The Hyperbolic Class of Quadratic Time-Frequency Representations—Part II: Subclasses, Intersection with the Affine and Power Classes, Regularity, and Unitarity

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Abstract—Part I of this paper introduced the hyperbolic class (HC) of quadratic/bilinear time-frequency representations (QTFR’s) as a new framework for constant-Q time-frequency analysis. The present Part II defines and studies the following four subclasses of the HC:

- The localized-kernel subclass of the HC is related to a time-frequency concentration property of QTFR’s. It is analogous to the localized-kernel subclass of the affine QTFR class.
- The affine subclass of the HC (affine HC) consists of all HC QTFR’s that satisfy the conventional time-shift covariance property. It forms the intersection of the HC with the affine QTFR class.
- The power subclasses of the HC consist of all HC QTFR’s that satisfy a “power time-shift” covariance property. They form the intersection of the HC with the recently introduced power classes.
- The power-warp subclass of the HC consists of all HC QTFR’s that satisfy a covariance to power-law frequency warpings. It is the HC counterpart of the shift-scale covariant subclass of Cohen’s class.

All of these subclasses are characterized by 1-D kernel functions. It is shown that the affine HC is contained in both the localized-kernel hyperbolic subclass and the localized-kernel affine subclass and that any affine HC QTFR can be derived from the Bertrand unitary distribution by a convolution.

We furthermore consider the properties of regularity (invertibility of a QTFR) and unitarity (preservation of inner products, Moyal’s formula) in the HC. The calculus of inverse kernels is developed, and important implications of regularity and unitarity are summarized. The results comprise a general method for least-squares signal synthesis and new relations for the Altes-Marinovich $Q$-distribution.

I. INTRODUCTION

THE QUADRATIC/BILINEAR time-frequency representations (QTFR’s) of the hyperbolic class (HC) have been proposed in Part I of this paper [1]. The HC comprises all QTFR’s $T_X(t,f)$ that are “covariant” to time-frequency (TF) scalings, $T_{C_aX}(t,f) = T_X(at,f/a)$ with $(C_aX)(f) = \frac{1}{\sqrt{a}}X(f/a)$, and covariant to hyperbolic time-shifts, $T_{H_cX}(t,f) = T_X(t-c/f, f)$ with $(H_cX)(f) = e^{-2\pi i c/f}X(f)$. Here, $X(f)$ is the Fourier transform of an analytic signal $x(t)$ (i.e., $X(f) = 0$ for $f < 0$), and $t$ and $f$ denote time and frequency, respectively. Any HC QTFR can be written as

$$T_X(t,f) = \frac{1}{f_c} \int_0^{\infty} \int_0^{\infty} \Gamma_T(f_1,f_2) e^{2\pi i f \ln \frac{f_1}{f_2}} X(f_1)X^*(f_2) df_1 df_2$$

$$= \int_0^{\infty} \int_0^{\infty} \Phi_T(f,f/f_r, -b, \beta) V_X(b, \beta) e^{2\pi i f f_r} db d\beta$$

$$= \int_0^{\infty} \int_{\infty}^{\infty} \Psi_T(\zeta, \beta) B_X(\zeta, \beta) e^{2\pi i \zeta \ln f_r} d\zeta d\beta, \quad f > 0$$

where $f_r > 0$ is a fixed reference or normalization frequency, the hyperbolic signal product is defined as

$$V_X(b, \beta) = f_r e^{bX(f_r e^{\beta/2})} X^*(f_r e^{-\beta/2})$$

and the hyperbolic ambiguity function [1–3] is

$$B_X(\zeta, \beta) = \int_{-\infty}^{\infty} V_X(b, \beta) e^{2\pi i \zeta \ln f_r} db$$

$$= \int_{-\infty}^{\infty} X(f_r e^{\beta/2}) X^*(f_r e^{-\beta/2}) e^{2\pi i \zeta \ln f_r} df_r.$$  \hspace{1cm} (4)

The 2-D kernels $\Gamma_T(b_1, b_2)$, $\Phi_T(b, \beta)$, and $\Psi_T(\zeta, \beta)$ uniquely characterize the HC QTFR $T$; they are related as

$$\Gamma_T(b_1, b_2) = \frac{1}{b_1 b_2} \Phi_T(-\ln \sqrt{b_1 b_2}, \ln b_1 \ln b_2)$$

and $\Psi_T(\zeta, \beta) = \int_{-\infty}^{\infty} \Phi_T(b, \beta) e^{2\pi i \zeta \ln f_r} db$.

Many HC QTFR’s produce a constant-Q TF analysis similar to the affine QTFR class and the wavelet transform [4]–[15], where higher frequencies are analyzed with better time resolution and poorer frequency resolution. The HC is connected

1For the sake of notational simplicity, we focus on quadratic auto-representations $T_X(t,f)$. The extension to bilinear cross-representations $T_{X,Y}(t,f)$ is straightforward [1].
to a hyperbolic TF geometry, which is particularly adapted to Doppler-invariant and self-similar signals [1], [3], [6], [16]. Fig. 1 shows that HC QTFR’s yield better TF resolution than conventional (Cohen’s class) QTFR’s if the signal under analysis is consistent with the hyperbolic TF geometry for which the HC is particularly suited.

The HC can be formally derived from the well-known Cohen’s class of TF shift-covariant QTFR’s [18]–[21], [4], [5] by a “constant-Q warping” transformation [1], [22], and it is thus “unitarily equivalent” [23], [24] to Cohen’s class. Two prominent members of the HC are the Altes–Marinovich Q-distribution [1], [2], [25]–[27] and the Bertrand unitary $P_0$-distribution [6]–[9], [26].

Fig. 1. Comparison of Cohen’s class QTFR and HC QTFR for a three-component signal consisting of two hyperbolic impulses and a Gaussian signal. (a) Smoothed pseudo Wigner distribution [5] from Cohen’s class. (b) Smoothed pseudo Altes–Marinovich Q-distribution from the HC [1]. Both QTFR’s employ a TF smoothing to attenuate cross terms [1], [17]. The smoothing in the HC QTFR in (b) is adapted to the hyperbolic TF geometry, which entails improved TF resolution in the two hyperbolic components at the cost of some TF distortion in the Gaussian component. The software used for computing the smoothed pseudo Q-distribution was obtained from [17].

where $G_p(\beta) = \frac{\beta^2}{2(\beta^2 + 1)}$, $A_p(\beta) = -\ln G_p(\beta)$. The implementation of HC QTFR’s is discussed in [17], [27], and [10].

The present Part II of this paper defines and studies four important subclasses of the HC (see Fig. 2), whose QTFR’s satisfy an additional property besides the scale and hyperbolic time-shift covariance properties satisfied by all HC QTFR’s. In Section II-A, we introduce the “localized-kernel subclass” of the HC, which is analogous to the localized-kernel subclass of the affine QTFR class [7], [9], [11]–[13]. It is closely related to a TF concentration property (see the “additional property” for the localized-kernel HC in Table I) and, hence, is particularly suited for signals with specific TF geometries. It forms a convenient framework for many important HC QTFR’s such as the $Q$-distribution and the unitary $P_0$-distribution. Section II-B discusses the “affine subclass” of the HC (affine HC) [8], [28], which consists of all HC QTFR’s satisfying the particularly important time-shift covariance property (see Table I). The affine HC forms the intersection of the HC and the affine class [4]–[8], and it is contained in both the localized-kernel HC and the localized-kernel affine subclass [7], [9], [11]–[13]. Section II-C considers a generalization of the affine HC, namely, the “power subclasses” of the HC (power HC’s), which consist of all HC QTFR’s satisfying the power time-shift covariance property [8], [29]. The power HC’s form the intersection of the HC with the power classes [29]–[31]. All power HC’s are contained in the localized-kernel HC. The intersection subclasses studied in Sections II-B and C (power HC’s with the affine HC as special case) emphasize the HC’s relation to other scale covariant classes. These classes are all based on the properties of i) scale covariance and ii) covariance to time shifts corresponding to a dispersive or nondispersive group delay law. Scale covariance is important in multiresolution analysis, and specific types of time-shift covariance are useful for accommodating the group delay characteristics of various systems and transmission media. Section II-D considers another HC subclass whose members satisfy a power-warp covariance property [1]. This “power-warp HC” is the counterpart of the shift-scale covariant subclass of Cohen’s class [32], [5], [20], [21], [33]. Fig. 2 summarizes pictorially the HC, the affine class, and the power classes, as well as the HC subclasses discussed in Section II.

