The Hyperbolic Class of Quadratic Time-Frequency Representations—Part II: Subclasses, Intersection with the Affine and Power Classes, Regularity, and Unitarity

Franz Hlawatsch, *Member, IEEE*, Antonia Papandreou-Suppappola, *Member, IEEE*, and G. Faye Boudreaux-Bartels, *Senior Member, IEEE*

***Abstract*— Part I of this paper introduced the *hyperbolic class* (HC) of quadratic/bilinear time-frequency representations (QTFR's) as a new framework for constant-Q time-frequency analysis. The present Part II defines and studies the following four subclasses of the HC:**

- **The *localized-kernel subclass* of the HC is related to a time-frequency concentration property of QTFR's. It is analogous to the localized-kernel subclass of the affine QTFR class.**
- **The *affine subclass* of the HC (affine HC) consists of all HC QTFR's that satisfy the conventional time-shift covariance property. It forms the intersection of the HC with the affine QTFR class.**
- **The *power subclasses* of the HC consist of all HC QTFR's that satisfy a "power time-shift" covariance property. They form the intersection of the HC with the recently introduced power classes.**
- **The *power-warp subclass* of the HC consists of all HC QTFR's that satisfy a covariance to power-law frequency warpings. It is the HC counterpart of the shift-scale covariant subclass of Cohen's class.**

All of these subclasses are characterized by 1-D kernel functions. It is shown that the affine HC is contained in both the localized-kernel hyperbolic subclass and the localized-kernel affine subclass and that any affine HC QTFR can be derived from the Bertrand unitary P_0-distribution by a convolution.

We furthermore consider the properties of *regularity* (invertibility of a QTFR) and *unitarity* (preservation of inner products, Moyal's formula) in the HC. The calculus of inverse kernels is developed, and important implications of regularity and unitarity are summarized. The results comprise a general method for least-squares signal synthesis and new relations for the Altes-Marinovich Q-distribution.

I. Introduction

THE QUADRATIC/BILINEAR time-frequency representations (QTFR's) of the *hyperbolic class* (HC) have been proposed in Part I of this paper [1]. The HC comprises all QTFR's $T_X(t,f)$ that are "covariant" to time-frequency (TF) scalings, $T_{\mathcal{C}_a X}(t,f) = T_X(at, f/a)$ with $(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{|a|}} X(f/a)$, and covariant to hyperbolic time-shifts, $T_{\mathcal{H}_c X}(t,f) = T_X(t - c/f, f)$ with $(\mathcal{H}_c X)(f) = e^{-j2\pi c \ln(f/f_r)} X(f)$. Here, $X(f)$ is the Fourier transform of an analytic signal $x(t)$ (i.e., $X(f) = 0$ for $f < 0$), and t and f denote time and frequency, respectively. Any HC QTFR can be written as[1]

$$\begin{aligned} T_X(t,f) &= \frac{1}{f}\int_0^\infty \int_0^\infty \Gamma_T\left(\frac{f_1}{f}, \frac{f_2}{f}\right) e^{j2\pi t f \ln \frac{f_1}{f_2}} \\ &\quad \times X(f_1) X^*(f_2) df_1 df_2 \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \Phi_T\left(\ln\frac{f}{f_r} - b, \beta\right) V_X(b,\beta) e^{j2\pi t f \beta} db\, d\beta \end{aligned} \tag{1}$$

$$\begin{aligned} &= \int_{-\infty}^\infty \int_{-\infty}^\infty \Psi_T(\zeta,\beta) B_X(\zeta,\beta) \\ &\quad \times e^{j2\pi\left(tf\beta - \zeta \ln \frac{f}{f_r}\right)} d\zeta\, d\beta, \quad f > 0 \end{aligned} \tag{2}$$

where $f_r > 0$ is a fixed reference or normalization frequency, the *hyperbolic signal product* is defined as

$$V_X(b,\beta) = f_r e^b X(f_r e^{b+\beta/2}) X^*(f_r e^{b-\beta/2}) \tag{3}$$

and the *hyperbolic ambiguity function* [1]–[3] is

$$\begin{aligned} B_X(\zeta,\beta) &= \int_{-\infty}^\infty V_X(b,\beta) e^{j2\pi\zeta b} db \\ &= \int_0^\infty X(fe^{\beta/2}) X^*(fe^{-\beta/2}) e^{j2\pi\zeta \ln\frac{f}{f_r}} df. \end{aligned} \tag{4}$$

The 2-D kernels $\Gamma_T(b_1,b_2)$, $\Phi_T(b,\beta)$, and $\Psi_T(\zeta,\beta)$ uniquely characterize the HC QTFR T; they are related as $\Gamma_T(b_1,b_2) = \frac{1}{\sqrt{b_1 b_2}} \Phi_T(-\ln\sqrt{b_1 b_2}, \ln\frac{b_1}{b_2})$ and $\Psi_T(\zeta,\beta) = \int_{-\infty}^\infty \Phi_T(b,\beta)\, e^{j2\pi\zeta b} db$.

