Oversampled Cosine Modulated Filter Banks with Linear Phase*

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Abstract—We introduce oversampled cosine modulated filter banks (CMFBs) and a new classification of (oversampled or critically sampled) CMFBs as odd-stacked and even-stacked. We propose the new class of even-stacked CMFBs which allows both perfect reconstruction (PR) and linear phase filters in all channels. We formulate conditions for PR and show that any PR CMFB corresponds to a PR DFT filter bank with twice the oversampling factor. We also show that the frame-theoretic properties of a CMFB and of the corresponding DFT filter bank are closely related.

I. INTRODUCTION

Recent interest in oversampled filter banks (FBs) [1]-[5] is mainly due to their increased design freedom and noise immunity [1],[3],[4]. Oversampled DFT FBs [6],[1],[3],[7] and oversampled cosine modulated FBs (CMFBs) [7],[8] allow efficient FFT- or DCT-based implementations. Here, CMFBs are advantageous as their subband signals are real if the input signal and the analysis prototype are real.

This paper introduces and studies oversampled CMFBs with perfect reconstruction (PR). Section II proposes a new classification* (odd-stacked/even-stacked) of oversampled or critically sampled CMFBs. The traditional CMFBs (previously defined for critical sampling only) [10]-[16] are of the odd-stacked type; their channel filters do not have linear phase even if the prototypes have linear phase. In contrast, the new class of even-stacked CMFBs [1] introduced in Section II allows both PR paraunitarity and linear phase filters in all channels; it contains the recently proposed Lin-Vaidyanathan CMFBs [17] and Wilson CMFBs [8] as special cases. Section III provides PR conditions and shows that PR CMFBs are associated to PR DFT FBs with twice the oversampling factor. Finally, Section IV shows that the frame bound ratio of a CMFB corresponding to a frame expansion [18],[1],[4] equals that of the associated DFT FB.

DFT FBs. For later use, we review DFT (or complex modulated) FBs [6],[3],[7],[19] with N channels and decimation factor M. The FB is critically sampled for N = M and oversampled for N > M. In an odd-stacked DFT FB [6], the analysis and synthesis filter transfer functions are $H_k^{DFT-o}(z) = H(zW_N^{-k+1/2})$ and $F_k^{DFT-o}(z) = F(zW_N^{-k+1/2})$ ($k = 0,1,\ldots,N-1$), respectively, with $W_N = e^{-j2\pi/N}$. The corresponding impulse responses are $h_k^{DFT-o}[n] = h[n]W_N^{-(k+1/2)n}$ and $f_k^{DFT-o}[n] = f[n]W_N^{-(k+1/2)n}$. In an even-stacked DFT FB [6], the transfer functions are $H_k^{DFT-e}(z) = H(zW_N^{k})$ and $F_k^{DFT-e}(z) = F(zW_N^{k})$ for $k = 0,1,\ldots,N-1$, and the impulse responses are $h_k^{DFT-e}[n] = h[n]W_N^{-kn}$ and $f_k^{DFT-e}[n] = f[n]W_N^{-kn}$. In both cases, $h[n] \leftrightarrow H(z)$ and $f[n] \leftrightarrow F(z)$ denote the analysis and synthesis prototypes, respectively.

A DFT FB satisfying PR with zero delay (i.e., $\hat{x}[n] = x[n]$, where $x[n]$ and $\hat{x}[n]$ denote the input and reconstructed signal, respectively) can be shown to correspond to the following expansion of the input signal,

$$x[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_k^{DFT} \rangle f_k^{DFT}[n].$$

Here, $h_{k,m}^{DFT}[n] = h^*[mM-n]W_N^{-(k+1/2)(n-mM)}$ and $f_{k,m}^{DFT}[n] = f[mM-n]W_N^{-(k+1/2)(n-mM)}$ in the odd-stacked case and $h_{k,m}^{DFT}[n] = h^*[mM-n]W_N^{-k(n-mM)}$ and $f_{k,m}^{DFT}[n] = f[mM-n]W_N^{-k(n-mM)}$ in the even-stacked case.

II. OVERSAMPLED COSINE MODULATED FILTER BANKS

We now introduce two different types of oversampled CMFBs, corresponding to a novel classification† of CMFBs as odd-stacked and even-stacked. A close relation to the odd-stacked/even-stacked classification of DFT FBs will be shown in Sections III and IV.