Besides the discussion of important HC subclasses, the second main contribution of this paper is a study of the QTFR properties of regularity and unitarity [14], [21] in the HC. These properties are fundamental on a theoretical level, and they form a basis for various methods for (statistical) TF signal processing [4], [34]–[43]. Section III considers the property of regularity (invertibility of a QTFR) and develops the calculus of inverse kernels. Important implications of a QTFR’s regularity include the recovery of the signal and the derivation of other quadratic signal representations from the QTFR. Section IV considers the property of unitarity (preservation of inner products, Moyal’s formula). Important implications of unitarity include a method for least-squares signal synthesis [38], [39], [42], [44]–[48]. The application of the results of Sections III and IV to the $Q$-distribution yields interesting new relations.
TABLE I

<table>
<thead>
<tr>
<th>HC SUBCLASS</th>
<th>ADDITIONAL PROPERTY</th>
<th>FORM OF KERNEL $\Phi_T(b, \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Localized-kernel</td>
<td>$T_X(x, f) = r^2(f) \delta(t - c \tau(f))$ with $X_c(f) = r(f) e^{-j2\pi \xi(f)}$</td>
<td>$B_T(\beta) \delta(b - A_T(\beta))$</td>
</tr>
<tr>
<td>HC</td>
<td></td>
<td>(see Theorem 1)</td>
</tr>
<tr>
<td>Affine HC</td>
<td>$T_{S_X}(x, f) = T_X(x - \tau, f)$ with $(S_f)X(f) = e^{-j2\pi \tau f} X(f)$</td>
<td>$B_T(\beta) \delta(b - \frac{1}{\pi} \ln \left( \frac{\sinh(\alpha/2 \beta)}{\beta^2} \right))$</td>
</tr>
<tr>
<td>Power-HC$_K$</td>
<td>$T_{D_{\alpha}X}(x, f) = T_X(t - c \tau_{\alpha}(f), f)$ with $\tau_{\alpha}(f) = \frac{1}{\beta} \left( \frac{f}{\beta} \right)^{\alpha-1}$ and $(D_{\alpha}(f)X(f)) = e^{-j2\pi \tau_{\alpha}(f)} X(f)$</td>
<td>$B_T(\beta) \delta(b - \frac{1}{\pi} \ln \left( \frac{\sinh(\alpha \beta/2)}{\beta^2} \right))$</td>
</tr>
<tr>
<td>Power-warp HC</td>
<td>$T_{P_{\gamma}X}(x, f) = T_X(x \left( \frac{f}{f_{\gamma}} \right)^{1/\gamma - 1}, f_{\gamma} \left( \frac{f}{f_{\gamma}} \right)^{1/\gamma})$ with $(P_{\gamma}(f)X(f)) = \frac{1}{\gamma} B_T(\beta) (-\beta)$</td>
<td>$\frac{1}{\gamma} B_T(\beta) (-\beta)$</td>
</tr>
</tbody>
</table>

II. HC SUBCLASSES

The HC subclasses discussed in this section are important since each of them is defined by a specific QTFR property (see Table I), which is satisfied in addition to the two basic properties of TF scale covariance and hyperbolic time-shift covariance satisfied by any HC QTFR. For example, it is often desirable to use a QTFR covariant to constant (nondispersive) time shifts. The subclass of HC QTFR’s that are covariant to such time shifts satisfies three covariance properties:

- scale covariance
- hyperbolic time-shift covariance
- constant time-shift covariance.

Since the first and third covariance properties define the affine QTFR class, these HC QTFR’s are simultaneously members of the affine class [8], [28]. They are specifically suited to signals that are consistent with the hyperbolic TF geometry except for unknown time shifts. Furthermore, the mathematical description of the HC subclasses considered is simplified since the 2-D kernels describing general HC QTFR’s are here parameterized in terms of 1-D functions.

A. The Localized-Kernel HC Subclass

The localized kernel idea was introduced by the Bertrands based on tomography [13] and then extended to the affine localized-kernel subclass [7], [9], [11], [12]. The “localized-kernel” HC subclass discussed in this section is conceptually analogous to the affine localized-kernel subclass. It consists of all HC QTFR’s whose kernel $\Phi_T(b, \beta)$ in (1) is perfectly localized along the curve $b = A_T(\beta)$ in the $(b, \beta)$-plane, i.e.,

$$\Phi_T(b, \beta) = B_T(\beta) \delta(b - A_T(\beta))$$  \hspace{1cm} (7)

with $A_T(\beta) \in \mathbb{R}$ and $B_T(\beta) \geq 0$ arbitrary functions. We note that (7) entails $\Psi_T(\xi, \beta) = B_T(\beta) e^{2\pi \xi A_T(\beta)}$ in (2). Inserting (7) into (1), we see that any QTFR of the localized-kernel HC can be written as

$$T_X(t, f) = \int_{-\infty}^{\infty} X_s \left( \ln \left( \frac{f}{f_{\gamma}} \right) - A_T(\beta) \right) B_T(\beta) e^{2\pi \xi f_{\gamma} \beta} d\beta$$

$$= f \int_{-\infty}^{\infty} X_s \left( f e^{-A_T(\beta) + \frac{\beta}{2}} \right) X^* \left( f e^{-A_T(\beta) - \frac{\beta}{2}} \right) B_T(\beta) e^{-A_T(\beta) - \frac{\beta}{2}} \beta \, d\beta \times B_T(\beta) e^{-A_T(\beta) - \frac{\beta}{2}} \beta \, d\beta.$$  \hspace{1cm} (8)

The 2-D kernels of a QTFR of the localized-kernel HC are parameterized in terms of the two 1-D functions (“kernels”)...
and $A_T(\beta)$ and $B_T(\beta)$, which uniquely characterize the QTFR $T$. This fact is important as it greatly simplifies the analysis of QTFR’s of the localized-kernel HC.

The localized-kernel HC is also important since, under certain assumptions, the localized-kernel structure is necessary for a class of TF concentration properties (cf. the discussion of the localized-kernel subclass of the affine class given in [12]). This is shown by considering the family of signals

$$X_c(f) = r(f) e^{-j2\pi \xi(f)c}, \quad c \in \mathbb{R}$$

with given amplitude function $r(f) \geq 0$ and given phase function $\xi(f) \in \mathbb{R}$. The group delay function of $X_c(f)$ is $
abla(f) = c \xi'(f)$, where $\xi'(f) = \frac{d}{df} \xi(f)$. We wish to find an HC QTFR $T$ such that, for the signal family $X_c(f)$ given above, $T$ is perfectly concentrated along the group delay curve $\tau(f) = c \xi'(f)$, i.e.,

$$T_{X_c}(t,f) = r^2(f) \delta(t - c \xi'(f)) \quad \text{for all } c \in \mathbb{R}. \quad (9)$$

The next theorem, which is proved in the Appendix, shows a relation of (9) with the localized-kernel structure (7).

**Theorem 1:** Let the functions $r(f) \geq 0$ and $\xi(f) \in \mathbb{R}$ be given, and assume that the function $\Xi_{f,\beta}(b) = \xi(f) e^{-j(\beta/2)b} - \xi(f) e^{-j(\beta/2)}/2$ is one-to-one and differentiable for any (fixed) $f, \beta$. Then, the concentration property (9) is satisfied by an HC QTFR $T$ if and only if the following three conditions are satisfied:

1. There exists a function $A_T(\beta)$ that is independent of $f$ and satisfies

$$\Xi_{f,\beta}(A_T(\beta)) = f \xi'(f) \beta \quad \text{for all } f, \beta \in \mathbb{R}. \quad (10)$$

2. The ratio $R_{f,\beta}(A_T(\beta)) = R_{f,\beta}(b) = r(f) e^{-j(\beta/2)b} - r(f) e^{-j(\beta/2)/2}$ is independent of $f$.