Many HC QTFR's produce a *constant-Q TF analysis* similar to the *affine QTFR class* and the *wavelet transform* [4]–[15], where higher frequencies are analyzed with better time resolution and poorer frequency resolution. The HC is connected

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F. Hlawatsch is with INTHFT, Vienna University of Technology, A-1040 Vienna, Austria (e-mail: fhlawats@email.tuwien.ac.at).

A. Papandreou-Suppappola and G. F. Boudreaux-Bartels are with the Department of Electrical and Computer Engineering, University of Rhode Island, Kingston, RI 02881 USA.

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[1] For the sake of notational simplicity, we focus on quadratic auto-representations $T_X(t,f)$. The extension to bilinear cross-representations $T_{X,Y}(t,f)$ is straightforward [1].

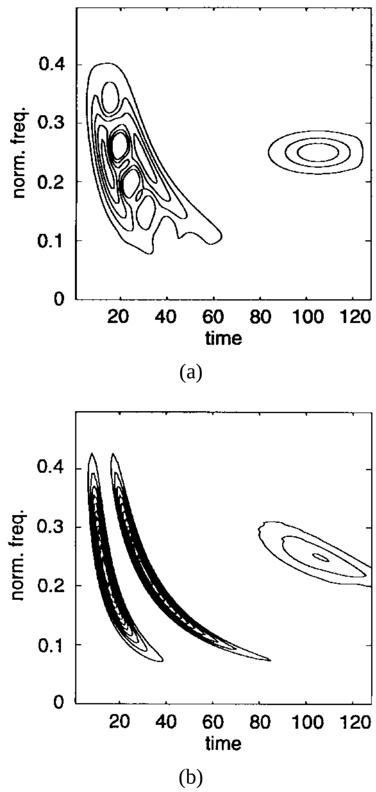

Fig. 1. Comparison of Cohen's class QTFR and HC QTFR for a three-component signal consisting of two hyperbolic impulses and a Gaussian signal. (a) Smoothed pseudo Wigner distribution [5] from Cohen's class. (b) Smoothed pseudo Altes–Marinovich Q-distribution from the HC [1]. Both QTFR's employ a TF smoothing to attenuate cross terms [1], [17]. The smoothing in the HC QTFR in (b) is adapted to the hyperbolic TF geometry, which entails improved TF resolution in the two hyperbolic components at the cost of some TF distortion in the Gaussian component. The software used for computing the smoothed pseudo Q-distribution was obtained from [17].

to a *hyperbolic TF geometry*, which is particularly adapted to Doppler-invariant and self-similar signals [1], [3], [6], [16]. Fig. 1 shows that HC QTFR's yield better TF resolution than conventional (Cohen's class) QTFR's if the signal under analysis is consistent with the hyperbolic TF geometry for which the HC is particularly suited.

The HC can be formally derived from the well-known Cohen's class of TF shift-covariant QTFR's [18]–[21], [4], [5] by a "constant-Q warping" transformation [1], [22], and it is thus "unitarily equivalent" [23], [24] to Cohen's class. Two prominent members of the HC are the *Altes-Marinovich Q-distribution* [1], [2], [25]

$$Q_X(t,f) = \int_{-\infty}^{\infty} V_X\left(\ln\frac{f}{f_r}, \beta\right) e^{j2\pi t f \beta}\, d\beta = f\int_{-\infty}^{\infty} X(fe^{\beta/2})\, X^*(fe^{-\beta/2})\, e^{j2\pi t f\beta}\, d\beta \tag{5}$$

and the *Bertrand unitary P_0-distribution* [6]–[9], [26]

$$P_X(t,f) = \int_{-\infty}^{\infty} V_X\left(\ln\frac{f}{f_r} - A_P(\beta),\, \beta\right) e^{j2\pi t f\beta}\, d\beta = f\int_{-\infty}^{\infty} X(f\,G_P(\beta)\,e^{\beta/2}) X^*(f\,G_P(\beta)\,e^{-\beta/2}) \times G_P(\beta) e^{j2\pi t f \beta}\, d\beta \tag{6}$$

where $G_P(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$, $A_P(\beta) = -\ln G_P(\beta)$. The implementation of HC QTFR's is discussed in [17], [27], and [10].