Odd-stacked CMFBs. Odd-stacked CMFBs are the traditional CMFB type previously defined for critical sampling [10]-[16]. In the general case of an odd-stacked CMFB with N channels and decimation factor M (note that the CMFB is oversampled for N > M), we define the analysis and synthesis filters respectively as

$$h_k^{CM-o}[n] = \sqrt{2} h[n] \cos \left( \frac{(k + 1/2)\pi}{N} n + \phi_k^o \right),$$

$$f_k^{CM-o}[n] = \sqrt{2} f[n] \cos \left( \frac{(k + 1/2)\pi}{N} - n - \phi_k^o \right)$$

for $k = 0,1,\ldots,N-1$. Here, $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, and the phases are defined as $\phi_k^o = -\alpha \frac{\pi}{2N} (k + 1/2) + r \frac{\pi}{2}$ with $\alpha \in \mathbb{Z}$ and $r \in \{0,1\}$ (this extends the phase definition given in [13] for the special case of critical sampling). Note that the channel frequencies are $\theta_k = \frac{k+1/2}{2N}$; in particular, the channel with index k = 0 is centered at frequency $\theta_0 = \frac{1}{2N}$, as depicted in Fig. 1(a). A disadvantage of odd-stacked CMFBs is that the channel filters do not have linear phase even if the prototypes have linear phase [10]-[16].

Even-stacked CMFBs. We now introduce the new class† of even-stacked CMFBs allowing both PR and linear
Figure 1. Transfer functions of the channel filters in (a) an N-channel odd-stacked CMFB and (b) a 2N-channel even-stacked CMFB.

Phase filters in all channels. The analysis FB in an even-stacked CMFB with 2N channels and decimation factor 2M (the CMFB is oversampled for N > M) consists of two partial FBs \( h_{CM}^M[n] = \{ h[n] \} \) for \( k = 0 \) and \( h_{CM}^M[n] = \{ h[n-sM](-1)^{n-sM}, k = N \) derived from an analysis prototype \( h[n] \) as

\[
\begin{align*}
    h_{CM}^M[n] &= \sqrt{2}h[n-M] \sin \left( \frac{k\pi}{N} (n-M) \right), \quad k = 1, \ldots, N-1, \\
    h_{CM}^M[n] &= \{ h[n] \}, \quad k = 0 \\
    h_{CM}^M[n] &= \{ \sqrt{2}h[n]\cos(\frac{k\pi}{N}n + \phi_k^M), \quad k = 1, \ldots, N-1 \\
    h_{CM}^M[n] &= \{ h[n-sM](-1)^{n-sM}, \quad k = N \}
\end{align*}
\]

for \( k = 1, \ldots, N-1 \). Similarly, the synthesis FB consists of partial FBs \( f_{CM}^M[n] \) defined in terms of a synthesis prototype \( f[n] \) as

\[
\begin{align*}
    f_{CM}^M[n] &= \{ f[n+2M], \quad k = 0 \\
    f_{CM}^M[n] &= \{ \sqrt{2}f[n]\cos(\frac{k\pi}{N}n - \phi_k^M), \quad k = 1, \ldots, N-1 \\
    f_{CM}^M[n] &= \{ f[n+sM](-1)^{n+sM}, \quad k = N \}
\end{align*}
\]

for \( k = 1, \ldots, N-1 \). We define the phases as \( \phi_k^M = -\alpha \frac{k\pi}{N} k + r \frac{\pi}{N} \) with \( \alpha \in \mathbb{Z} \); furthermore, \( r, s \in \{0,1\} \) with \( s = r \) for \( \alpha \) even and \( s = 1 - r \) for \( \alpha \) odd. Note that there are \( 2N \) channels but only \( N+1 \) different channel frequencies \( \theta_k = \frac{2\pi k}{2N} \) (\( k = 0, \ldots, N \)), as depicted in Fig. 1(b). In particular, the \( k = 0 \) channel is centered at frequency \( \theta_0 = 0 \). For any choice of \( \alpha \in \mathbb{Z} \) and \( r \in \{0,1\} \), all filters have linear phase if the prototypes have linear phase.

