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# Oversampled Cosine Modulated Filter Banks with Linear Phase*

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*Abstract*— **We introduce *oversampled* cosine modulated filter banks (CMFBs) and a new classification of (oversampled or critically sampled) CMFBs as odd-stacked and even-stacked. We propose the new class of *even-stacked* CMFBs which allows both perfect reconstruction (PR) and linear phase filters in all channels. We formulate conditions for PR and show that any PR CMFB corresponds to a PR DFT filter bank with twice the oversampling factor. We also show that the frame-theoretic properties of a CMFB and of the corresponding DFT filter bank are closely related.**

## I. Introduction

Recent interest in *oversampled* filter banks (FBs) [1]-[5] is mainly due to their increased design freedom and noise immunity [1],[3],[4]. Oversampled DFT FBs [6],[1],[3],[7] and oversampled cosine modulated FBs (CMFBs) [7],[8] allow efficient FFT- or DCT-based implementations. Here, CMFBs are advantageous as their subband signals are real if the input signal and the analysis prototype are real.

This paper introduces and studies *oversampled* CMFBs with perfect reconstruction (PR). Section II proposes a new classification[1] (odd-stacked/even-stacked) of oversampled or critically sampled CMFBs. The traditional CMFBs (previously defined for critical sampling only) [10]-[16] are of the *odd-stacked* type; their channel filters do not have linear phase even if the prototypes have linear phase. In contrast, the new class of *even-stacked* CMFBs introduced[1] in Section II allows both PR/paraunitarity and linear phase filters in all channels; it contains the recently proposed Lin-Vaidyanathan CMFBs [17] and Wilson CMFBs [8] as special cases. Section III provides PR conditions and shows that PR CMFBs are associated to PR DFT FBs with twice the oversampling factor. Finally, Section IV shows that the frame bound ratio of a CMFB corresponding to a frame expansion [18],[1],[4] equals that of the associated DFT FB.

**DFT FBs.** For later use, we review DFT (or complex modulated) FBs [6],[3],[7],[19] with $N$ channels and decimation factor $M$. The FB is critically sampled for $N=M$ and oversampled for $N>M$. In an *odd-stacked* DFT FB [6], the analysis and synthesis filter transfer functions are $H_k^{\text{DFT-o}}(z) = H\big(zW_N^{k+1/2}\big)$ and $F_k^{\text{DFT-o}}(z) = F\big(zW_N^{k+1/2}\big)$ $(k = 0,1,..,N-1)$, respectively, with $W_N = e^{-j2\pi/N}$. The corresponding impulse responses are $h_k^{\text{DFT-o}}[n] = h[n]\,W_N^{-(k+1/2)n}$ and $f_k^{\text{DFT-o}}[n] = f[n]\,W_N^{-(k+1/2)n}$. In an *even-stacked* DFT FB [6], the transfer functions are $H_k^{\text{DFT-e}}(z) = H\big(zW_N^{k}\big)$ and $F_k^{\text{DFT-e}}(z) = F\big(zW_N^{k}\big)$ for $k = 0,1,...,N-1$, and the impulse responses are $h_k^{\text{DFT-e}}[n] = h[n]\,W_N^{-kn}$ and $f_k^{\text{DFT-e}}[n] = f[n]\,W_N^{-kn}$. In both cases, $h[n] \leftrightarrow H(z)$ and $f[n] \leftrightarrow F(z)$ denote the analysis and synthesis prototypes, respectively.

A DFT FB satisfying PR with zero delay (i.e., $\hat{x}[n] = x[n]$, where $x[n]$ and $\hat{x}[n]$ denote the input and reconstructed signal, respectively) can be shown to correspond to the following expansion of the input signal,

$$x[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{\text{DFT}} \rangle\, f_{k,m}^{\text{DFT}}[n]\,. \tag{1}$$

Here, $h_{k,m}^{\text{DFT}}[n] = h^*[mM-n]\,W_N^{-(k+1/2)(n-mM)}$ and $f_{k,m}^{\text{DFT}}[n] = f[n-mM]\,W_N^{-(k+1/2)(n-mM)}$ in the odd-stacked case and $h_{k,m}^{\text{DFT}}[n] = h^*[mM-n]\,W_N^{-k(n-mM)}$ and $f_{k,m}^{\text{DFT}}[n] = f[n-mM]\,W_N^{-k(n-mM)}$ in the even-stacked case.