The present Part II of this paper defines and studies four important subclasses of the HC (see Fig. 2), whose QTFR's satisfy an additional property besides the scale and hyperbolic time-shift covariance properties satisfied by all HC QTFR's. In Section II-A, we introduce the "localized-kernel subclass" of the HC, which is analogous to the localized-kernel subclass of the affine QTFR class [7], [9], [11]–[13]. It is closely related to a *TF concentration property* (see the "additional property" for the localized-kernel HC in Table I) and, hence, is particularly suited for signals with specific TF geometries. It forms a convenient framework for many important HC QTFR's such as the Q-distribution and the unitary P_0-distribution. Section II-B discusses the "affine subclass" of the HC (affine HC) [8], [28], which consists of all HC QTFR's satisfying the particularly important *time-shift covariance property* (see Table I). The affine HC forms the intersection of the HC and the *affine class* [4]–[8], and it is contained in both the localized-kernel HC and the localized-kernel affine subclass [7], [9], [11]–[13]. Section II-C considers a generalization of the affine HC, namely, the "power subclasses" of the HC (power HC's), which consist of all HC QTFR's satisfying the *power time-shift covariance property* [8], [29]. The power HC's form the intersection of the HC with the *power classes* [29]–[31]. All power HC's are contained in the localized-kernel HC. The intersection subclasses studied in Sections II-B and C (power HC's with the affine HC as special case) emphasize the HC's relation to other scale covariant classes. These classes are all based on the properties of i) scale covariance and ii) covariance to time shifts corresponding to a dispersive or nondispersive group delay law. Scale covariance is important in multiresolution analysis, and specific types of time-shift covariance are useful for accommodating the group delay characteristics of various systems and transmission media. Section II-D considers another HC subclass whose members satisfy a *power-warp covariance property* [1]. This "power-warp HC" is the counterpart of the *shift-scale covariant subclass of Cohen's class* [32], [5], [20], [21], [33]. Fig. 2 summarizes pictorially the HC, the affine class, and the power classes, as well as the HC subclasses discussed in Section II.

Besides the discussion of important HC subclasses, the second main contribution of this paper is a study of the QTFR properties of regularity and unitarity [14], [21] in the HC. These properties are fundamental on a theoretical level, and they form a basis for various methods for (statistical) TF signal processing [4], [34]–[43]. Section III considers the property of *regularity* (invertibility of a QTFR) and develops the calculus of inverse kernels. Important implications of a QTFR's regularity include the recovery of the signal and the derivation of other quadratic signal representations from the QTFR. Section IV considers the property of *unitarity* (preservation of inner products, Moyal's formula). Important implications of unitarity include a method for least-squares signal synthesis [38], [39], [42], [44]–[48]. The application of the results of Sections III and IV to the Q-distribution yields interesting new relations.

TABLE I

SUBCLASSES OF THE HC. EACH HC SUBCLASS CONTAINS HC QTFR'S THAT SATISFY THE "ADDITIONAL PROPERTY" LISTED IN THE SECOND COLUMN, IN ADDITION TO THE SCALE AND HYPERBOLIC TIME-SHIFT COVARIANCE PROPERTIES SATISFIED BY ALL HC QTFR'S. THE 2-D KERNEL $\Phi_T(b,\beta)$ CAN BE EXPRESSED IN TERMS OF 1-D KERNELS $(A_T(\beta), B_T(\beta), s_T(\eta))$ AS GIVEN IN THE THIRD COLUMN. NOTE THAT $f_r > 0$ IS A FIXED REFERENCE FREQUENCY

