}}^{(\alpha)}$ is tightest for $\alpha = 0$, in which case $C_{\mathbf{H}}^{(0)} = 2\pi\, m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$. For $|\alpha| \le 1/2$, $c_\alpha = 1$ and thus $C_{\mathbf{H}}^{(\alpha)} = 2\pi\big(m_{\mathbf{H}}^{(1,0)} m_{\mathbf{H}}^{(0,1)} + 2|\alpha|\, m_{\mathbf{H}}^{(1,1)}\big) \le 2\pi\big(m_{\mathbf{H}}^{(1,0)} m_{\mathbf{H}}^{(0,1)} + m_{\mathbf{H}}^{(1,1)}\big)$.

3.4 Approximate Eigenvalues and Eigenfunctions

The response of an LTI system to $e^{j2\pi f_0 t}$ (signal with perfect frequency concentration) is $G(f_0)\, e^{j2\pi f_0 t}$, and the response of an LFI system to $\delta(t - t_0)$ (signal with perfect time concentration) is $m(t_0)\,\delta(t - t_0)$. We now ask if the response of an LTV system to an input signal $s_{t_0,f_0}(t) = s(t - t_0)\, e^{j2\pi f_0 t}$ that is well concentrated about the TF point (t_0, f_0) is approximately $L_{\mathbf{H}}^{(\alpha)}(t_0,f_0)\, s_{t_0,f_0}(t)$, i.e., if $s_{t_0,f_0}(t)$ is an "approximate eigenfunction" of $\mathbf{H}$ with $L_{\mathbf{H}}^{(\alpha)}(t_0,f_0)$ the associated "approximate eigenvalue."

Theorem 3.4 *For any LTV system $\mathbf{H}$, any TF point (t_0, f_0), and any normalized signal $s(t)$ (i.e., $\|s\|_2 = 1$), the difference $\Delta_5(t) = \big(\mathbf{H} s_{t_0,f_0}\big)(t) - L_{\mathbf{H}}^{(\alpha)}(t_0,f_0)\, s_{t_0,f_0}(t)$ is bounded as*

$$\frac{\|\Delta_5\|_2}{\|S_{\mathbf{H}}\|_1} \le D_{\mathbf{H},s}^{(\alpha)} \triangleq \sqrt{C_{\mathbf{H}}^{(\alpha)} + 2 m_{\mathbf{H}}^{(\phi_s)} + m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}}, \quad (6)$$

with the weighting function $\phi_s(\tau,\nu) = |1 - A_s^{(\alpha)}(\tau,\nu)|$ where $A_s^{(\alpha)}(\tau,\nu) = \int_t s\big(t + \big(\frac{1}{2} - \alpha\big)\tau\big)\, s^*\big(t - \big(\frac{1}{2} + \alpha\big)\tau\big)\, e^{-j2\pi\nu t}\, dt$ is the generalized ambiguity function [15] of $s(t)$, and where $C_{\mathbf{H}}^{(\alpha)}$ has been defined in (5).

Hence, for an underspread system $\mathbf{H}$ where $D_{\mathbf{H},s}^{(\alpha)}$ can be made small, we have the "approximate eigenvalue relation"

$$\big(\mathbf{H} s_{t_0,f_0}\big)(t) \approx L_{\mathbf{H}}^{(\alpha)}(t_0,f_0)\, s_{t_0,f_0}(t),$$

which implies that the GWS $L_{\mathbf{H}}^{(\alpha)}(t,f)$ can be interpreted as an approximate eigenvalue distribution over the TF plane. In particular, small $m_{\mathbf{H}}^{(\phi_s)}$ and $m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}$ requires that, on the effective supports of $|S_{\mathbf{H}}(\tau,\nu)|$ and of $|S_{\mathbf{H}^+\mathbf{H}}(\tau,\nu)|$, there is $\phi_s(\tau,\nu) \approx 0$ and thus $A_s^{(\alpha)}(\tau,\nu) \approx A_s^{(\alpha)}(0,0) \equiv 1$. The latter condition can be satisfied only if these effective support regions are small, i.e., if $\mathbf{H}$ is underspread. We note that the bound $D_{\mathbf{H},s}^{(\alpha)}$ is tightest for $\alpha = 0$.