3. The kernel $\Phi_T(b, \beta)$ of the HC QTFR $T$ is

$$\Phi_T(b, \beta) = B_T(\beta) \delta(b - A_T(\beta))$$

with

$$B_T(\beta) = \frac{r^2(f)}{R_{f,\beta}(A_T(\beta))} \geq 0.$$ 

We note that Condition 1 restricts the admissible phase functions $\xi(f)$: If no $A_T(\beta)$ satisfying (10) exists, then there does not exist any HC QTFR satisfying the concentration property (9) for the given phase function $\xi(f)$. If a function $A_T(\beta)$ satisfying (10) exists, then we insert it in $r^2(f)/R_{f,\beta}(A_T(\beta))$ in order to check Condition 2. This condition restricts the admissible amplitude functions $r(f)$; it will always be satisfied if $r(f) = r_0(f)$ with $\alpha \in \mathbb{R}$, in which case $B_T(\beta) = e^{j(2\alpha+1)\beta}$. Table II lists some phase and amplitude functions for which the concentration property (9) is satisfied by an HC QTFR. Finally, **Condition 3 shows that the HC QTFR $T$ must necessarily be a member of the localized-kernel HC.**

The localized-kernel HC provides a convenient framework for many important HC QTFR’s, including all HC members that are also members of the affine class or of the power classes (see Sections II-B and C). Prominent members of the localized-kernel HC are the $P_0$-distribution $P_0X(t,f)$ in (6), and the “power unitary $P_0$-distribution” $P_0^{(x)}(t,f)$ [22] (see Section II-C). The 1-D kernels of these QTFR’s are

$$A_Q(\beta) = 0, \quad B_Q(\beta) = 1;$$

$$A_{Q{x}}(\beta) = \alpha \beta, \quad B_{Q{x}}(\beta) = 1;$$

$$A_{R_0}(\beta) = \frac{\sinh(\beta/2)}{\beta/2}, \quad B_{R_0}(\beta) = 1, \quad \kappa \neq 0. \quad (11)$$

**B. The Affine HC Subclass**

In general, HC QTFR’s need not satisfy the constant (nondispersive) time-shift covariance property

$$T_{S_0}(t,f) = T_X(t - \tau, f) \quad (12)$$

with

$$(S_0X)(i) = e^{-j2\pi\tau f} X(f).$$

Motivated by the exceptional importance of this property, we define the **affine subclass** of the HC (affine HC) to consist of all HC QTFR’s satisfying (12). Thus, all QTFR’s of the affine HC satisfy the following three QTFR properties:

- time-shift covariance
- scale covariance
- hyperbolic time-shift covariance.

Since time-shift covariance and scale covariance are the axioms of the affine class [4–14], the QTFR’s of the affine HC are also members of the affine class. In fact, the affine HC forms the intersection of the affine class and the HC as illustrated in Fig. 2. As both the affine class and the HC are frameworks for constant-$Q$ TF analysis, it is not surprising that these classes overlap.

The affine HC is identical to the class of $P_0$-distributions previously introduced in [8] as a subclass of the affine class.

<table>
<thead>
<tr>
<th>AMPLITUDE FUNCTION</th>
<th>PHASE FUNCTION</th>
<th>GROUP DELAY FUNCTION</th>
<th>KERNEL</th>
<th>KERNEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(f)$</td>
<td>$\xi(f)$</td>
<td>$\tau(f) = \xi'(f)$</td>
<td>$A_T(\beta)$</td>
<td>$B_T(\beta)$</td>
</tr>
<tr>
<td>$r(f)^k$</td>
<td>$\xi^{(k)}(f)$</td>
<td>$\xi'(f)^{k-1}$</td>
<td>$\kappa$</td>
<td>$\kappa$</td>
</tr>
</tbody>
</table>
TABLE III

PROPERTY CONSTRAINTS FORMULATED WITH THE KERNELS $A_T(\beta)$, $B_T(\beta)$ OF THE LOCALIZED-KERNEL HC, THE KERNEL $B_T(\beta)$ OF THE AFFINE HC, AND THE KERNEL $\sigma_T(\gamma)$ (WITH FOURIER TRANSFORM $S_T(\beta)$) OF THE POWER-WARP HC. THE PROPERTIES ARE DEFINED IN TABLE II OF PART I [1]. THE RELATIONSHIPS OF $A_T(\beta)$, $B_T(\beta)$, AND $S_T(\gamma)$ WITH THE HC KERNEL $\Phi_T(b, \beta)$ ARE PROVIDED IN THE LAST COLUMN OF TABLE I. FOR THE AFFINE HC, $A_T(\beta) = \ln(\sinh(\beta/2)/\beta^2)$ = $A_T(\beta)$. THE CONSTRAINTS FOR THE POWER HC’S ARE DISCUSSED IN SECTION II-C. WE NOTE THAT $S_T'(0) = \frac{d}{d\gamma} A_T(\beta)|_{\beta=0}$ ETC.

<table>
<thead>
<tr>
<th>QFTR PROPERTY</th>
<th>LOCALIZED-KERNEL HC KERNEL CONSTRAINT</th>
<th>AFFINE HC KERNEL CONSTRAINT</th>
<th>POWER-WARP HC KERNEL CONSTRAINT</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1: hyp. time-shift covar.</td>
<td>always satisfied</td>
<td>always satisfied</td>
<td>always satisfied</td>
</tr>
<tr>
<td>P2: scale covariance</td>
<td>always satisfied</td>
<td>always satisfied</td>
<td>always satisfied</td>
</tr>
<tr>
<td>P3: real-valuedness</td>
<td>$A_T(-\beta) = A_T(\beta)$, $B_T^2(-\beta) = B_T(\beta)$</td>
<td>$B_T^2(-\beta) = B_T(\beta)$</td>
<td>$S_T(\gamma) \in \mathbb{R}$</td>
</tr>
<tr>
<td>P4: energy distribution</td>
<td>$B_T(0) = 1$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$, $S_T(0) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P5: frequency marginal</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P6: Moyal’s formula/unitarity</td>
<td>$</td>
<td>B_T(\beta)</td>
<td>= 1$</td>
</tr>
<tr>
<td>P7: group delay</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>$B_T(0) = 0$, $S_T(0) = 1$, $S_T(0) = 1$</td>
<td>$S_T(0) = 0$</td>
</tr>
<tr>
<td>P8: finite frequency support</td>
<td>$B_T(\beta) = 0$ for all $\beta$</td>
<td>$B_T(\beta) = 1$, $S_T(\gamma) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P9: Dirac freq. localization</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P10: hyperbolic marginal</td>
<td>$B_T(\beta) = 1$</td>
<td>$B_T(\beta) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P11: hyperbolic localization</td>
<td>$B_T(\beta) = 1$</td>
<td>$B_T(\beta) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P12: finite hyperbolic support</td>
<td>$\int_{-\infty}^{\infty} B_T(\beta) e^{2\pi i \epsilon(z + t) A_T(\beta)} d\beta = 0$</td>
<td>$\int_{-\infty}^{\infty} B_T(\beta) e^{2\pi i \epsilon(z + t) A_T(\beta)} d\beta = 0$</td>
<td>$S_T(0) = 0$</td>
</tr>
<tr>
<td>P13: hyperbolic moments</td>
<td>$B_T(0) = 1$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$, $S_T(0) = 0$</td>
<td></td>
</tr>
<tr>
<td>P14: power-warp covariance</td>
<td>$A_T(0) = a, B_T(0) = b$</td>
<td>not satisfied</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P15: time-shift covariance</td>
<td>$A_T(\beta) = e^{-i \beta}$</td>
<td>not satisfied</td>
<td>always satisfied</td>
</tr>
<tr>
<td>P16: hyperbolic axis reversal</td>
<td>$A_T(-\beta) = A_T(\beta)$, $B_T(-\beta) = B_T(\beta)$</td>
<td>not satisfied</td>
<td>not satisfied</td>
</tr>
<tr>
<td>P17: hyp. freq. moments</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P18: hyp. instantaneous frequency</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>$B_T(0) = 1$</td>
<td>$S_T(0) = 1$</td>
</tr>
<tr>
<td>P19: hyperbolic convolution</td>
<td>$B_T(\beta) = 1$</td>
<td>$B_T(\beta) = 1$</td>
<td>$S_T(\gamma) = e^{-i \gamma}$</td>
</tr>
<tr>
<td>P20: hyp. multiplication</td>
<td>$A_T(\beta) = a \beta$, $B_T(\beta) = e^{i \beta}$</td>
<td>not satisfied</td>
<td>$S_T(\gamma) = e^{-i \gamma}$</td>
</tr>
<tr>
<td>P21: convolution</td>
<td>$B_T(\beta) = e^{i A_T(\beta)}$</td>
<td>not satisfied</td>
<td>$S_T(\gamma) = 1$</td>
</tr>
<tr>
<td>P22: hyp. chirp localization</td>
<td>$A_T(0) = 0$, $B_T(0) = 1$</td>
<td>not satisfied</td>
<td>$S_T(\gamma) = 1$</td>
</tr>
</tbody>
</table>

The unitary $P_1$-distribution in (6) is a special case, as discussed below. The $Q$-distribution is not a member of the affine HC since it is not time-shift covariant.