## II. Oversampled Cosine Modulated Filter Banks

We now introduce two different types of oversampled CMFBs, corresponding to a novel classification[1] of CMFBs as odd-stacked and even-stacked. A close relation to the odd-stacked/even-stacked classification of DFT FBs will be shown in Sections III and IV.

**Odd-stacked CMFBs.** *Odd-stacked* CMFBs are the traditional CMFB type previously defined for critical sampling [10]-[16]. In the general case of an odd-stacked CMFB with $N$ channels and decimation factor $M$ (note that the CMFB is oversampled for $N > M$), we define the analysis and synthesis filters respectively as

$$\begin{aligned} h_k^{\text{CM-o}}[n] &= \sqrt{2}\,h[n]\,\cos\Big(\frac{(k+1/2)\pi}{N}n + \phi_k^{\text{o}}\Big), \\ f_k^{\text{CM-o}}[n] &= \sqrt{2}\,f[n]\,\cos\Big(\frac{(k+1/2)\pi}{N}n - \phi_k^{\text{o}}\Big) \end{aligned}$$

for $k = 0,1,...,N-1$. Here, $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, and the phases are defined as $\phi_k^{\text{o}} = -\alpha\frac{\pi}{2N}(k+1/2) + r\frac{\pi}{2}$ with $\alpha \in \mathbb{Z}$ and $r \in \{0,1\}$ (this extends the phase definition given in [13] for the special case of critical sampling). Note that the channel frequencies are $\theta_k = \frac{k+1/2}{2N}$; in particular, the channel with index $k=0$ is centered at frequency $\theta_0 = \frac{1}{4N}$, as depicted in Fig. 1(a). A disadvantage of odd-stacked CMFBs is that the channel filters do not have linear phase even if the prototypes have linear phase [10]-[16].

**Even-stacked CMFBs.** We now introduce the new class[1] of *even-stacked* CMFBs allowing both PR and linear

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[1] After completion of this manuscript, we realized that for the special case of *critical sampling* this classification and, in particular, the class of even-stacked CMFBs had previously been introduced in [9].

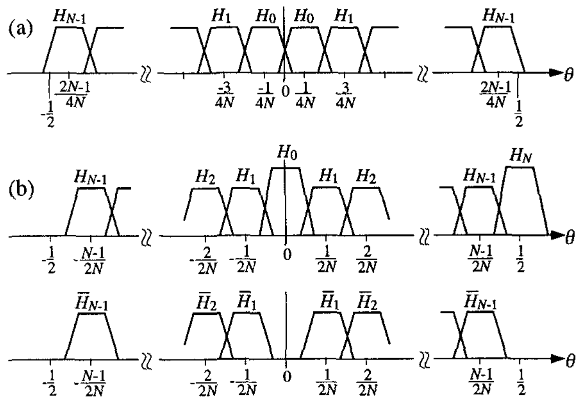


*Figure 1. Transfer functions of the channel filters in (a) an N-channel odd-stacked CMFB and (b) a 2N-channel even-stacked CMFB.*

phase filters in all channels. The analysis FB in an even-stacked CMFB with $2N$ channels and decimation factor $2M$ (the CMFB is oversampled for $N > M$) consists of two partial FBs $\{h_k^{\text{CM-e}}[n]\}_{k=0,..,N}$ and $\{\bar{h}_k^{\text{CM-e}}[n]\}_{k=1,..,N-1}$ derived from an analysis prototype $h[n]$ as