Two special even-stacked CMFBs are the CMFB recently introduced (for critical sampling) by Lin and Vaidyanathan (LV) in [17] and the Wilson-type (WI) CMFB (corresponding to the discrete-time Wilson expansion [20]) recently introduced by the authors in [8]. The parameters of these two even-stacked CMFBs and of two variants (abbreviated LV' and WI') are summarized in Table 1. In particular, the analysis filters of an LV CMFB are

\[
\begin{align*}
    h_{LV}^M[n] &= \sqrt{2}h[n-M] \sin \left( \frac{k\pi}{N} (n-M) \right), \quad k = 1, \ldots, N-1, \\
    h_{LV}^M[n] &= \{ h[n] \}, \quad k = 0 \\
    h_{LV}^M[n] &= \{ \sqrt{2}h[n]\cos(\frac{k\pi}{N}n + \phi_k^M), \quad k = 1, \ldots, N-1 \\
    h_{LV}^M[n] &= \{ h[n-sM](-1)^{n-sM}, \quad k = N \}
\end{align*}
\]

and the analysis filters of a WI FB are

\[
\begin{align*}
    h_{WI}^M[n] &= \{ h[n] \}, \quad k = 0 \\
    h_{WI}^M[n] &= \{ \sqrt{2}h[n]\cos(\frac{k\pi}{N}n - \frac{N-1}{2}), \quad k = 1, \ldots, N-1 \\
    h_{WI}^M[n] &= \{ h[n-sM](-1)^{n-sM}, \quad k = N \}
\end{align*}
\]

for \( k = 1, \ldots, N-1 \). In particular, the \( k = 0 \) channel is centered at frequency \( \theta_0 = 0 \). For any choice of \( \alpha, r \in \{0,1\} \), all filters have linear phase if the prototypes have linear phase.

We shall next provide PR conditions for oversampled and critically sampled CMFBs. For both odd-stacked and even-stacked CMFBs, the following decomposition of the reconstructed signal can be shown [21],

\[
\hat{x}[n] = \frac{1}{2} \left[ (S_{DFST}^{(h,f)} x)[n] + (T_{DFST}^{(h,f)} x)[n] \right]
\]

where the operators \( S_{DFST}^{(h,f)} \) and \( T_{DFST}^{(h,f)} \) are defined as

\[
\begin{align*}
    (S_{DFST}^{(h,f)} x)[n] &= \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{DFST} \rangle f_{k,m}^{DFST}[n], \\
    (T_{DFST}^{(h,f)} x)[n] &= \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{i2\phi_k^M} c_m \langle x, h_{k,m}^{DFST} \rangle f_{k,m}^{DFST}[n].
\end{align*}
\]

Here, \( h_{k,m}^{DFST}[n] \) and \( f_{k,m}^{DFST}[n] \) are defined as before, \( f_{k,m}^{DFST}[n] = f_{2N-k-1,m}[n] \), \( \phi_k = \phi_k^M \), and \( c_m = 1 \) for an odd-stacked CMFB, and \( f_{k,m}^{DFST}[n] = f_{2N-k,m}[n] \), \( \phi_k = \phi_k^M \), and \( c_m = (-1)^m \) for an even-stacked CMFB. Note that \( (S_{DFST}^{(h,f)} x)[n] \) is the DFT FB expansion in (1) with \( N \) replaced by \( 2N \).

For PR with zero delay, \( \hat{x}[n] = x[n] \), it is necessary and sufficient [21] that

\[
S_{DFST}^{(h,f)} = 2I \quad \text{and} \quad T_{DFST}^{(h,f)} = 0,
\]

where \( I \) and \( 0 \) denote the identity and zero operator, respectively. The operator \( S_{DFST}^{(h,f)} \) can be expressed as

\[
(S_{DFST}^{(h,f)} x)[n] = 2N \sum_{l=-\infty}^{\infty} d_l x[n-2lN] \sum_{m=-\infty}^{\infty} f[n-mM] h[-n+mM+2lN]
\]

Table 1. Parameters of special even-stacked CMFBs.

<table>
<thead>
<tr>
<th>FB</th>
<th>( \alpha )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LV</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>LV'</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>WI</td>
<td>1</td>
<td>0</td>
<td>0(1)</td>
</tr>
<tr>
<td>WI'</td>
<td>1</td>
<td>1</td>
<td>0(0)</td>
</tr>
</tbody>
</table>

III. Perfect Reconstruction Conditions

We shall next provide PR conditions for oversampled and critically sampled CMFBs. For both odd-stacked and even-stacked CMFBs, the following decomposition of the reconstructed signal can be shown [21],
with \(d_t = (-1)^t\) in the odd-stacked case and \(d_t = 1\) in the even-stacked case. Hence, the first PR condition in (3), 
\[
S_{\text{DFT}}^{(h,f)} = 2I, 
\]
notes the number of channels. The frame bound ratio \(B/A\) characterizes important numerical properties of the signal expansion [18], and thus also of the corresponding FB. The above frame condition can also be written as
\[
A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|x\|^2, 
\]
where \(S\) is the frame operator defined as
\[
(Sx)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m} \rangle f_{k,m}[n]. 
\]
The frame bounds \(A\) and \(B\) are the infimum and supremum, respectively, of the eigenvalues of \(S\).