HC SUBCLASS	ADDITIONAL PROPERTY	FORM OF KERNEL $\Phi_T(b,\beta)$
Localized-kernel HC	$T_{X_c}(t,f) = r^2(f)\,\delta\Big(t - c\xi'(f)\Big)$ with $X_c(f) = r(f)\,e^{-j2\pi c\xi(f)}$	$B_T(\beta)\,\delta\Big(b - A_T(\beta)\Big)$ (see Theorem 1)
Affine HC	$T_{\mathcal{S}_\tau X}(t,f) = T_X(t-\tau, f)$ with $(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f}\,X(f)$	$B_T(\beta)\,\delta\Big(b - \ln\Big(\frac{\sinh(\beta/2)}{\beta/2}\Big)\Big)$
Power-HC$_\kappa$	$T_{\mathcal{D}_c^{(\kappa)}X}(t,f) = T_X\Big(t - c\,\tau_\kappa(f), f\Big)$ with $\tau_\kappa(f) = \frac{\kappa}{f_r}\left(\frac{f}{f_r}\right)^{\kappa-1}$ and $(\mathcal{D}_c^{(\kappa)}X)(f) = e^{-j2\pi c(f/f_r)^\kappa}\,X(f)$	$B_T(\beta)\,\delta\Big(b - \frac{1}{\kappa}\ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big)\Big)$
Power-warp HC	$T_{\mathcal{P}_a X}(t,f) \;=\; T_X\left(\frac{at}{\left(\frac{f}{f_r}\right)^{\frac{1}{a}-1}},\; f_r\left(\frac{f}{f_r}\right)^{\frac{1}{a}}\right)$ with $(\mathcal{P}_a X)(f) = \sqrt{\frac{1}{a}\left(\frac{f}{f_r}\right)^{1/a-1}}\,X\Big(f_r\Big(\frac{f}{f_r}\Big)^{1/a}\Big)$	$\frac{1}{\lvert\beta\rvert}\,s_T\Big(-\frac{b}{\beta}\Big)$

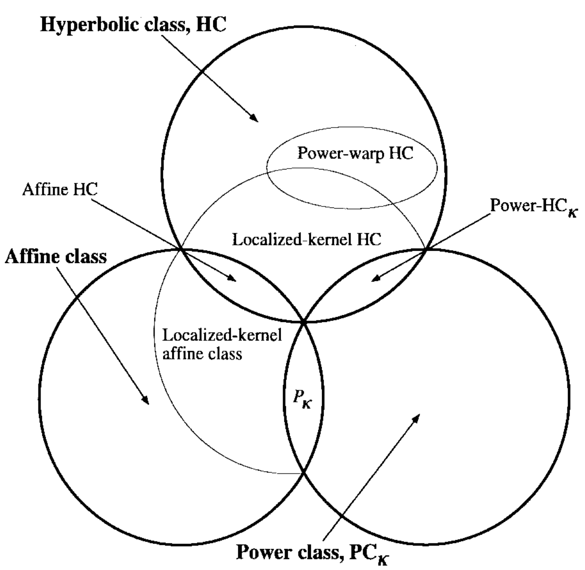

Fig. 2. QTFR classes for scale-covariant TF analysis. The HC, affine class, and power class PC$_\kappa$ contain all QTFR's that are scale-covariant and covariant to a specific type of dispersive or nondispersive time shifts. The *localized-kernel HC* is related to a TF concentration property. Similarly, the *localized-kernel affine class* is a subclass of the affine class related to a TF concentration property [12]. The *affine HC* forms the intersection of the HC with the affine class. The *power HC$_\kappa$* forms the intersection of the HC with the power class PC$_\kappa$. The *power-warp HC* contains all QTFR's that satisfy the power-warp covariance property. The intersection between the affine class and the power class PC$_\kappa$ (which is not discussed in this paper) contains the general form of the Bertrand P$_\kappa$-distributions [8].

II. HC SUBCLASSES

The HC subclasses discussed in this section are important since each of them is defined by a specific QTFR property (see Table I), which is satisfied in addition to the two basic properties of TF scale covariance and hyperbolic time-shift covariance satisfied by any HC QTFR. For example, it is often desirable to use a QTFR covariant to constant (nondispersive) time shifts. The subclass of HC QTFR's that are covariant to such time shifts satisfies three covariance properties:

scale covariance
hyperbolic time-shift covariance
constant time-shift covariance.

Since the first and third covariance properties define the affine QTFR class, these HC QTFR's are simultaneously members of the affine class [8], [28]. They are specifically suited to signals that are consistent with the hyperbolic TF geometry except for unknown time shifts. Furthermore, the mathematical description of the HC subclasses considered is simplified since the 2-D kernels describing general HC QTFR's are here parameterized in terms of 1-D functions.