Following [8], we shall now derive a condition for the time-shift covariance (12) within the HC. Inserting (1) into (12), it is easily shown that (12) is satisfied by an HC QFT $T$ if and only if

$$\Phi_T(b, \beta) \exp\{-i 2\pi \epsilon \left(e^{\beta} - e^{-\beta}ight) - \beta\} = \Phi_T(b, \beta),$$

for $c, b, \beta$. This will be true if and only if the argument of the exponential in the left-hand side is zero, i.e., $e^{\beta} - e^{-\beta} = 0$, which yields the condition $b = \frac{\pi}{\sinh(\beta/2)}$, expressing an explicit dependence of $b$ on $\beta$. The kernel $\Phi_T(b, \beta)$ must be such that $b = \frac{\pi}{\sinh(\beta/2)}$ is enforced, which leads to

$$\Phi_T(b, \beta) = B_T(\beta) \delta(b - A_T(\beta))$$

with

$$A_T(\beta) = \frac{\sinh(\beta/2)}{\beta/2}$$

and $B_T(\beta)$ arbitrary.

This form of $\Phi_T(b, \beta)$ is necessary and sufficient for the time-shift covariance (12), and thus characterizes the affine HC (cf. Table I). Comparing (13) with (7), we see that the affine HC is a subclass of the localized-kernel HC, with kernel $A_T(\beta) = A_T(\beta) = \frac{\sinh(\beta/2)}{\beta/2}$, which is the “$A$” kernel of the unitary $P_0$-distribution, whereas $B_T(\beta)$ is arbitrary. Combining (8)
and (13), we can write any affine HC member as
\[ T_X(t, f) = \int_{-\infty}^{\infty} X(f \Gamma_R(\beta) e^{\beta/2}) X^*(f \Gamma_R(\beta) e^{-\beta/2}) \times \Gamma_R(\beta) B_T(\beta) e^{2\pi f \beta} \, d\beta \]  
with \( \Gamma_R(\beta) = \frac{\beta^2}{\sinh(\beta/2)} \) and \( B_T(\beta) \) arbitrary.\(^3\)

Since the kernel \( A_T(\beta) \) is fixed, only \( B_T(\beta) \) remains to be chosen. Hence, members of the affine HC are characterized in terms of the single 1-D kernel \( B_T(\beta) \). The kernel constraints for desirable QTFR properties can now be reformulated in terms of \( B_T(\beta) \), as listed in the third column of Table III.

A prominent member of the affine HC is the unitary \( P_0 \)-distribution \( P_X(t, f) \) in (6), for which \( B_T(\beta) = \Gamma_R(\beta) = 1 \). From Table III, it follows that the unitary \( P_0 \)-distribution is the only QTFR of the affine HC that satisfies the hyperbolic marginal property \( P_{\beta} \) (or the hyperbolic localization property \( P_{\beta} \)). Thus, this property uniquely specifies the unitary \( P_0 \)-distribution inside the affine HC. Furthermore, using (14) we can show that any member of the affine HC can be derived from the unitary \( P_0 \)-distribution through a scaled convolution with respect to the time variable
\[ T_X(t, f) = \int_{-\infty}^{\infty} b_T(f(t-t')) P_X(t', f) \, dt' \]
where \( b_T(c) \) is the inverse Fourier transform of the kernel \( B_T(\beta) \). (This result is important since it permits the implementation of affine HC QTFR’s using existing algorithms for the unitary \( P_0 \)-distribution [10].) For these reasons, the unitary \( P_0 \)-distribution can be considered to be the “central” QTFR of the affine HC.

In [7], [9], [11], and [12], the localized-kernel subclass of the affine QTFR class is defined as the subclass of all affine QTFR’s \( T_X^{(A)}(t, f) \) that can be written in the form
\[ T_X^{(A)}(t, f) = \int_{-\infty}^{\infty} X(f(\Gamma_T(\beta) + \beta/2)) \times X^*(f(\Gamma_T(\beta) - \beta/2)) \times \Gamma_T(\beta) e^{2\pi f \beta} \, d\beta \]
with \( \Gamma_T(\beta) \) and \( \Gamma_R(\beta) \) are two 1-D kernel functions. It is easily checked that any QTFR of the affine HC in (14) can be written in the form (16), with
\[ F_T(\beta) = F_R(\beta) = \frac{\beta}{2} \coth \frac{\beta}{2}, \]
\[ G_T(\beta) = \frac{\beta^2}{\sinh(\beta/2)} B_T(\beta) = \Gamma_R(\beta) B_T(\beta), \]
(17)

This shows that the affine HC is also a subclass of the localized-kernel affine subclass, with kernel \( F_T(\beta) = F_R(\beta) = \frac{\beta}{2} \coth \frac{\beta}{2} \) equal to the kernel of the unitary \( P_0 \)-distribution in the framework of the localized-kernel affine subclass [11]. The second kernel \( G_T(\beta) \) is arbitrary but related to the corresponding HC kernel \( B_T(\beta) \) according

\(^3\)Equation (14) is Equation (38) in [8] for the class of \( P_0 \)-distributions, with \( \lambda_0(\beta) \) and \( \mu(\beta) \) used in [8] related to \( G_R(\beta) \) and \( B_T(\beta) \) as \( \lambda_0(\beta) = G_R(\beta) e^{\beta/2} \) and \( \mu(\beta) = G_R(\beta) B_T(\beta) \). The unitary \( P_0 \)-distribution is obtained when \( \mu(\beta) = G_R(\beta) \) [8].

C. The Power HC Subclasses

The affine HC concept can be extended by considering a generalization of the affine class. The power class \( PC_\kappa \) associated with the power parameter \( \kappa \in \mathbb{R}, \kappa \neq 0 \) has been introduced in [29] and [30] as the class of all QTFR’s \( T \) satisfying the scale covariance property and the “power time-shift covariance property”
\[ T_D^{(\kappa)}(t, f) = T_X(t - c \tau_\kappa(f), f) \]
with
\[ (D_c^{(\kappa)} X(f)) = e^{-\kappa c^2 f^2 \xi_\kappa(f)} X(f). \]
Here, \( \xi_\kappa(f) = (\frac{f}{\kappa})^\kappa \) and \( \tau_\kappa(f) = \frac{\kappa f}{2} e^{\kappa f} \]
with \( f > 0 \) and \( \kappa \neq 0 \) (cf. Tables I and II). The operator \( D_c^{(\kappa)} \) is an allpass filter with power-law group delay \( c \tau_\kappa(f) \). The affine class is equivalent to the power class when the power parameter \( \kappa = 1 \) and is, thus, a special case of the family \( PC_\kappa \) of power classes.

For a given power \( \kappa \), we define the power subclass of the HC (which is abbreviated power-HC\(\kappa \)) to consist of all HC QTFR’s satisfying the power time-shift covariance (18) [22], [29], [30]; this is an important property when analyzing signals propagating through dispersive systems. The power-HC\(\kappa \) is a subclass of both the HC and the power class \( PC_\kappa \); in fact, it forms the intersection of the two classes. The power-HC\(\kappa \) with \( \kappa = 1 \) equals the affine HC. It can be shown that any QTFR of the power-HC\(\kappa \) is a member of the localized-kernel HC in (8), with kernels
\[ A_T(\beta) = A_R(\beta) = \frac{1}{\kappa} \ln \left( \frac{\sinh(\kappa \beta/2)}{\kappa \beta/2} \right) \]
and
\[ B_T(\beta) \]
where \( A_R(\beta) \) is the “A” kernel of the \( \kappa \)th power unitary \( P_0 \)-distribution (cf. (11)). Thus, the power-HC\(\kappa \) is a subclass of the localized-kernel HC parameterized in terms of the 1-D kernel \( B_T(\beta) \). With (8), it follows that any QTFR of the power-HC\(\kappa \) can be written as a generalization of (14)
\[ T_X(t, f) = \int_{-\infty}^{\infty} X(f G_R(\beta) e^{\beta/2}) X^*(f G_R(\beta) e^{-\beta/2}) \times G_R(\beta) B_T(\beta) e^{2\pi f \beta} \, d\beta \]
with \( G_R(\beta) = (\frac{\kappa \beta^2}{\sinh(\kappa \beta/2)})^{1/\kappa} = e^{-A_R(\beta)} \) and \( B_T(\beta) \) arbitrary. Note that (14) is obtained when \( \kappa = 1 \).