3.5 Input-Output Relation

For LTI systems, the Fourier transform of $(\mathbf{H}x)(t)$ equals the Fourier transform of $x(t)$ multiplied by the transfer function $G(f)$. Similarly, for LFI systems $(\mathbf{H}x)(t)$ equals $m(t)\,x(t)$. In the case of LTV systems, one may desire a similar "input-output relation" stating that a suitably defined TF representation of $(\mathbf{H}x)(t)$ equals the TF representation of $x(t)$ multiplied by the TF transfer function $L_{\mathbf{H}}^{(\alpha)}(t,f)$. In the following, we use as TF representation the short-time Fourier transform (STFT) [16] defined as

$$\mathrm{STFT}_x^{(w)}(t,f) = \int_{t'} x(t')\, w^*(t' - t)\, e^{-j2\pi f t'}\, dt',$$

where $w(t)$ is a normalized window function.

Theorem 3.5 *For any LTV system $\mathbf{H}$, the difference $\Delta_6(t,f) = \mathrm{STFT}_{\mathbf{H}x}^{(w)}(t,f) - L_{\mathbf{H}}^{(\alpha)}(t,f)\, \mathrm{STFT}_x^{(w)}(t,f)$ with arbitrary normalized STFT window $w(t)$ is bounded as*

$$\frac{|\Delta_6(t,f)|}{\|S_{\mathbf{H}}\|_1 \|x\|_2} \le D_{\mathbf{H},w}^{(\alpha)}, \qquad \frac{\|\Delta_6\|_2}{\|\mathbf{H}\|_2 \|x\|_2} \le \sqrt{2}\, M_{\mathbf{H}}^{(\phi_w)},$$

with the weighting function $\phi_w(\tau,\nu) = \sqrt{1 - \mathrm{Re}\{A_w^{(\alpha)}(\tau,\nu)\}}$ *and* $D_{\mathbf{H},w}^{(\alpha)}$ *as defined in (6).*

Hence, if $D_{\mathbf{H},w}^{(\alpha)}$ and $M_{\mathbf{H}}^{(\phi_w)}$ can be made small by suitable choice of $w(t)$, we obtain the approximate input-output relation

$$\mathrm{STFT}_{\mathbf{H}x}^{(w)}(t,f) \approx L_{\mathbf{H}}^{(\alpha)}(t,f)\, \mathrm{STFT}_x^{(w)}(t,f).$$

Small $M_{\mathbf{H}}^{(\phi_w)}$ requires that $\mathrm{Re}\{A_w^{(\alpha)}(\tau,\nu)\} \approx A_w^{(\alpha)}(0,0) \equiv 1$ on the effective support of $|S_{\mathbf{H}}(\tau,\nu)|$, thus implying that this effective support is small, i.e., that $\mathbf{H}$ is underspread.

3.6 Minimum and Maximum Gain

The minimum and maximum system gain are defined as

$$\gamma_{\mathbf{H}} \triangleq \inf_x \frac{\|\mathbf{H}x\|_2}{\|x\|_2}, \qquad \Gamma_{\mathbf{H}} \triangleq \sup_x \frac{\|\mathbf{H}x\|_2}{\|x\|_2}.$$

For LTI and LFI systems, $\gamma_{\mathbf{H}}$ and $\Gamma_{\mathbf{H}}$ equal the infimum and supremum, respectively, of the magnitude of the transfer function. The squared magnitude of the transfer function of $\mathbf{H}$ in turn equals the transfer function of $\mathbf{H}^+\mathbf{H}$. For general LTV systems, on the other hand, $\gamma_{\mathbf{H}}$ and $\Gamma_{\mathbf{H}}$ are *not* related to $L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t,f)$. In the following, we restrict to $\alpha = 0$ since $L_{\mathbf{H}^+\mathbf{H}}^{(0)}(t,f)$ is real-valued, and we consider

$$L_{\mathbf{H}^+\mathbf{H}}^{\inf} \triangleq \inf_{t,f} L_{\mathbf{H}^+\mathbf{H}}^{(0)}(t,f), \qquad L_{\mathbf{H}^+\mathbf{H}}^{\sup} \triangleq \sup_{t,f} L_{\mathbf{H}^+\mathbf{H}}^{(0)}(t,f).$$