A. The Localized-Kernel HC Subclass

The localized kernel idea was introduced by the Bertrands based on tomography [13] and then extended to the affine localized-kernel subclass [7], [9], [11], [12]. The "localized-kernel" HC subclass discussed in this section is conceptually analogous to the affine localized-kernel subclass. It consists of all HC QTFR's whose kernel $\Phi_T(b,\beta)$ in (1) is perfectly localized along the curve $b = A_T(\beta)$ in the (b,β)-plane, i.e.,

$$\Phi_T(b,\beta) = B_T(\beta)\delta(b - A_T(\beta)) \tag{7}$$

with $A_T(\beta) \in \mathbb{R}$ and $B_T(\beta) \geq 0$ arbitrary functions. We note that (7) entails $\Psi_T(\zeta,\beta) = B_T(\beta)e^{j2\pi\zeta A_T(\beta)}$ in (2). Inserting (7) into (1), we see that any QTFR of the localized-kernel HC can be written as

$$\begin{aligned} T_X(t,f) &= \int_{-\infty}^{\infty} V_X\left(\ln\frac{f}{f_r} - A_T(\beta), \beta\right) B_T(\beta) e^{j2\pi t f \beta}\, d\beta \\ &= f\int_{-\infty}^{\infty} X\big(f e^{-A_T(\beta)+\frac{\beta}{2}}\big) X^*\big(f\, e^{-A_T(\beta)-\frac{\beta}{2}}\big) \\ &\quad \times B_T(\beta) e^{-A_T(\beta)} e^{j2\pi t f\beta} d\beta. \end{aligned} \tag{8}$$

The 2-D kernels of a QTFR of the localized-kernel HC are parameterized in terms of the two 1-D functions ("kernels")

$A_T(\beta)$ and $B_T(\beta)$, which uniquely characterize the QTFR T. This fact is important as it greatly simplifies the analysis of QTFR's of the localized-kernel HC.

The localized-kernel HC is also important since, under certain assumptions, the localized-kernel structure is necessary for a class of TF concentration properties (cf. the discussion of the localized-kernel subclass of the affine class given in [12]). This is shown by considering the family of signals

$$X_c(f) = r(f)e^{-j2\pi c\xi(f)}, \quad c \in \mathbb{R}$$

with given amplitude function $r(f) \geq 0$ and given phase function $\xi(f) \in \mathbb{R}$. The group delay function of $X_c(f)$ is $\tau(f) = c\xi'(f)$, where $\xi'(f) = \frac{d}{df}\xi(f)$. We wish to find an HC QTFR T such that, for the signal family $X_c(f)$ given above, T is perfectly concentrated along the group delay curve $\tau(f) = c\xi'(f)$, i.e.,

$$T_{X_c}(t,f) = r^2(f)\delta(t - c\xi'(f)) \quad \text{for all } c \in \mathbb{R}. \tag{9}$$

The next theorem, which is proved in the Appendix, shows a relation of (9) with the localized-kernel structure (7).

Theorem 1: Let the functions $r(f) \geq 0$ and $\xi(f) \in \mathbb{R}$ be given, and assume that the function $\Xi_{f,\beta}(b) \triangleq \xi(f\,e^{-b+\beta/2}) - \xi(f\,e^{-b-\beta/2})$ is one-to-one and differentiable for any (fixed) f, β. Then, the concentration property (9) is satisfied by an HC QTFR T if and only if the following three conditions are satisfied:

1) There exists a function $A_T(\beta)$ that is independent of f and satisfies

$$\Xi_{f,\beta}(A_T(\beta)) = f\,\xi'(f)\beta \quad \text{for all } f, \beta \in \mathbb{R}. \tag{10}$$

2) The ratio $\frac{r^2(f)}{R_{f,\beta}(A_T(\beta))}$ with $R_{f,\beta}(b) = r(f\,e^{-b+\beta/2})r\,(f\,e^{-b-\beta/2})\;e^{-b}$ is independent of f.

3) The kernel $\Phi_T(b,\beta)$ of the HC QTFR T is

$$\Phi_T(b,\beta) = B_T(\beta)\delta(b - A_T(\beta))$$

with

$$B_T(\beta) = \frac{r^2(f)}{R_{f,\beta}(A_T(\beta))} \;\geq\; 0.$$

We note that Condition 1 restricts the admissible phase functions $\xi(f)$: If no $A_T(\beta)$ satisfying (10) exists, then there does not exist *any* HC QTFR satisfying the concentration property (9) for the given phase function $\xi(f)$. If a function $A_T(\beta)$ satisfying (10) exists, then we insert it in $\frac{r^2(f)}{R_{f,\beta}(A_T(\beta))}$ in order to check Condition 2. This condition restricts the admissible amplitude functions $r(f)$; it will always be satisfied if $r(f) = r_0 f^\alpha$ with $\alpha \in \mathbb{R}$, in which case $B_T(\beta) = e^{(2\alpha+1)\,A_T(\beta)}$. Table II lists some phase and amplitude functions for which the concentration property (9) is satisfied by an HC QTFR. Finally, *Condition 3 shows that the HC QTFR T must necessarily be a member of the localized-kernel HC.*