Since power-HC\(\kappa \) QTFR’s have \( A_T(\beta) \) fixed, they are characterized by the single 1-D kernel \( B_T(\beta) \). The kernel constraints on \( B_T(\beta) \) for desirable QTFR properties equal those for the affine HC in the third column of Table III, except for the following:

i) In \( P_{12} \), \( A_T(\beta) \) is replaced by \( A_R(\beta) \).

ii) \( P_{15} \) is replaced by the power time-shift covariance property (18).
iii) The constraint for $P_{21}$ becomes $B_T(\beta) = \left(\sinh(2\beta/\pi)\right)^{-1}.$

A "central" QTFR in the power-HC is the power unitary $P$-distribution $P_X^{(S)}(t,f)$ defined by $B_T(\beta) = B_T(S(\beta)) = 1$ [22]. Inside the power-HC, the power unitary $P$-distributions are uniquely specified by the hyperbolic marginal property $P_{10}$ (or the hyperbolic localization property $P_{12}$). It can be shown that any member of the power-HC can be derived from the power unitary $P$-distribution $P_X^{(S)}(t,f)$ through a scaled convolution with respect to the time variable

$$T_X(t,f) = \frac{d}{d\tau} \int_{-\infty}^{\infty} b_T(\tau^2(t-t')) P_X^{(S)}(t',f) \, dt'$$

(19)

where $b_T(\tau)$ is the inverse Fourier transform of $B_T(\beta)$. Note that this relation generalizes (15).

D. The Power-Warp HC Subclass

An important subclass of Cohen's class is the shift-scale covariant subclass [32], [5], [20], [21], [33], which consists of all Cohen's class QTFR's satisfying the scale covariance property. Prominent members of this class are the generalized Wigner distribution [32] and the Choi–Williams distribution [33], [49].

We define the power-warp subclass of the HC (power-warp HC) as the HC subclass whose members satisfy the power-warp covariance property introduced in Section IV-D of Part I [1]

$$T_{\mathcal{P}_aX}(t,f) = T_X \left( \frac{\alpha}{\alpha} f, \frac{\alpha}{\alpha} f \right)^{z}$$

with

$$(\mathcal{P}_aX)(f) = \sqrt{a} \left( \frac{f}{f_{r}} \right)^{\alpha} X \left( \frac{f}{f_{r}} \right)^{\alpha}, \quad \alpha > 0.\quad (20)$$

Since this property corresponds to the scale covariance property in Cohen's class, the power-warp HC is the HC counterpart of the shift-scale covariant subclass of Cohen's class. The operator $\mathcal{P}_a$ produces a power-law frequency warping $f \to a f_{r}(f_{r})^{1/\alpha}$; the factor $\sqrt{a} \left( \frac{f}{f_{r}} \right)^{\alpha/2}$ assures the unitarity of $\mathcal{P}_a$ [23], [29]. With the kernel relations in Equations (20) and (21) of Part I [1], the theory of the shift-scale covariant subclass of Cohen's class [32], [20], [21] can be reformulated for its HC counterpart: the power-warp HC.

III. Regularity

Besides the discussion of four subclasses of the HC, the second contribution of this paper is a study of the fundamental QTFR properties of regularity and unitarity [14], [21] within the HC. These properties are important for various methods of (statistical) TF signal processing [4], [34]–[43]. The regularity property considered in this section expresses the QTFR's "invertibility" in the sense that no essential information about the signal is lost in the signal's QTFR. Regularity has several further important implications.

A. Inverse Kernels

The expressions (1) and (2) state that an HC QTFR $T_X(t,f)$ can be derived from $V_X(b,\beta)$ and $B_X(\zeta,\beta)$ by characteristic linear transformations. An HC QTFR $T_X(t,f)$ is said to be regular if these linear transformations are invertible. Inversion of (1) and (2) can be shown to result in

$$V_X(b,\beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_T^{-1} \left( \frac{\beta}{\beta} f, \frac{\beta}{\beta} f \right) T_X(t,f)e^{-2\pi j \beta f} \, dt \, df$$

(23)
where the “inverse kernels” $\Phi_T^{-1}(b, \beta)$ and $\Psi_T^{-1}(\zeta, \beta)$ are related as $\Psi_T^{-1}(\zeta, \beta) = \int_{-\infty}^{\infty} \Phi_T^{-1}(b, \beta) e^{2\pi i \beta b} \, db$. Inverse kernels and “direct” kernels are related as

$$ \int_{-\infty}^{\infty} \Phi_T^{-1}(b-u, \beta) \Psi_T(u, \beta) \, du = \delta(b), $$

$$ \Psi_T^{-1}(\zeta, \beta) = 1. $$

The last relation implies $\Psi_T^{-1}(\zeta, \beta) = 1/\Psi_T(\zeta, \beta)$; therefore, an HC QTFR is regular if and only if $\Psi_T(\zeta, \beta)$ is essentially nonzero. In general, such a simple expression does not exist for the other inverse kernels, except for unitary QTFR’s (see Section IV). It is easily shown that an HC QTFR is regular if and only if the Cohen’s class QTFR corresponding to it, via the constant-Q warping mapping [1], is regular. The (generalized) $Q$-distribution, unitary $R_0$-distribution, and power unitary $R_0$-distribution are all regular.

**Example:** The inverse kernels of the $Q$-distribution $Q_X(t, f)$ are $\Phi_Q^{-1}(b, \beta) = \Phi_Q(b, \beta) = \delta(b)$ and $\Psi_Q^{-1}(\zeta, \beta) = \Psi_Q(\zeta, \beta) = 1$. Specializing (23) to the $Q$-distribution by inserting $\Phi_Q^{-1}(b, \beta) = \delta(b)$ yields the relation $V_X(b, \beta) = \int_{-\infty}^{\infty} Q_X(\zeta, f, \beta) e^{-2\pi i \beta \zeta} \, d\zeta$, which can be reformulated as

$$ X(f_1) X^*(f_2) = \int_{-\infty}^{\infty} Q_X(t, \sqrt{f_1 f_2}) e^{-2\pi \sqrt{f_1 f_2}(\ln \frac{f_2}{f_1})t} \, dt. $$

**B. Implications of Regularity**

The regularity of an HC QTFR has far-reaching implications, which are summarized in the following.

**Implication 1:** From a regular HC QTFR, the signal can be recovered up to a constant phase. In order to recover the signal, we first calculate $V_X(b, \beta)$ via (23). Then, it follows from (3) that the signal, up to an unknown constant phase $\phi$, can be derived from $V_X(b, \beta)$ as

$$ X(f) = \frac{V_X\left(\ln \frac{f}{f_0}, \ln \frac{f}{f_0}\right)}{\sqrt{f V_X(\ln \frac{f}{f_0}, 0)}} e^{i \phi}. $$

The frequency $\hat{f}$ can be chosen arbitrarily apart from the requirement that $V_X(\ln \frac{\hat{f}}{f_0}, 0)$ be nonzero.

**Implication 2:** From a regular HC QTFR, any quadratic signal representation can be derived by a linear transformation. Any quadratic signal representation of an analytic signal can be written as [21]

$$ T_X(\Theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_T(\Theta; f_1, f_2) X(f_1) X^*(f_2) \, df_1 \, df_2. $$

Here, $\Theta$ is a parameter vector such as $(t, f)$ in the case of a QTFR, and $K_T(\Theta; f_1, f_2)$ is a kernel characterizing $T$. Expressing $X(f_1) X^*(f_2)$ in terms of $V_X(b, \beta)$ (see (3)) and inserting (23), it is seen that $T_X(\Theta)$ can be derived from a regular HC QTFR $T_X(t, f)$ via a linear transformation

$$ T_X(\Theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_T(\Theta; t', f') T_X(t', f') \, dt' \, df'. $$

where the kernel $L_T(\Theta; t', f')$ is constructed as

$$ L_T(\Theta; t', f') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_T(\Theta; f' e^{\beta/2}, f' e^{-\beta/2}) \times \Phi_T^{-1}\left(\ln \frac{f'}{f}, \beta\right) e^{-2\pi i \beta t'} \, df' \, d\beta. $$

If the quadratic signal representation $\hat{T}_X(\Theta)$ is itself an HC QTFR, then (26) simplifies to

$$ \hat{T}_X(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda_T\left(f - t', f'\right) T_X(t', f') \, dt' \, df' $$

with

$$ \lambda_T(c, b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_T(b - u, \beta) \psi^{-1}(u, \beta) e^{2\pi i \beta c} \, du \, d\beta. $$

An important degenerate special case of a quadratic signal representation (25) is a quadratic form

$$ \Lambda_X = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X^*(f_1) D(f_1, f_2) X(f_2) \, df_1 \, df_2 $$

for which $K(\Theta; f_1, f_2) = D(f_2, f_1)$. Specializing (26) and (27), the quadratic form can be expressed as a weighted TF integral of a regular HC QTFR $T_X(t, f)$

$$ \Lambda_X = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(\Theta) T_X(t, f) \, dt \, df $$

where

$$ L(\Theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(f', f' e^{\beta/2}, f' e^{-\beta/2}) \Phi_T^{-1}\left(\ln \frac{f'}{f}, \beta\right) \times e^{-2\pi i \beta t'} \, df' \, d\beta. $$

Quadratic forms are important as they occur in many applications, especially in statistical signal processing. For example, the optimum detection statistic for the “Gauss-Gauss” detection problem involves a quadratic form [50]. The expression (31) can here be interpreted as a TF correlator using a TF reference (weighting) function $L(\Theta)$ that incorporates the statistical a priori information [34]. Although this TF formulation does not yield an efficient implementation, it often facilitates the detector’s interpretation [34]–[37].

