EXTENDING THE TRANSFER FUNCTION CALCULUS OF TIME-VARYING LINEAR SYSTEMS: A GENERALIZED UNDERSPREAD THEORY*

Gerald Matz and Franz Hlawatsch

INTHFT, Vienna University of Technology, Gusshausstrasse 25/389, A-1040 Wien, Austria
phone: +43 1 58801 3515, fax: +43 1 5870583, email: gmatz@aurora.nt.tuwien.ac.at
web: http://www.nt.tuwien.ac.at/dspgroup/time.html

ABSTRACT

We extend the approximate transfer function calculus of "underspread" linear time-varying (LTV) systems introduced by W. Kozek. Our extension is based on a new, generalized definition of underspread LTV systems that does not assume finite support of the systems' spreading function. We establish explicit bounds on various error quantities associated with the transfer function approximation. Our results yield a simple and convenient transfer function calculus for a significantly wider and practically more relevant class of LTV systems than that previously considered.

1 INTRODUCTION

Background. Linear time-varying (LTV) systems model a variety of phenomena as diverse as speech production and mobile radio channels. The input-output relation for an LTV system (linear operator \[H\]) is given by

\[ (Hz)(t) = \int h(t, t') z(t') dt', \]

\[ h(t, t') \] is the impulse response (kernel) of \[H\].

Unfortunately, general LTV systems are much more difficult to analyze and characterize than linear time-invariant (LTI) systems, i.e., systems with convolution-type impulse response of the form \[h(t, t') = g(t - t')\]. For an LTI system, the transfer function (frequency response) \[G(f) = \int g(\tau) e^{-j2\pi ft} d\tau\] is an extremely simple and efficient system description. This is due to the following properties:

- The complex sinusoids \[e^{j2\pi ft}\] are the eigenfunctions of any LTI system, with \[G(f)\] the associated eigenvalue.
- Thus, the response of an LTI system to a complex sinusoid \[e^{j2\pi ft}\] equals \[e^{j2\pi ft}\] multiplied by \[G(f)\].
- The Fourier transform of \((Hz)(t)\) equals the Fourier transform of \(z(t)\) multiplied by \(G(f)\).
- The transfer function of the series connection (composition) of two LTI systems \[H_1\] and \[H_2\] equals \[G_1(f)G_2(f)\].
- The transfer function of the adjoint of an LTI system equals the complex conjugate of \(G(f)\).
- The minimum and maximum system gain are reflected by the infimum and supremum, respectively, of \(|G(f)|\).

A similar simple characterization exists for "linear frequency-invariant" (LFI) systems which have an impulse response of the type \[h(t, t') = m(t) \delta(t - t')\]. Here, the factor \(m(t)\) plays the role of a "temporal transfer function."

In contrast to LTI or LFI systems, general LTV systems do not allow a simple and efficient description via a universal "transfer function" with properties similar to those listed above.

Outline of paper. This paper shows that the generalized Weyl symbol (GWS) introduced by W. Kozek \cite{Kozek2} is an approximate transfer function for a practically important class of LTV systems. The GWS of an LTV system \[H\] is a (generally complex-valued) function of time \(t\) and frequency \(f\) defined as

\[ L^0_{H}(t, f) \triangleq \int h^{(\alpha)}(t, \tau) e^{-j2\pi f \tau} d\tau, \]

where

\[ h^{(\alpha)}(t, \tau) = h\left(t + \frac{1 - \alpha}{2}, \tau - \frac{1 + \alpha}{2}\right) \]

with \(\alpha\) a real-valued parameter. For \(\alpha = 0, 1/2,\) and \(-1/2,\) the GWS reduces to respectively the Weyl symbol \cite{Kozek1, Kozek2, Zadeh1, Nirenberg1}, Zadeh's time-varying transfer function \cite{Zadeh2}, and the Kohn-Nirenberg symbol \cite{KohnNirenberg1, KohnNirenberg2} (equivalently Bello's frequency-dependent modulation function \cite{Bello}). For LTI and LFI systems, the GWS simplifies to the spectral and temporal transfer function, respectively.

Our results extend the pioneering work of W. Kozek who developed a GWS-based approximate transfer function calculus for a class of "underspread" LTV systems whose spreading function (see Section 2) has compact support of area \(\ll 1\) \cite{Kozek3, Kozek4}. In Section 2 of this paper, we shall extend the concept of underspread systems using weighted integrals and moments of the spreading function. Subsequently, in Section 3 we will employ these integrals/moments to formulate explicit bounds on the errors incurred by the transfer function approximation, thereby extending the GWS-based transfer function calculus to a significantly wider and practically more relevant class of LTV systems than that considered in \cite{Kozek3, Kozek4}. We note that a different approach to related topics is taken in the theory of pseudo-differential operators \cite{Hörmander1, Hörmander2, Hörmander3}.