In particular, the frame operator of a CMFB is given by
\[
(S_{\text{CM}}x)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m} \rangle f_{k,m}[n], 
\]
where in the odd-stacked case \(N' = N\) and \(f_{k,m}[n] = f_{k,m}^{\text{CM}}[n-mM]\), and in the even-stacked case \(N' = N + 1\) and
\[
f_{k,m}^{\text{CM}}[n] = \left\{ \begin{array}{ll}
f_{k,m}^{\text{CM}}[n-2\mu M], & m = 2\mu, \ k = 0, \ldots, N \\
f_{k,m}^{\text{CM}}[n-2\mu M], & m = 2\mu - 1, \ k = 1, \ldots, N - 1. 
\end{array} \right.
\]
It can be shown [21] that the frame operator of both odd-stacked and even-stacked CMFBs can be decomposed as
\[
S_{\text{CM}} = \frac{1}{2}(S_{\text{DFT}} + T_{\text{DFT}}), 
\]
where \(S_{\text{DFT}}\) is the frame operator of a DFT FB with 2\(N\) channels and decimation factor \(M\),
\[
(S_{\text{DFT}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m}^{\text{DFT}} \rangle f_{k,m}^{\text{DFT}}[n], 
\]
and \(T_{\text{DFT}}\) is defined as
\[
(T_{\text{DFT}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\pi km} c_m \langle x, f_{k,m}^{\text{DFT}} \rangle f_{k,m}^{\text{DFT}}[n]. 
\]

IV. Frame Properties

Any PR FB corresponds to an expansion of the input signal \(x[n]\) into a set of “synthesis functions” \(f_{k,m}[n]\) [1], [4]. These synthesis functions are called a frame in \(l^2(\mathbb{Z})\), the space of square-summable discrete-time signals, if
\[
A\|x\|^2 \leq \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} |\langle x, f_{k,m} \rangle|^2 \leq B\|x\|^2 \quad \forall x[n] \in l^2(\mathbb{Z}) 
\]
with the frame bounds \(A > 0\) and \(B < \infty\) [18] \((N'\) denotes the number of channels). The frame bound ratio \(B/A\) characterizes important numerical properties of the signal expansion [18], and thus also of the corresponding FB. The above frame condition can also be written as
\[
A\|x\|^2 \leq \langle Sx, x \rangle \leq B\|x\|^2, 
\]
where \(S\) is the frame operator defined as
\[
(Sx)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m} \rangle f_{k,m}[n]. 
\]
The frame bounds \(A\) and \(B\) are the infimum and supremum, respectively, of the eigenvalues of \(S\).

In particular, the frame operator of a CMFB is given by
\[
(S_{\text{CM}}x)[n] = \sum_{k=0}^{N'-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m} \rangle f_{k,m}[n], 
\]
where in the odd-stacked case \(N' = N\) and \(f_{k,m}[n] = f_{k,m}^{\text{CM}}[n-mM]\), and in the even-stacked case \(N' = N + 1\) and
\[
f_{k,m}^{\text{CM}}[n] = \left\{ \begin{array}{ll}
f_{k,m}^{\text{CM}}[n-2\mu M], & m = 2\mu, \ k = 0, \ldots, N \\
f_{k,m}^{\text{CM}}[n-2\mu M], & m = 2\mu - 1, \ k = 1, \ldots, N - 1. 
\end{array} \right.
\]
It can be shown [21] that the frame operator of both odd-stacked and even-stacked CMFBs can be decomposed as
\[
S_{\text{CM}} = \frac{1}{2}(S_{\text{DFT}} + T_{\text{DFT}}), 
\]
where \(S_{\text{DFT}}\) is the frame operator of a DFT FB with 2\(N\) channels and decimation factor \(M\),
\[
(S_{\text{DFT}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, f_{k,m}^{\text{DFT}} \rangle f_{k,m}^{\text{DFT}}[n], 
\]
and \(T_{\text{DFT}}\) is defined as
\[
(T_{\text{DFT}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\pi km} c_m \langle x, f_{k,m}^{\text{DFT}} \rangle f_{k,m}^{\text{DFT}}[n]. 
\]
is again a Weyl-Heisenberg frame, and thus corresponds to a PR analysis DFT FB. Note that the above holds for both even-stacked and odd-stacked DFT FBs. The following theorem [21] states an important relation between the frame and PR properties of DFT FBs and CMFBs.