**Example:** Since the $Q$-distribution is regular, any quadratic signal representation can be derived from the $Q$-distribution via a linear transformation. Specializing (27) and (29) to the $Q$-distribution by inserting $\Phi_Q^{-1}(b, \beta) = \delta(b)$, we obtain, respectively, the transformation kernels

$$ L_Q(\Theta; t', f') = f' \int_{-\infty}^{\infty} K_T(\Theta; f' e^{\beta/2}, f' e^{-\beta/2}) \times e^{-2\pi i \beta t'} \, df' \, d\beta, $$

$$ \lambda_Q(c, b) = \int_{-\infty}^{\infty} \Phi_Q(b, \beta) e^{2\pi i \beta c} \, d\beta. $$

By way of example, the following relations show how the hyperbolic ambiguity function $B_X(\zeta, \beta)$ in (4), the unitary $R_0$-distribution $P_X(t, f)$ in (6), the generalized $Q$-distribution $Q_X^{(c)}(t, f)$ in (22), the
hyperbologram \( Y_X(t, f) = \frac{\beta}{f} \int_0^\infty X(f') \Gamma^4(f, f') e^{j2\pi t f' \ln(f'/f)} df' \), and the wideband ambiguity function \( \chi_X(\tau, \alpha) = \int_0^\infty X(\alpha) X^*(\alpha) e^{j2\pi t\tau} df \) [1] can be derived from the Q-distribution:

\[
B_X(\zeta, \beta) = \int_{-\infty}^\infty \int_0^\infty Q_X(t, f) e^{-j2\pi t f - j\ln(f/f)} dt df,
\]

\[
P_X(t, f) = \int_{-\infty}^\infty \int_0^\infty Q_X(f) \left( \int f G_T(\beta), f G_T(\beta) \right) e^{j2\pi (f-\beta)\delta} df d\beta,
\]

with \( G_T(\beta) = \frac{\beta^2}{2\sinh(\beta/2)} \),

\[
Q_X^2(t, f) = \int_{-\infty}^\infty \int_0^\infty Q_X(f) \left( \int f G_T(\beta), f G_T(\beta) \right) e^{j2\pi (f-\beta)\delta} df d\beta,
\]

\[
Y_X(t, f) = \int_{-\infty}^\infty \int_0^\infty Q_T \left( \frac{1}{f} \frac{\delta}{\beta}, \frac{\delta}{\beta} \right) \times Q_X(t', f') dt' df',
\]

\[
\chi_X(\tau, \alpha) = \int_{-\infty}^\infty \int_0^\infty Q_X(t, f) e^{-j2\pi (f - \alpha)\ln(f/f)} dt df.
\]

Furthermore, any quadratic form (30) can be expressed as a weighted integral of the Q-distribution

\[
L_\Lambda Q(t, f) = \int_{-\infty}^\infty D(\delta^{3/2}, \delta) e^{-2\pi t f \ln(\delta)} \delta \beta d\beta
\]

where \( D(f_1, f_2) \) is given in (30). We note that \( L_\Lambda Q(t, f) \) is the HC counterpart of the Weyl symbol [51]–[53].

Implication 3: For a regular HC QTFR, a linear transformation of the signal results in a linear transformation of the QTFR. If \( T \) is a regular HC QTFR, then the QTFR \( T_X(t, f) \) of a linearly transformed signal \( \hat{X}(f) = \int_0^\infty H(f, f') X(f') df' \) is a linearly transformed QTFR of the original signal

\[
T_X(t, f) = \int_{-\infty}^\infty \int_0^\infty L_H T(t, f; t', f') T_X(t', f') dt' df'.
\]

where the kernel \( L_H T(t, f; t', f') \) is constructed as [21]

\[
L_H T(t, f; t', f') = \int_0^\infty \int_0^\infty \int_0^\infty H(f_1 \delta^{3/2}, f_2 \delta^{3/2}) \times H^*(f_1 \delta^{3/2}, f_2 \delta^{3/2}) \times \Phi_T \left( \frac{f}{f_1 \delta^{3/2}}, \frac{f_1}{f_2 \delta^{3/2}} \right) \times \Phi_T \left( \frac{f}{f_1 \delta^{3/2}}, \frac{f_1}{f_2 \delta^{3/2}} \right) \times e^{j2\pi (t - t')f \ln(f/f')} df_1 df_2.
\]

This is a mathematical framework for “covariance properties.”

Example: For the Q-distribution (i.e., \( T = Q \)), the kernel (32) becomes

\[
L_H Q(t, f; t', f') = \int_0^\infty \int_0^\infty \int_0^\infty H(f \delta^{3/2}, f' \delta^{3/2}) \times H^*(f \delta^{3/2}, f' \delta^{3/2}) \times e^{j2\pi (t - t')f \ln(f/f')} df \delta d\beta.
\]

For example, with \( H(f, f') = G(f) \delta(f' - f) \), the following covariance property of the Q-distribution is obtained: \( \hat{X}(f) = G(f) X(f) \Rightarrow \hat{Y}(f) = \int_0^\infty G(t - t', f) \chi_X(t', f) dt' \).

Further special cases involving the Q-distribution are the covariance properties P1, P2, P14, P16, P19, and P20 in Table II of Part I [1].

Implication 4: In the case of a regular QTFR, a signal basis induces a QTFR basis. If the signals \( Y_k(f) \) with \( k = 1, \ldots, \infty \) are linearly independent and complete in the linear space of all square-integrable (finite-energy) analytic signals, and if \( T \) is a regular HC QTFR, then the signals’ auto- and cross-QTFR’s \( T_{Y_k, Y_l}(t, f) \) with \( k = 1, \ldots, \infty \) and \( l = 1, \ldots, \infty \) are linearly independent and complete in \( L_2(\mathbb{R} \times \mathbb{R}^+) \), which is the space of all square-integrable 2-D functions \( F(t, f) \) defined for \( f > 0 \) [21]. Hence, any function \( F(t, f) \) in \( L_2(\mathbb{R} \times \mathbb{R}^+) \) can be expanded into the basis \( \{ T_{Y_k, Y_l}(t, f) \} \), and the expansion coefficients are uniquely determined. Such a “TF expansion” is useful for TF optimization problems [38]–[42], [44]–[46], [54].

C. Regularity in the HC Subclasses

The calculus of inverse kernels is greatly simplified in the various subclasses of the HC.

1) Localized-Kernel HC: The inverse kernels of a regular localized-kernel HC QTFR can be written as

\[
\Phi_T^{-1}(b, \beta) = \frac{1}{B_T(\beta)} \delta(b + A_T(\beta)),
\]

\[
\Psi_T^{-1}(c, \beta) = \frac{1}{B_T(\beta)} e^{j2\pi C_T(\beta)}
\]

where \( A_T(\beta) \) and \( B_T(\beta) \) were defined in (7). Hence, if \( T \) is regular if and only if its kernel \( B_T(\beta) \) is essentially nonzero. Note that this condition does not constrain the kernel \( A_T(\beta) \).