2 EXTENDED CONCEPT OF UNDERSPREAD SYSTEMS

In contrast to LTI or LFI systems (which cause only time or frequency shifts, respectively), general LTV systems shift the input signal with respect to both time and frequency. Indeed, the output signal in (1) can be written as \cite{Zadeh1, Zadeh2, Kozek2, Kozek3, Kozek4}

\[(Hz)(t) = \int \int c^{(\alpha)}_{H}(\tau, \nu) x^{(\alpha)}(t) d\tau d\nu.\]

Here, \[c^{(\alpha)}_{H}(\tau, \nu) = a(t - \tau) e^{i2\pi \nu t} e^{i\phi(\alpha, \nu)}\] is the signal \(z(t)\) shifted by \(\tau\) in time and by \(\nu\) in frequency, with the parameter \(\alpha \in \mathbb{R}\) expressing a freedom in defining joint time-frequency (TF) shifts, and \[c^{(\alpha)}_{H}(\tau, \nu)\] is the generalized spreading function (GSF) of \[H\], defined as \cite{Zadeh1, Zadeh2, Kozek2, Kozek3, Kozek4}

\[ c^{(\alpha)}_{H}(\tau, \nu) \triangleq \int h^{(\alpha)}(t, \tau) e^{-j2\pi \nu t} dt, \]
with \( h^{(0)}(\tau, \tau) \) as in (3). The GSF is the 2-D Fourier transform of the GWS in (2). It can be shown that \( S^{(0)}(\tau, \nu) = S^{(0)}(\tau, \nu) e^{2\pi i (\alpha_1 - \alpha_2)} \) so that \( \left| S^{(0)}(\tau, \nu) \right|^2 = \left| S^{(0)}(\tau, \nu) \right|^2 \). Hence, we will write \( |S_{H}(\tau, \nu)| \) instead of \( |S^{(0)}(\tau, \nu)|. \)

Conceptually, an LTV system is underspread if its GSF is concentrated in a small region about the origin of the \((\tau, \nu)\)-plane, which indicates that the system introduces only small TF shifts \( \tau, \nu \). In [10, 11], the GSF of an underspread system was required to be exactly zero outside a small support region about the origin. In practice, however, this condition is often not satisfied exactly but only effectively. This poses the problem of how to choose the effective support region and how the resulting modeling error affects the validity of the results based on the finite support model.

To circumvent these problems, we here propose to characterize an underspread system by means of the following (\( \alpha \)-independent) normalized weighted GSF integrals:

\[
\begin{align*}
\mu_H^{(\phi)} & \triangleq \frac{1}{\|S_H\|_1} \int \int \phi(\tau, \nu) |S_H(\tau, \nu)| \, d\tau \, d\nu, \\
\mu_H^{(\psi)} & \triangleq \frac{1}{\|H\|_2^2} \int \int \phi(\tau, \nu) |S_H(\tau, \nu)|^2 |\kappa|^{1/2} \, d\tau \, d\nu.
\end{align*}
\]

Here, \( \phi(\tau, \nu) \) with \( \phi(\tau, \nu) \geq \phi(0,0) = 0 \) is a weighting function which penalizes GSF contributions that are far away from the origin. Fig. 1 shows some weighting functions: (a) \( \phi(\tau, \nu) = |\nu|^4 \), (b) \( \phi(\tau, \nu) = |\nu|^6 \), (c) \( \phi(\tau, \nu) = |\nu|^8 \), (d) \( \phi(\tau, \nu) = [1 - A^{(0)}(\tau, \nu)]^2 \) with \( A^{(0)}(\tau, \nu) \) the ambiguity function of a normalized Gaussian function (cf. Subsection 3.4). We note that darker shades correspond to larger values.

The parameter \( \alpha \). However, this dependence is bounded according to the following theorem.