**Theorem.** Let \( h[n] \) and \( f[n] \) denote the analysis and synthesis prototype, respectively, in an odd-stacked CMFB with \( N \) channels and decimation factor \( M \), or in an even-stacked CMFB with \( 2N \) channels and decimation factor \( 2M \). Let \( f[n] \) be such that it generates a Weyl-Heisenberg frame of the same stacking type in \( l^2(Z) \), i.e.,

\[
A_{\text{DFT}} \|x\|^2 \leq \langle S_{\text{DFT}} x, x \rangle \leq B_{\text{DFT}} \|x\|^2.
\]

Furthermore, let \( f[n] \) be such that \( T_{\text{DFT}} = 0 \). Then,

1. if the CMFB synthesis functions \( \{j_{k,m}^C[n]\} \) are a frame in \( l^2(Z) \) with frame bounds \( A_{\text{CM}} = A_{\text{DFT}} / 2 \) and \( B_{\text{CM}} = B_{\text{DFT}} / 2 \), i.e.,

\[
A_{\text{DFT}} \|x\|^2 / 2 \leq \langle S_{\text{CM}} x, x \rangle \leq B_{\text{DFT}} \|x\|^2 / 2;
\]

(ii) for \( h[n] = 2h[n] \), where \( h[n] = (S_{\text{DFT}}^{-1} f^*[n])[-n] \) denotes the minimum norm analysis prototype of the DFT FB, the analysis CMFB constructed from \( h[n] \) is the PR analysis CMFB with minimum norm filters.

The following interpretations and conclusions apply:

- If \( T_{\text{DFT}} = 0 \) as assumed above, the CMFB becomes equivalent to a DFT FB, and therefore the design of a CMFB reduces to that of a DFT FB of the same stacking type and twice the oversampling factor.
- The CMFB frame bounds \( A_{\text{CM}} = A_{\text{DFT}} / 2 \) and \( B_{\text{CM}} = B_{\text{DFT}} / 2 \) are trivially related to the frame bounds \( A_{\text{DFT}} \) and \( B_{\text{DFT}} \) of the corresponding DFT FB. Since \( B_{\text{CM}} / A_{\text{CM}} = B_{\text{DFT}} / A_{\text{DFT}} \), the CMFB inherits the numerical properties of the corresponding DFT FB.
- In particular, if the DFT FB is paraunitary (\( A_{\text{DFT}} = B_{\text{DFT}} \) [1],[4]), then the corresponding CMFB is paraunitary as well.
- The minimum norm PR analysis prototype in the CMFB is equal (up to a constant factor of 2) to the minimum norm PR analysis prototype in the DFT FB.

A simple condition guaranteeing \( T_{\text{DFT}} = 0 \) exists for integer oversampling, i.e., \( N/M = K \in \mathbb{N} \). For odd-stacked CMFBs with arbitrary integer oversampling factor \( K \), and for even-stacked CMFBs with odd \( K \), the condition

\[
f[n] = f^*[-\alpha - (2l+1)KM - n] \quad \text{(with some } l \in \mathbb{Z})
\]

(9) can be shown to be sufficient for \( T_{\text{DFT}} = 0 \). This condition implies that \( f[n] \) has linear phase. Thus, PR (with linear phase filters in the case of an even-stacked CMFB) is achieved by choosing \( f[n] \) according to (9) and identifying the analysis prototype with \( 2h[n] \).

**V. Conclusion**

We introduced and studied oversampled cosine modulated filter banks (CMFBs) with perfect reconstruction (PR), and we defined the new class of even-stacked CMFBs allowing both PR and linear phase filters in all channels. The CMFB recently introduced for critical sampling by Lin and Vaidyanathan has been extended to the oversampled case and shown to be a special case of even-stacked CMFBs. We derived PR conditions for oversampled CMFBs and demonstrated that, concerning both PR and frame-theoretic properties, CMFBs are closely related to DFT filter banks of the same stacking type and with twice the oversampling factor.

**References**