If \( T \) and \( \bar{T} \) are members of the localized-kernel HC, and if \( \bar{T} \) is regular, then the conversion kernel \( \lambda_T(\beta, b) \) in (28), (29) is given by

\[
\lambda_T(b, \beta) = \int_{-\infty}^\infty \frac{B_T(\beta)}{B_T(\beta)} \delta(b - [A_T(\beta) - A_T(\beta)]) e^{j2\pi \beta \delta} d\beta.
\]

If \( A_T(\beta) = A_T(\beta) \), which is satisfied within the affine HC or the power HC, then \( \lambda_T(b, \beta) = \omega_T(\delta) \delta(b) \) with \( \omega_T(\delta) = \int_{-\infty}^\infty \frac{B_T(\beta)}{B_T(\beta)} e^{j2\pi \beta \delta} d\beta \), and (28) simplifies to the convolution (extending (15) and (19))

\[
\bar{T}_X(t, f) = \int_{-\infty}^\infty \omega_T(\delta(t - t')) T_X(t', f') dt'.
\]

2) Affine HC and Power HC’s: Since the affine HC is a subclass of the localized-kernel HC, it follows from (33) that the inverse kernels of a regular QTFR of the affine HC can be written as

\[
\Phi_T^{-1}(b, \beta) = \frac{1}{B_T(\beta)} \delta(b + A_T(\beta)),
\]

\[
\Psi_T^{-1}(c, \beta) = \frac{1}{B_T(\beta)} e^{j2\pi C_T(\beta)}
\]

where \( A_T(\beta) = \ln(\sinh(\beta/2)^2) \). Hence, if \( T \) is regular if and only if \( B_T(\beta) \) is nonzero. From a regular affine HC QTFR
any other affine HC QTFR \( \hat{T} \) can be derived via the convolution in (34) (generalizing (15)). This discussion can be generalized to all power HC’s (simply replace \( A_T(\beta) \) by \( A_{T(b)}(\beta) = \frac{1}{2} \ln \left( \frac{\cosh(b \beta)}{\cosh(3\beta/2)} \right) \)).

3) Power-Warp HC: The inverse kernels of a regular QTFR of the power-warp HC can be written as
\[
\Phi_T^{-1}(b, \beta) = \frac{1}{|\beta|} s_T^{-1}(b), \quad \Psi_T^{-1}(\zeta, \beta) = s_T^{-1}(\zeta, \beta)
\]
where the inverse 1-D kernels \( s_T^{-1}(\eta) \) and \( s_T^{-1}(\gamma) \) are a Fourier transform pair and are related to the power-warp HC kernels in (20) according to \( s_T(\eta) = \delta(\eta) \) and \( s_T^{-1}(\gamma) s_T(\gamma) = 1 \). From \( s_T^{-1}(\gamma) = 1/|s_T(\gamma)| \), it follows that \( T \) is regular if and only if \( s_T(\gamma) \) is nonzero. If \( T \) and \( \hat{T} \) are members of the power-warp HC, and if \( T \) is regular, then the conversion kernel \( \lambda_{TT}(c, b) \) in (28) and (29) is given by
\[
\lambda_{TT}(c, b) = \int_{-\infty}^{\infty} \mu_{TT}(\eta) \frac{1}{|\eta|} e^{-\pi \eta \bar{c}_b} d\eta
\]
with
\[
\mu_{TT}(b) = \int_{-\infty}^{\infty} s_T(\eta - \eta') s_T^{-1}(\eta') d\eta'.
\]

IV. UNITARITY

The QTFR property of unitarity [21], or validity of Moyal’s formula, expresses a preservation of inner products and norms. A unitary QTFR is also regular, with the inverse kernels being essentially equal to the “direct” kernels. The unitarity property has some further important implications.

A. Moyal’s Formula, Kernel Constraints, and Inverse Kernels

An HC QTFR is unitary if the linear transformations underlying the general expressions (1) and (2) preserve inner products, i.e., if \( \langle X, \hat{T} Y \rangle = \langle X, Y \rangle \) and \( \langle X, \hat{T} Y \rangle = \langle B_X, B_Y \rangle \), where \( V_X(b, \beta) \) and \( B_X(\zeta, \beta) \) are defined in (3) and (4), respectively. These equations are equivalent and can be rewritten as
\[
\langle X, \hat{T} Y \rangle = \langle X, Y \rangle
\]
which is known as Moyal’s formula [55]. Thus, a QTFR is unitary if and only if it satisfies Moyal’s formula. An important special case of Moyal’s formula is \( ||T_X|| = ||X|| \). It can be shown that an HC QTFR is unitary if and only if the corresponding Cohen’s class QTFR is unitary. Examples of unitary HC QTFR’s are the (generalized) Q-distribution, the unitary Q-distribution, and the power unitary Q-distribution.

The following equivalent kernel constraints can be shown to be necessary and sufficient for an HC QTFR to be unitary [14], [21]:
\[
\int_{-\infty}^{\infty} \Phi_T(\eta - \eta', \beta) d\eta' = \delta(\eta), \quad \Psi_T(\zeta, \beta) = 1.
\]

The last relation shows that \( \Psi_T(\zeta, \beta) \) must be unimodular, \( ||\Psi_T(\zeta, \beta)|| = 1 \). Comparing (35) with (24), it follows that a unitary HC QTFR is regular, with inverse kernels \( \Phi_T^{-1}(b, \beta) = \Phi_T(\beta, -b) \) and \( \Psi_T^{-1}(\zeta, \beta) = \Psi_T(\beta, -\zeta) \). Hence, the inverse kernels of a unitary QTFR are easily calculated from the direct kernels.

B. Implications of Unitarity

Since a unitary QTFR is regular, all results discussed in Section III-B apply. Moreover, unitarity has some important implications of its own, three of which are summarized in the following.

Implication 1: The squared magnitude of any quadratic signal representation can be written as a quadratic form involving an arbitrary unitary HC QTFR. For the general quadratic signal representation \( \bar{T}_X(\Theta) \) in (25), \( \bar{T}_X(\Theta) \) can be written as a quadratic form
\[
\bar{T}_X(\Theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_X(t, f) \bar{T}_X(\Theta; f) dT_X(t, f) d\Theta
\]
where \( T \) is an arbitrary unitary HC QTFR, and \( \bar{T}_X(\Theta; f, f', t') \) is given by [21]
\[
L_{TT}(f, f', t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_T(\Theta; f e^{3\beta/2}, f' e^{3\beta/2})
\]
\[
\times K_T^*(\Theta; f f' e^{-3\beta/2}, f f' e^{-3\beta/2})
\]
\[
\times \Phi_T(\ln \frac{f}{f'}, \beta) \Phi_T^*(\ln \frac{f'}{f}, \beta)
\]
\[
\times e^{2\pi(f f' - f' f') \beta} d\beta d\beta d\beta.
\]

Example: Since the Q-distribution \( Q_X(t, f) \) is unitary, the squared magnitude of any quadratic signal representation can be written as a quadratic form of \( Q_X(t, f) \). Specializing (37) to \( T = Q \) yields the kernel
\[
L_{QQ}(f, f', t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_X(t, f) Q_X^*(f', f) dT_X(t, f) dT_X^*(f', f')
\]
\[
\times K_T(\Theta; f e^{3\beta/2}, f' e^{3\beta/2}) K_T^*(\Theta; f' e^{-3\beta/2}, f e^{-3\beta/2}) e^{2\pi(f f' - f' f') \beta} d\beta d\beta d\beta d\beta.
\]

Two special cases of (36) involving the Q-distribution are the relations
\[
[Q_X(t, f)]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_X(t, f) Q_X(t, f) dT_X(t, f) dt dT_X^*(t, f)
\]
\[
\times Q_X(t, f) Q_X(t, f) d\beta d\beta
\]
\[
[B_X(\zeta, \beta)]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_X(\zeta, \beta) B_X(\zeta, \beta) d\zeta d\beta
\]
\[
\times Q_X(t, f) Q_X(t, f) dt d\zeta.
\]
The first relation (derived independently in [12]) is the HC counterpart of the interference formula of the Wigner distribution [56].
Implication 2: In the case of a unitary HC QTFR, an orthonormal signal basis induces an orthonormal QTFR basis. If the signals $Y_k(f)$ with $k = 1, \ldots, \infty$ are orthonormal and complete in the space of finite-energy analytic signals, and if $T$ is unitary, then the auto- and cross-QTFR’s $T_{Y_kY_l}(t, f)$ with $k = 1, \ldots, \infty$ and $l = 1, \ldots, \infty$ are orthonormal and complete in $L_2(\mathbb{R} \times \mathbb{R}_+)$ [21]. Thus, unitarity assures that the orthornormality of the basis signals $Y_k(f)$ carries over to the 2-D basis functions $T_{Y_kY_l}(t, f)$. This will not be the case if the QTFR $T$ is merely regular (cf. Section III-B).