$$h_k^{\text{CM-e}}[n] = \begin{cases} h[n-rM], & k=0 \\ \sqrt{2}\, h[n]\, \cos\!\left(\frac{k\pi}{N} n + \phi_k^{\text{e}}\right), & k=1,..,N-1 \\ h[n-sM]\,(-1)^{n-sM}, & k=N \end{cases}$$

and

$$\bar{h}_k^{\text{CM-e}}[n] = \sqrt{2}\, h[n-M]\, \sin\!\left(\frac{k\pi}{N}(n-M) + \phi_k^{\text{e}}\right)$$

for $k=1,..,N-1$. Similarly, the synthesis FB consists of partial FBs $\{f_k^{\text{CM-e}}[n]\}_{k=0,..,N}$ and $\{\bar{f}_k^{\text{CM-e}}[n]\}_{k=1,..,N-1}$ defined in terms of a synthesis prototype $f[n]$ as

$$f_k^{\text{CM-e}}[n] = \begin{cases} f[n+rM], & k=0 \\ \sqrt{2}\, f[n]\, \cos\!\left(\frac{k\pi}{N} n - \phi_k^{\text{e}}\right), & k=1,..,N-1 \\ f[n+sM]\,(-1)^{n+sM}, & k=N \end{cases}$$

and

$$\bar{f}_k^{\text{CM-e}}[n] = -\sqrt{2}\, f[n+M]\, \sin\!\left(\frac{k\pi}{N}(n+M) - \phi_k^{\text{e}}\right)$$

for $k=1,..,N-1$. We define the phases as $\phi_k^{\text{e}} = -\alpha\frac{\pi}{2N}k + r\frac{\pi}{2}$ with $\alpha \in \mathbb{Z}$; furthermore $r,s \in \{0,1\}$ with $s=r$ for $\alpha$ even and $s=1-r$ for $\alpha$ odd. Note that there are $2N$ channels but only $N+1$ different channel frequencies $\theta_k = \frac{k}{2N}$ $(k=0,...,N)$, as depicted in Fig. 1(b). In particular, the $k=0$ channel is centered at frequency $\theta_0 = 0$. For *any* choice of $\alpha \in \mathbb{Z}$ and $r \in \{0,1\}$, all filters have linear phase if the prototypes have linear phase.

Two special even-stacked CMFBs are the CMFB recently introduced (for critical sampling) by Lin and Vaidyanathan (LV) in [17] and the Wilson-type (WI) CMFB (corresponding to the discrete-time Wilson expansion [20]) recently introduced by the authors in [8]. The parameters of these two even-stacked CMFBs and of two variants (abbreviated LV' and WI') are summarized in Table 1. In particular, the analysis filters of an LV CMFB are

$$h_k^{\text{LV}}[n] = \begin{cases} h[n], & k=0 \\ \sqrt{2}\, h[n]\, \cos\!\left(\frac{k\pi}{N} n\right), & k=1,..,N-1 \\ h[n]\,(-1)^n, & k=N, \end{cases}$$

$$\bar{h}_k^{\text{LV}}[n] = \sqrt{2}\, h[n-M]\, \sin\!\left(\frac{k\pi}{N}(n-M)\right), \quad k=1,..,N-1,$$

and the analysis filters of a WI FB are

$$h_k^{\text{WI}}[n] = \begin{cases} h[n], & k=0 \\ \sqrt{2}\, h[n]\, \cos\!\left(\frac{k\pi}{N}\left(n-\frac{N}{2}\right)\right), & k=1,..,N-1 \\ h[n-sM]\,(-1)^{n-sM}, & k=N, \end{cases}$$

$$\bar{h}_k^{\text{WI}}[n] = \sqrt{2}\, h[n-M]\, \sin\!\left(\frac{k\pi}{N}\left(n-M-\frac{N}{2}\right)\right), \quad k=1,..,N-1$$

with $s=0$ for $N$ even and $s=1$ for $N$ odd. Similar expressions exist for the synthesis filters.