**Theorem 3.1** For any LTV system \( H \), the difference \( \Delta_2(t, f) = L_H^{(\alpha_2)}(t, f) - L_H^{(\alpha_1)}(t, f) \) between two GWSs with parameters \( \alpha_1 \) and \( \alpha_2 \) is bounded as

\[
\frac{\|\Delta_2(t, f)\|_{L_1}}{\|S_H\|_1} \leq 2\pi\alpha_1 \alpha_2 |\mu_H^{(\phi)}|^{1/2}, \quad \frac{\|\Delta_2(t, f)\|_{L_2}}{\|H\|_2} \leq 4\pi\alpha_1 \alpha_2 |\mu_H^{(\psi)}|^{1/2}.
\]

For an underspread system whose moments \( \mu_H^{(\phi)} \) and \( \mu_H^{(\psi)} \) will be small, these bounds show that the GWS is approximately independent of \( \alpha \), i.e.,

\[
\frac{L_H^{(\alpha_1)}(t, f)}{L_H^{(\alpha_2)}(t, f)} \approx L_H^{(\alpha_1)}(t, f).
\]

Hence, the TF transfer function of an underspread system is approximately unique. We note that small \( \mu_H^{(\phi)} \) or \( \mu_H^{(\psi)} \) requires the GSF to be concentrated along the \( \tau \) or \( \nu \) axes, i.e., not oriented in oblique directions.

### 3.2 Adjoint Systems

The transfer function of the adjoint [1] of an LTI system is \( G^*(f) \), and similarly for an LFI system. In contrast, the GWS of the adjoint \( H^* \) of an LTV system \( H \) is \( L_H^{(\alpha_2)}(t, f) \), which does not equal \( L_H^{(\alpha_1)}(t, f) \) unless \( \alpha = 0 \). However, Theorem 3.1 leads to the following bounds.

**Corollary 3.2** For any LTV system \( H \), the difference \( \Delta_2(t, f) = L_H^{(\alpha_2)}(t, f) - L_H^{(\alpha_1)}(t, f) \) is bounded as

\[
\frac{\|\Delta_2(t, f)\|_{L_1}}{\|S_H\|_1} \leq 4\pi\alpha_1 \alpha_2 |\mu_H^{(\phi)}|^{1/2}, \quad \frac{\|\Delta_2(t, f)\|_{L_2}}{\|H\|_2} \leq 4\pi\alpha_1 \alpha_2 |\mu_H^{(\psi)}|^{1/2}.
\]

For an underspread system, these bounds show that

\[
\frac{L_H^{(\alpha_1)}(t, f)}{L_H^{(\alpha_2)}(t, f)} \approx L_H^{(\alpha_1)}(t, f).
\]

Unlike in the LTI or LFI case, the GWS of a self-adjoint (Hermitian) system (i.e., a system satisfying \( H^* = H \)) is not real-valued for \( \alpha \neq 0 \). Corollary 3.2 implies that the imaginary part of the GWS of a self-adjoint system, \( I(t, f) = \frac{L_H^{(\alpha_1)}(t, f)}{L_H^{(\alpha_2)}(t, f)} \), is
\[ \frac{1}{2} [L^{(\alpha)}_{H^+}(t,f) - L^{(\alpha)}_{H^-}(t,f)] = \frac{1}{2} [L^{(\alpha)}_{H^+}(t,f) - L^{(\alpha)}_{H^-}(t,f)] = \frac{1}{2} \Delta_2(t,f), \]

is bounded as

\[ \frac{[\Delta_2(t,f)]}{||S_{H^+}||} \leq 2\pi|\alpha|m_{H^-}^{(1,1)}, \quad \frac{[\Delta_2(t,f)]}{||S_{H^-}||} \leq 2\pi|\alpha|m_{H^+}^{(1,1)}. \]

Hence, the GWS of an underspread, self-adjoint system is approximately real-valued even if \( \alpha \neq 0 \).

### 3.3 Composition of Systems

The transfer function of the series connection (composition) of two LTI systems \( H_1 \) and \( H_2 \) equals the product of the individual transfer functions, \( G \frac{f}{G_2}(f) \), and similarly for LFI systems. In contrast, the GWS of the composition \( H_2 H_1 \) of two general LTV systems is not equal to the product of the individual GWSs of \( H_1 \) and \( H_2 \).

**Theorem 3.3** For any two LTV systems \( H_1, H_2 \), the error \( \Delta_3(t,f) = L^{(\alpha)}_{H_2H_1}(t,f) - L^{(\alpha)}_{H_1}(t,f) \) is bounded as

\[ \| \Delta_3(t,f) \|_{S_{H_1}} \|_{S_{H_2}} \leq 2\pi B^{(\alpha)}_{H_1H_2}, \quad (4) \]

with

\[ B^{(\alpha)}_{H_1H_2} \triangleq \alpha + \frac{1}{2} m_{H_1}^{(0,1)} m_{H_2}^{(1,0)} + \alpha - \frac{1}{2} m_{H_1}^{(1,0)} m_{H_2}^{(0,1)}. \]