Implication 3: For a unitary QTFR, there exist standard solutions to the problem of least-squares signal synthesis. The least-squares signal synthesis problem is solved using the following equation:

$$X_{\text{opt}}(f) \triangleq \arg \min_{X} ||M - TX||,$$

where $M$ is the matrix of the basis functions.

Signal synthesis methods are useful for the isolation of signal components. The solution to this problem is obtained by minimizing the squared difference between the observed and predicted signals.

If $T$ is a unitary HC QTFR, then the signal synthesis result $X_{\text{opt}}(f)$ can be derived as follows [38] (see [38], [39], [44], [54] for alternative formulations for an orthonormal signal basis):

1. The transformation (23) is applied to $M(t, f)$:
   $$\hat{V}(b, \beta) = \int_{-\infty}^{\infty} \Phi_T(f) \left( \frac{f}{\beta} - b \right) M(t, f) e^{j2\pi f t} dt df.$$

2. The function $V(b, \beta)$ is converted as $J(f_1, f_2) = \sqrt{f_1 f_2} \hat{V}(\ln \frac{\sqrt{f_1 f_2}}{f_1}, \ln \frac{\sqrt{f_1 f_2}}{f_2}).$

3. The Hermite part of $J(f_1, f_2)$ is formed, $J_h(f_1, f_2) = \frac{1}{\sqrt{\tau_1}} J(f_1, f_2) + J^*(f_2, f_1)$.

4. The largest eigenvalue $\lambda_1$ and the corresponding eigenfunction $E_1(f)$ of $J_h(f_1, f_2)$ are calculated.

5. If $\lambda_1 > 0$, the signal synthesis result is given by
   $$X_{\text{opt}}(f) = \sqrt{\lambda_1} E_1(f),$$
   where $\hat{\phi}$ is an arbitrary constant phase. If $\lambda_1 \leq 0$, the signal synthesis result is $X_{\text{opt}}(f) = 0$.

Example: The signal synthesis problem $X_{\text{opt}}(f) = \arg \min_X ||M - TX||$ for the distribution problem is solved using the method summarized above. The kernel $J(f_1, f_2)$ involved in this method is given by

$$J(f_1, f_2) = \int_{-\infty}^{\infty} M(t, \sqrt{f_1 f_2}) e^{-j2\pi \sqrt{f_1 f_2} \ln \frac{\sqrt{f_1 f_2}}{f_1}} dt.$$

C. Unitarity in the HC Subclasses

A QTFR of the localized-kernel HC, affine HC, or power-HC$_R$ is unitary if and only if the kernel $B_T(\beta)$ in (7) is unimodular (see $P_2$ in Table III)

$$|B_T(\beta)| = 1$$

which implies $\frac{1}{\beta B_T(\beta)} = B_T^*(\beta)$. Note that the condition for unitarity does not constrain the kernel $A_T(\beta)$.

A QTFR of the power-warp HC is unitary if and only if the kernel $S_T(\gamma)$ in (20) is unimodular

$$|S_T(\gamma)| = 1.$$

Implication 4: For a unitary power-warp HC QTFR, the inverse kernels are simply $S_T^{-1}(\eta) = S_T^*(\eta)$ and $S_T^{-1}(\gamma) = S_T^*(\gamma)$.

V. Conclusion

The hyperbolic class (HC) comprises quadratic time-frequency representations (QTFR’s) with constant-Q characteristic. In this paper, we have defined and discussed several important subclasses of the HC in which the description of an HC QTFR in terms of 2-D kernel functions is simplified. Furthermore, we have studied the properties of regularity and unitarity in the HC and its subclasses.

Motivated by the localized-kernel affine subclass, we have introduced the localized-kernel subclass of the HC via a parameterization of the 2-D kernels in terms of two 1-D functions. The localized-kernel structure was shown to be related to a time-frequency concentration property.

The affine subclass of the HC is the intersection of the HC with the affine QTFR class. It is part of the localized-kernel HC as well as part of the localized-kernel affine class. Any QTFR of the affine HC can be derived from the unitary $\tilde{R}_0$-distribution through a 1-D convolution. In a similar manner, the power subclasses of the HC have been defined as intersections of the HC and the power QTFR classes.

A further HC subclass is the power-warp subclass, which consists of all HC QTFR’s satisfying the power-warp covariance property. The power-warp HC is the HC counterpart of the shifted-scale covariant subclass of Cohen’s class, and its members are superpositions of generalized $Q$-distributions.

We have discussed two fundamental QTFR properties and their implications. The first property—regularity—expresses, basically, the QTFR’s “invertibility.” The regularity property has several further important implications. In particular, any quadratic signal representation can be derived from a regular QTFR, and linear signal transformations are equivalent to linear QTFR transformations.

A more restrictive property than regularity is a QTFR’s unitarity (validity of Moyal’s formula), which allows a simple calculation of the inverse QTFR kernels. The squared magnitude of any quadratic signal representation can be expressed in terms of a unitary QTFR. In addition, there exist standard solutions to the signal synthesis problem in the case of a unitary QTFR. Specialization of general results from the regularity/unitarity theory yielded interesting new relations involving the $Q$-distribution.

APPENDIX

Proof of Theorem 1

Inserting (1) into the left-hand side of the concentration property (9), performing a Fourier transform with respect to $t$ and an inverse Fourier transform with respect to $c$ on both sides of the resulting equation, and simplifying yields the equation (equivalent to (9))

$$\int_{-\infty}^{\infty} \Phi_T(b, \beta) R_{\tilde{f}_\beta}(b) \hat{\phi}(\Xi_{f_\beta}(b) - \gamma) db = \gamma^2 (f \hat{\phi}(\gamma - f_\beta^2(f) \beta)).$$

(38)
where $R_{f,\beta}(b)$ and $\Xi_{f,\beta}(b)$ have been defined in Theorem 1. If (as assumed in Theorem 1) $\Xi_{f,\beta}(b)$ is one-to-one and differentiable for all fixed $f, \beta$, then the left-hand side of (38) can be written as

$$
\Phi_T(b, \beta) \frac{R_{f,\beta}(b)}{|\Xi_{f,\beta}(b)|} \delta(b - \Xi_{f,\beta}^{-1}(\gamma)) \, db
$$

$$
= \Phi_T(\Xi_{f,\beta}^{-1}(\gamma), \beta) \frac{R_{f,\beta}(\Xi_{f,\beta}^{-1}(\gamma))}{|\Xi_{f,\beta}^{-1}(\gamma)|},
$$

where $\Xi_{f,\beta}^{-1}(\gamma)$ is the function inverse to $\Xi_{f,\beta}(\gamma)$. Substituting $\gamma = \Delta_T(f, \beta)$, which is Condition 1 in Theorem 1, we have now to make sure that

$$
\Phi_T(b, \beta) = \frac{\varphi^2(f)}{R_{f,\beta}(\Delta_T(f, \beta))} \delta(b - \Delta_T(f, \beta)),
$$

or equivalently

$$
\Phi_T(b, \beta) = \frac{\varphi^2(f)}{R_{f,\beta}(|\Delta_T(f, \beta)|)} \delta(b - \Delta_T(f, \beta)).
$$

(39)

Obviously, the right-hand side of (39) must not depend on $f$. This requires, first of all, that $\Xi_{f,\beta}^{-1}(\Delta_T(f, \beta))$ is independent of $f$, so that $\Xi_{f,\beta}^{-1}(\Delta_T(f, \beta)) = \Delta_T(f, \beta)$ with some real-valued function $\Delta_T(f, \beta)$. Thus, there must exist a function $A_T(\beta)$ such that

$$
\Xi_{f,\beta}^{-1}(\Delta_T(f, \beta)) = A_T(\beta)
$$
or equivalently,

$$
\Delta_T(f, \beta) = \Xi_{f,\beta}(A_T(\beta))
$$
for all $f, \beta \in \mathbb{R}$, which is Condition 1 in Theorem 1. If this first condition is satisfied, (39) simplifies to

$$
\Phi_T(b, \beta) = \frac{\varphi^2(f)}{R_{f,\beta}(\Delta_T(\beta))} \delta(b - \Delta_T(\beta)),
$$

and we have now to make sure that $\frac{\varphi^2(f)}{R_{f,\beta}(\Delta_T(\beta))}$ is independent of $f$, which is Condition 2. If this second condition is satisfied too, then we can write

$$
\frac{\varphi^2(f)}{R_{f,\beta}(\Delta_T(\beta))} = B_T(\beta) = B_T(\beta) \delta(b - \Delta_T(\beta)),
$$

which is Condition 3.

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