| FB | $\alpha$ | $r$ | $s$ |
|---|---|---|---|
| LV | 0 | 0 | 0 |
| LV' | 0 | 1 | 1 |
| WI | $N$ | 0 | 0 (1) for $N$ even (odd) |
| WI' | $N$ | 1 | 1 (0) for $N$ even (odd) |

*Table 1. Parameters of special even-stacked CMFBs.*

## III. Perfect Reconstruction Conditions

We shall next provide PR conditions for oversampled and critically sampled CMFBs. For both odd-stacked and even-stacked CMFBs, the following decomposition of the reconstructed signal can be shown [21],

$$\hat{x}[n] = \frac{1}{2}\left[(\mathbf{S}_{\text{DFT}}^{(h,f)} x)[n] + (\mathbf{T}_{\text{DFT}}^{(h,f)} x)[n]\right], \tag{2}$$

where the operators $\mathbf{S}_{\text{DFT}}^{(h,f)}$ and $\mathbf{T}_{\text{DFT}}^{(h,f)}$ are defined as

$$(\mathbf{S}_{\text{DFT}}^{(h,f)} x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{\text{DFT}} \rangle\, f_{k,m}^{\text{DFT}}[n],$$

$$(\mathbf{T}_{\text{DFT}}^{(h,f)} x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \langle x, h_{k,m}^{\text{DFT}} \rangle\, \hat{f}_{k,m}^{\text{DFT}}[n].$$

Here, $h_{k,m}^{\text{DFT}}[n]$ and $f_{k,m}^{\text{DFT}}[n]$ are defined as before, $\hat{f}_{k,m}^{\text{DFT}}[n] = f_{2N-k-1,m}^{\text{DFT}}[n]$, $\phi_k = \phi_k^{\text{o}}$, and $c_m = 1$ for an odd-stacked CMFB, and $\hat{f}_{k,m}^{\text{DFT}}[n] = f_{2N-k,m}^{\text{DFT}}[n]$, $\phi_k = \phi_k^{\text{e}}$, and $c_m = (-1)^m$ for an even-stacked CMFB. Note that $(\mathbf{S}_{\text{DFT}}^{(h,f)} x)[n]$ is the DFT FB expansion in (1) with $N$ replaced by $2N$.

For PR with zero delay, $\hat{x}[n] = x[n]$, it is necessary and sufficient [21] that

$$\mathbf{S}_{\text{DFT}}^{(h,f)} = 2\mathbf{I} \qquad \text{and} \qquad \mathbf{T}_{\text{DFT}}^{(h,f)} = \mathbf{0}, \tag{3}$$

where $\mathbf{I}$ and $\mathbf{0}$ denote the identity and zero operator, respectively. The operator $\mathbf{S}_{\text{DFT}}^{(h,f)}$ can be expressed as

$$(\mathbf{S}_{\text{DFT}}^{(h,f)} x)[n] = 2N \sum_{l=-\infty}^{\infty} d_l\, x[n-2lN] \sum_{m=-\infty}^{\infty} f[n-mM] \cdot h[-n+mM+2lN] \tag{4}$$

with $d_l = (-1)^l$ in the odd-stacked case and $d_l = 1$ in the even-stacked case. Hence, the first PR condition in (3), $\mathbf{S}_{\mathrm{DFT}}^{(h,f)} = 2\mathbf{I}$, is satisfied if and only if

$$\sum_{m=-\infty}^{\infty} f[n-mM]\, h[-n+mM+2lN] = \frac{1}{N}\,\delta[l]\,. \quad (5)$$