The theorem shows that if \( H_1 \) and \( H_2 \) are such that \( m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} \) and \( m_{H_1}^{(0,1)} m_{H_2}^{(1,0)} \) are both small, we have

\[ L^{(\alpha)}_{H_2H_1}(t,f) \approx L^{(\alpha)}_{H_1}(t,f) L^{(\alpha)}_{H_2}(t,f). \]

Small \( m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} \) and \( m_{H_1}^{(0,1)} m_{H_2}^{(1,0)} \) requires that the GSFs of \( H_1 \) and \( H_2 \) are both concentrated about the origin of the \((\tau, \nu)\)-plane, with similar orientation parallel to the \( \tau \) or \( \nu \) axis. That is, \( |S_{H_1}(\tau, \nu)| \) and \( |S_{H_2}(\tau, \nu)| \) may not be oriented in oblique directions or significantly different directions. For \( \alpha = 0 \), the bound in (4) simplifies since \( B^{(\alpha)}_{H_1H_2} = \frac{1}{2} (m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} + m_{H_1}^{(0,1)} m_{H_2}^{(1,0)}) \). It can here be shown that (4) remains valid if \( B^{(\alpha)}_{H_1H_2} \) is replaced by \( B^{(\alpha)}_{H_1H_2} \) with \( H_1 = U H_1 U^+ \) and \( H_2 = U H_2 U^+ \), where \( U \) is any unitary operator corresponding to a rotation or some other symplectic coordinate transform of the \((\tau, \nu)\)-plane [6]. Thus, for \( \alpha = 0 \), \( \Delta_3(t,f) \) may be small even if the GSFs of \( H_1 \) and \( H_2 \) are oriented in (similar) oblique directions. For \( \alpha = 1/2 \), we obtain \( B^{(\alpha)}_{H_1H_2} = m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} \), which may be small even if \( |S_{H_1}(\tau, \nu)| \) is located along the \( \tau \) axis and \( |S_{H_2}(\tau, \nu)| \) is located along the \( \nu \) axis. Similarly, for \( \alpha = -1/2 \) we have \( B^{(\alpha)}_{H_1H_2} = m_{H_1}^{(0,1)} m_{H_2}^{(1,0)} \), which may be small even if \( |S_{H_1}(\tau, \nu)| \) is located along the \( \nu \) axis and \( |S_{H_2}(\tau, \nu)| \) is located along the \( \tau \) axis.

Of particular interest are the cases of \( \alpha \) equal to 0 or 1. \( \Delta_3(t,f) \) may be small even if the GSFs of \( H_1 \) and \( H_2 \) are oriented in (similar) oblique directions. For \( \alpha = 0 \), the bound in (4) simplifies since \( B^{(\alpha)}_{H_1H_2} = \frac{1}{2} (m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} + m_{H_1}^{(0,1)} m_{H_2}^{(1,0)}) \). It can here be shown that (4) remains valid if \( B^{(\alpha)}_{H_1H_2} \) is replaced by \( B^{(\alpha)}_{H_1H_2} \) with \( H_1 = U H_1 U^+ \) and \( H_2 = U H_2 U^+ \), where \( U \) is any unitary operator corresponding to a rotation or some other symplectic coordinate transform of the \((\tau, \nu)\)-plane [6]. Thus, for \( \alpha = 0 \), \( \Delta_3(t,f) \) may be small even if the GSFs of \( H_1 \) and \( H_2 \) are oriented in (similar) oblique directions. For \( \alpha = 1/2 \), we obtain \( B^{(\alpha)}_{H_1H_2} = m_{H_1}^{(1,0)} m_{H_2}^{(0,1)} \), which may be small even if \( |S_{H_1}(\tau, \nu)| \) is located along the \( \tau \) axis and \( |S_{H_2}(\tau, \nu)| \) is located along the \( \nu \) axis. Similarly, for \( \alpha = -1/2 \) we have \( B^{(\alpha)}_{H_1H_2} = m_{H_1}^{(0,1)} m_{H_2}^{(1,0)} \), which may be small even if \( |S_{H_1}(\tau, \nu)| \) is located along the \( \nu \) axis and \( |S_{H_2}(\tau, \nu)| \) is located along the \( \tau \) axis.

Of particular interest are the cases of \( \alpha \) equal to 0 or 1.
with the weighting function $\phi_\alpha(t, \nu) = \sqrt{1 - \text{Re}(A^{(0)}_\alpha(t, \nu))}$ and $D^{(0)}_{\alpha, 0}$ as defined in (6).