Theorem 3.6 *For any LTV system* $\mathbf{H}$, *the difference between the infimum/supremum of* $L_{\mathbf{H}^+\mathbf{H}}^{(0)}(t,f)$ *and the squared minimum/maximum system gain is bounded as*

$$\frac{\left|L_{\mathbf{H}^+\mathbf{H}}^{\inf} - \gamma_{\mathbf{H}}^2\right|}{\|S_{\mathbf{H}}\|_1^2} \le m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}, \qquad \frac{\left|L_{\mathbf{H}^+\mathbf{H}}^{\sup} - \Gamma_{\mathbf{H}}^2\right|}{\|S_{\mathbf{H}}\|_1^2} \le m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)},$$

with the weighting function $\phi_s(\tau,\nu) = \left|1 - \frac{1}{A_s^{(0)}(\tau,\nu)}\right|$, *where* $s(t)$ *is an arbitrary normalized function.*

Hence, if $m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}$ can be made small by suitable choice of the function $s(t)$, we have

$$L_{\mathbf{H}^+\mathbf{H}}^{\inf} \approx \gamma_{\mathbf{H}}^2, \qquad L_{\mathbf{H}^+\mathbf{H}}^{\sup} \approx \Gamma_{\mathbf{H}}^2.$$

Small $m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}$ requires that $A_s^{(0)}(\tau,\nu) \approx A_s^{(0)}(0,0) \equiv 1$ on the effective support of $|S_{\mathbf{H}^+\mathbf{H}}(\tau,\nu)|$, thus implying that the effective support of $|S_{\mathbf{H}^+\mathbf{H}}(\tau,\nu)|$ is small, i.e., that $\mathbf{H}$ is underspread. In that case, we also have $L_{\mathbf{H}^+\mathbf{H}}^{(0)}(t,f) \approx \left|L_{\mathbf{H}}^{(0)}(t,f)\right|^2$ according to (5) and thus we finally obtain

$$\inf_{t,f} \left|L_{\mathbf{H}}^{(0)}(t,f)\right| \approx \gamma_{\mathbf{H}}, \qquad \sup_{t,f} \left|L_{\mathbf{H}}^{(0)}(t,f)\right| \approx \Gamma_{\mathbf{H}}.$$

Due to the approximate uniqueness of the GWS (cf. Subsection 3.1), this approximation will also hold for $\alpha \neq 0$.

4 CONCLUSION

We have introduced an extended class of "underspread" linear time-varying systems. For this type of systems, the generalized Weyl symbol is an approximate time-frequency transfer function that is similarly simple to use as the conventional transfer function of linear time-invariant systems. We have provided quantitative bounds on the errors incurred by this approximate transfer function calculus. These bounds are based on weighted integrals and moments of the generalized spreading function and do not require the generalized spreading function to have finite support.

APPENDIX: PROOF OF THEOREM 3.1

Using $S_{\mathbf{H}}^{(\alpha_2)}(\tau,\nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)\, e^{j2\pi\Delta\alpha\tau\nu}$ where $\Delta\alpha = \alpha_1 - \alpha_2$, the Fourier transform of $\Delta_1(t,f)$ is given by $\hat{\Delta}_1(\tau,\nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau,\nu)\left(1 - e^{j2\pi\Delta\alpha\tau\nu}\right)$. The first bound is then shown as

$$\begin{aligned}
|\Delta_1(t,f)| &\le \|\hat{\Delta}_1\|_1 = \int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)| \left|1 - e^{j2\pi\Delta\alpha\tau\nu}\right| d\tau\, d\nu \\
&= 2\int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|\, |\sin(\pi\Delta\alpha\tau\nu)|\, d\tau\, d\nu \\
&\le 2\pi|\Delta\alpha| \int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|\, |\tau|\, |\nu|\, d\tau\, d\nu \\
&= 2\pi|\Delta\alpha|\, \|S_{\mathbf{H}}\|_1\, m_{\mathbf{H}}^{(1,1)}.
\end{aligned}$$

Using $\|\Delta_1\|_2 = \|\hat{\Delta}_1\|_2$ and $\sin^2 x \le x^2$, the second bound can be shown in a similar manner.

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