Note that this condition is independent of the stacking type. The operator $\mathbf{T}_{\mathrm{DFT}}^{(h,f)}$ can be expressed as

$$\left(\mathbf{T}_{\mathrm{DFT}}^{(h,f)}x\right)[n] = (-1)^r\, 2N \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} d_l\, a_m \quad (6)$$
$$\cdot\, x[-n+2mM-2lN-\alpha]\, f[n-mM]\, h[n-mM+\alpha+2lN]$$

with $d_l$ as above, $a_m = 1$ in the odd-stacked case, and $a_m = (-1)^m$ in the even-stacked case. For integer oversampling, i.e., $N = KM$ with $K \in \mathbb{N}$, it can be shown that the second PR condition in (3), $\mathbf{T}_{\mathrm{DFT}}^{(h,f)} = \mathbf{0}$, is satisfied if and only if

$$\sum_{i=-\infty}^{\infty} b_i\, f[n-iKM]\, h[n+iKM+\alpha] = 0, \quad (7)$$

where $b_i = (-1)^i$ in the odd-stacked case and $b_i = (-1)^{iK}$ in the even-stacked case. Note that critical sampling, $N = M$, is a special case with $K = 1$. For the more general case of rational oversampling, time-domain formulations of $\mathbf{T}_{\mathrm{DFT}}^{(h,f)} = \mathbf{0}$ are provided in [21].

It is important to note that (5) is the PR condition for a DFT FB with $2N$ channels and decimation factor $M$ [6]. Thus, if $\mathbf{T}_{\mathrm{DFT}}^{(h,f)} = \mathbf{0}$ (or, for integer oversampling, the equivalent condition (7)) is satisfied, the PR condition for an odd-stacked CMFB with $N$ channels and decimation factor $M$, or for an even-stacked CMFB with $2N$ channels and decimation factor $2M$, reduces to the PR condition for a DFT FB with $2N$ channels and decimation factor $M$; this PR condition is *the same* for odd-stacked and even-stacked CMFBs. Note that the oversampling factor of the DFT FB is twice that of the corresponding CMFB.

Moreover, for $\mathbf{T}_{\mathrm{DFT}}^{(h,f)} = \mathbf{0}$ the CMFB's input-output relation (2) reduces to $\hat{x}[n] = \frac{1}{2}\left(\mathbf{S}_{\mathrm{DFT}}^{(h,f)}x\right)[n]$, which is the input-output relation of a DFT FB with $2N$ channels and decimation factor $M$. This DFT FB is odd-stacked (even-stacked) for an odd-stacked (even-stacked) CMFB. Thus, we conclude that CMFBs correspond to DFT FBs of the *same* stacking type and with twice the oversampling factor. In particular, it can be shown that paraunitary CMFBs correspond to paraunitary DFT FBs (here, $f[n] = h^*[-n]$).

## IV. Frame Properties

Any PR FB corresponds to an expansion of the input signal $x[n]$ into a set of "synthesis functions" $f_{k,m}[n]$ [1], [4]. These synthesis functions are called a *frame* in $l^2(\mathbb{Z})$, the space of square-summable discrete-time signals, if

$$A\|x\|^2 \le \sum_{k=0}^{N'-1}\sum_{m=-\infty}^{\infty} \left|\langle x, f_{k,m}\rangle\right|^2 \le B\|x\|^2 \quad \forall\, x[n] \in l^2(\mathbb{Z})$$

with the *frame bounds* $A > 0$ and $B < \infty$ [18] ($N'$ denotes the number of channels). The frame bound ratio $B/A$ characterizes important numerical properties of the signal expansion [18], and thus also of the corresponding FB. The above frame condition can also be written as

$$A\|x\|^2 \le \langle \mathbf{S}x, x\rangle \le B\|x\|^2\,,$$

where $\mathbf{S}$ is the *frame operator* defined as

$$(\mathbf{S}x)[n] = \sum_{k=0}^{N'-1}\sum_{m=-\infty}^{\infty} \langle x, f_{k,m}\rangle\, f_{k,m}[n]\,.$$

The frame bounds $A$ and $B$ are the infimum and supremum, respectively, of the eigenvalues of $\mathbf{S}$.