Hence, if $L^{(0)}_{H, 0}$ and $M^{(0)}_{H, 0}$ can be made small by suitable choice of $w(t)$, we obtain the approximate input-output relation

$$\text{STFT}_L^{(0)}(t, f) \approx L^{(0)}_{H, 0}(t, f) \text{STFT}_{L, w}(t, f).$$

Small $M^{(0)}_{H, 0}$ requires that $\text{Re}(A^{(0)}_{\alpha, 0}(t, \nu)) \approx A^{(0)}_{\alpha, 0}(0, 0) \equiv 1$ on the effective support of $[S_H(t, \nu) + I]$, thus implying that this effective support is small, i.e., that $H$ is underspread.

### 3.6 Minimum and Maximum Gain

The minimum and maximum system gain are defined as

$$\gamma_H \triangleq \inf_a \frac{\|Hx\|_2}{\|x\|_2}, \quad \Gamma_H \triangleq \sup_a \frac{\|Hx\|_2}{\|x\|_2}.$$ 

For LTI and LFI systems, $\gamma_H$ and $\Gamma_H$ equal the infimum and supremum, respectively, of the magnitude of the transfer function. The squared magnitude of the transfer function of $H$ on the effective support of $I$ is given by

$$[S_H(t, \nu) + I^2]$$

Small $M^{(0)}_{H, 0}$ requires that $\text{Re}(A^{(0)}_{\alpha, 0}(t, \nu)) \approx A^{(0)}_{\alpha, 0}(0, 0) \equiv 1$ on the effective support of $[S_H(t, \nu) + I]$, thus implying that this effective support is small, i.e., that $H$ is underspread.

For any LTV system $H$, we restrict to $\alpha = 0$ since $L^{(0)}_{H, 0}(t, f)$ is real-valued, and we consider

$$\frac{L^{(0)}_{H, 0}(t, f)}{\|S_H(t, f)\|_2} \leq \frac{\|L^{(0)}_{H, 0}(t, f)\|_2}{\|S_H(t, f)\|_2} \leq m^{(0)}_{H, 0}$$

with the weighting function $\phi_0(t, \nu) = 1 - \frac{1}{\text{Re}(A^{(0)}_0(t, \nu))}$, where $s(t)$ is an arbitrary normalized function.

Hence, if $m^{(0)}_{H, 0}$ can be made small by suitable choice of the function $s(t)$, we have

$$L^{(0)}_{H, 0}(t, f) \approx \gamma_H,$$

Small $m^{(0)}_{H, 0}$ requires that $A^{(0)}_{\alpha, 0}(t, \nu) \approx A^{(0)}_{\alpha, 0}(0, 0) \equiv 1$ on the effective support of $[S_H(t, \nu) + I]$, thus implying that the effective support of $[S_H(t, \nu) + I]$, is small, i.e., that $H$ is underspread. In that case, we also have

$$L^{(0)}_{H, 0}(t, f) \approx \gamma_H,$$

Due to the approximate uniqueness of the GWS (cf. Subsection 3.1), this approximation will also hold for $\alpha \neq 0$.

### 4 CONCLUSION

We have introduced an extended class of “underspread” linear time-varying systems. For this type of systems, the generalized Weyl symbol is an approximate time-frequency transfer function that is similarly simple to use as the conventional transfer function of linear time-invariant systems. We have provided quantitative bounds on the errors incurred by this approximate transfer function calculus. These bounds are based on weighted integrals and moments of the generalized spreading function and do not require the generalized spreading function to have finite support.

---

**APPENDIX: PROOF OF THEOREM 3.1**

Using $S^{(0)}_H(t, \nu) = S^{(0)}_H(t, \nu) e^{i2\pi \Delta \alpha t \nu}$ where $\Delta \alpha = \alpha_1 - \alpha_2$, the Fourier transform of $\Delta_1(t, \nu)$ is given by

$$\Delta_1(t, \nu) = S^{(0)}_H(t, \nu) e^{i2\pi \Delta \alpha t \nu}.$$ 

The first bound is then shown as

$$|\Delta_1(t, \nu)| \leq \|\Delta_1\|_1 = \int_{-\nu/2}^{\nu/2} \|S_H(t, \nu)| - e^{i2\pi \Delta \alpha t \nu} |d\nu$$

$$\leq 2\|S_H(t, \nu)| |\sin(\pi \Delta \alpha t \nu)| d\nu$$

$$\leq 2\|\Delta_1\|_1 \|S_H(t, \nu)| |\nu| d\nu$$

Using $|\Delta_1|_1 = \|\Delta_1\|_1$ and $\sin^2 x \leq x^2$, the second bound can be shown in a similar manner.

**REFERENCES**