In particular, the frame operator of a CMFB is given by

$$(\mathbf{S}_{\mathrm{CM}}x)[n] = \sum_{k=0}^{N'-1}\sum_{m=-\infty}^{\infty} \langle x, f_{k,m}^{\mathrm{CM}}\rangle\, f_{k,m}^{\mathrm{CM}}[n]$$

where in the odd-stacked case $N' = N$ and $f_{k,m}^{\mathrm{CM}}[n] = f_k^{\mathrm{CM\text{-}o}}[n-mM]$, and in the even-stacked case $N' = N+1$ and

$$f_{k,m}^{\mathrm{CM}}[n] = \begin{cases} f_k^{\mathrm{CM\text{-}e}}[n-2\mu M], & m = 2\mu,\ \ k = 0,..,N \\ \tilde{f}_k^{\mathrm{CM\text{-}e}}[n-2\mu M], & m = 2\mu-1,\ \ k = 1,..,N-1. \end{cases}$$

It can be shown [21] that the frame operator of both odd-stacked and even-stacked CMFBs can be decomposed as

$$\mathbf{S}_{\mathrm{CM}} = \frac{1}{2}\left(\mathbf{S}_{\mathrm{DFT}} + \mathbf{T}_{\mathrm{DFT}}\right),$$

where $\mathbf{S}_{\mathrm{DFT}}$ is the frame operator of a DFT FB with $2N$ channels and decimation factor $M$,

$$(\mathbf{S}_{\mathrm{DFT}}x)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty} \langle x, f_{k,m}^{\mathrm{DFT}}\rangle\, f_{k,m}^{\mathrm{DFT}}[n]\,,$$

and $\mathbf{T}_{\mathrm{DFT}}$ is defined as

$$(\mathbf{T}_{\mathrm{DFT}}x)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m\, \langle x, f_{k,m}^{\mathrm{DFT}}\rangle\, \hat{f}_{k,m}^{\mathrm{DFT}}[n]$$

with $f_{k,m}^{\mathrm{DFT}}[n]$, $\hat{f}_{k,m}^{\mathrm{DFT}}[n]$, $\phi_k$, and $c_m$ as defined in Section III. Note that $\mathbf{S}_{\mathrm{DFT}}$ is the frame operator of a DFT FB with the same stacking type and twice the oversampling factor of the CMFB. The operators $\mathbf{S}_{\mathrm{DFT}}$ and $\mathbf{T}_{\mathrm{DFT}}$ can be expressed as in (4) and (6), respectively with $h[n]$ replaced by $f^*[-n]$.

Frames corresponding to DFT FBs are called *Weyl-Heisenberg frames* [18]. The frame property provides a means for achieving PR. Specifically, if the set of synthesis functions $\{f_{k,m}^{\mathrm{DFT}}[n]\}$ derived from a synthesis prototype $f[n]$ is a Weyl-Heisenberg frame for $l^2(\mathbb{Z})$, then the PR analysis prototype with minimum energy (norm) is [18]

$$\tilde{h}[n] = \left(\mathbf{S}_{\mathrm{DFT}}^{-1} f\right)^*[-n]\,, \quad (8)$$

where $\mathbf{S}_{\mathrm{DFT}}^{-1}$ is the inverse of the frame operator $\mathbf{S}_{\mathrm{DFT}}$. The minimum norm analysis prototype $\tilde{h}[n]$ leads to a set of (minimum norm) PR analysis functions $\{\tilde{h}_{k,m}^{\mathrm{DFT}}[n]\}$ that is again a Weyl-Heisenberg frame, and thus corresponds to a PR analysis DFT FB. Note that the above holds for both even-stacked and odd-stacked DFT FBs. The following theorem [21] states an important relation between the frame and PR properties of DFT FBs and CMFBs.

**Theorem.** Let $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, in an odd-stacked CMFB with $N$ channels and decimation factor $M$, or in an even-stacked CMFB with $2N$ channels and decimation factor $2M$. Let $f[n]$ be such that it generates a Weyl-Heisenberg frame of the same stacking type in $l^2(\mathbb{Z})$, i.e.,

$$A_{\mathrm{DFT}}\|x\|^2 \leq \langle \mathbf{S}_{\mathrm{DFT}}x, x\rangle \leq B_{\mathrm{DFT}}\|x\|^2.$$

Furthermore, let $f[n]$ be such that $\mathbf{T}_{\mathrm{DFT}} = \mathbf{0}$. Then,

(i) the CMFB synthesis functions $\{f_{k,m}^{\mathrm{CM}}[n]\}$ are a frame in $l^2(\mathbb{Z})$ with frame bounds $A_{\mathrm{CM}} = A_{\mathrm{DFT}}/2$ and $B_{\mathrm{CM}} = B_{\mathrm{DFT}}/2$, i.e.,

$$\frac{A_{\mathrm{DFT}}}{2}\|x\|^2 \leq \langle \mathbf{S}_{\mathrm{CM}}x, x\rangle \leq \frac{B_{\mathrm{DFT}}}{2}\|x\|^2;$$

(ii) for $h[n] = 2\,\tilde{h}[n]$, where $\tilde{h}[n] = (\mathbf{S}_{\mathrm{DFT}}^{-1}f)^*[-n]$ denotes the minimum norm analysis prototype of the DFT FB, the analysis CMFB constructed from $h[n]$ is the PR analysis CMFB with minimum norm filters.

The following interpretations and conclusions apply:

- If $\mathbf{T}_{\mathrm{DFT}} = \mathbf{0}$ as assumed above, the CMFB becomes equivalent to a DFT FB, and therefore the design of a CMFB reduces to that of a DFT FB of the same stacking type and twice the oversampling factor.
- The CMFB frame bounds $A_{\mathrm{CM}} = A_{\mathrm{DFT}}/2$ and $B_{\mathrm{CM}} = B_{\mathrm{DFT}}/2$ are trivially related to the frame bounds $A_{\mathrm{DFT}}$ and $B_{\mathrm{DFT}}$ of the corresponding DFT FB. Since $B_{\mathrm{CM}}/A_{\mathrm{CM}} = B_{\mathrm{DFT}}/A_{\mathrm{DFT}}$, the CMFB inherits the numerical properties of the corresponding DFT FB.
- In particular, if the DFT FB is paraunitary ($A_{\mathrm{DFT}} = B_{\mathrm{DFT}}$ [1],[4]), then the corresponding CMFB is paraunitary as well.
- The minimum norm PR analysis prototype in the CMFB is equal (up to a constant factor of 2) to the minimum norm PR analysis prototype in the DFT FB.

A simple condition guaranteeing $\mathbf{T}_{\mathrm{DFT}} = \mathbf{0}$ exists for integer oversampling, i.e., $N/M = K \in \mathbb{N}$. For odd-stacked CMFBs with arbitrary integer oversampling factor $K$, and for even-stacked CMFBs with odd $K$, the condition

$$f[n] = f^*[-\alpha-(2l+1)KM-n] \quad (\text{with some } l \in \mathbb{Z}) \quad (9)$$

can be shown to be sufficient for $\mathbf{T}_{\mathrm{DFT}} = \mathbf{0}$. This condition implies that $f[n]$ has linear phase. Thus, PR (with linear phase filters in the case of an even-stacked CMFB) is achieved by choosing $f[n]$ according to (9) and identifying the analysis prototype with $2\tilde{h}[n]$.

## V. Conclusion

We introduced and studied *oversampled* cosine modulated filter banks (CMFBs) with perfect reconstruction (PR), and we defined the new class of *even-stacked* CMFBs allowing both PR and linear phase filters in all channels. The CMFB recently introduced for critical sampling by Lin and Vaidyanathan has been extended to the oversampled case and shown to be a special case of even-stacked CMFBs. We derived PR conditions for oversampled CMFBs and demonstrated that, concerning both PR and frame-theoretic properties, CMFBs are closely related to DFT filter banks of the same stacking type and with twice the oversampling factor.

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