The localized-kernel HC provides a convenient framework for many important HC QTFR's, including all HC members that are also members of the affine class or of the power classes (see Sections II-B and C). Prominent members of the localized-kernel HC are the Q-distribution $Q_X(t,f)$ in (5), the generalized Q-distribution $Q_X^{(\alpha)}(t,f)$ (see Section II-D), the unitary P_0-distribution $P_X(t,f)$ in (6), and the "power unitary P_0-distribution" $P_X^{(\kappa)}(t,f)$ [22] (see Section II-C). The 1-D kernels of these QTFR's are[2]

TABLE II
SOME AMPLITUDE AND PHASE FUNCTIONS FOR WHICH THE TF CONCENTRATION PROPERTY IN (9) IS SATISFIED BY ONE OR MORE HC QTFF'S, ALONG WITH THE FUNCTIONS $A_T(\beta)$ AND $B_T(\beta)$ OBTAINED AS SOLUTIONS OF CONDITIONS 1 AND 2 IN THEOREM 1. THE GROUP DELAY FUNCTIONS IN THE THIRD COLUMN HAVE BEEN CONSIDERED IN [6] AND [8]

AMPLITUDE FUNCTION $r(f)$	PHASE FUNCTION $\xi(f)$	GROUP DELAY FUNCTION $\tau(f) = c\xi'(f)$	KERNEL $A_T(\beta)$	KERNEL $B_T(\beta)$
$r_0\,f^\alpha$	$\ln\frac{f}{f_r}$	$c\,\frac{1}{f}$	arbitrary	$e^{(2\alpha+1)A_T(\beta)}$
$r_0\,f^\alpha$	$\left(\frac{f}{f_r}\right)^\kappa \quad (\kappa \neq 0)$	$c\,\frac{\kappa}{f_r}\left(\frac{f}{f_r}\right)^{\kappa-1}$	$\frac{1}{\kappa}\,\ln\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right)$	$\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right)^{\frac{2\alpha+1}{\kappa}}$
$r_0\,f^\alpha$	$\frac{1}{2}\left(\ln\frac{f}{f_r}\right)^2$	$c\,\frac{1}{f}\,\ln\frac{f}{f_r}$	0	1

$$A_Q(\beta) = 0, \qquad B_Q(\beta) = 1;$$
$$A_{Q^{(\alpha)}}(\beta) = \alpha\beta, \qquad B_{Q^{(\alpha)}}(\beta) = 1;$$
$$A_P(\beta) = \ln\left(\frac{\sinh(\beta/2)}{\beta/2}\right), \qquad B_P(\beta) = 1;$$
$$A_{P^{(\kappa)}}(\beta) = \frac{1}{\kappa}\ln\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right), \qquad B_{P^{(\kappa)}}(\beta) = 1, \quad \kappa \neq 0. \tag{11}$$

In Table II of Part I, we listed desirable QTFR properties and the associated constraints on the 2-D HC kernels [1]. For a member of the localized-kernel HC, these constraints can be reformulated in terms of the 1-D kernels $A_T(\beta)$ and $B_T(\beta)$, as listed in the second column of Table III.

B. The Affine HC Subclass

In general, HC QTFR's need not satisfy the constant (nondispersive) time-shift covariance property

$$T_{\mathcal{S}_\tau X}(t,f) = T_X(t-\tau, f) \tag{12}$$

with

$$(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f}X(f).$$

Motivated by the exceptional importance of this property, we define the *affine subclass* of the HC (affine HC) to consist of all HC QTFR's satisfying (12). Thus, all QTFR's of the affine HC satisfy the following three QTFR properties:

time-shift covariance
scale covariance
hyperbolic time-shift covariance.

Since time-shift covariance and scale covariance are the axioms of the *affine class* [4]–[14], the QTFR's of the affine HC are also members of the affine class. In fact, the affine HC forms the intersection of the affine class and the HC as illustrated in Fig. 2. As both the affine class and the HC are frameworks for constant-Q TF analysis, it is not surprising that these classes overlap.

The affine HC is identical to the class of P_0-*distributions* previously introduced in [8] as a subclass of the affine class.

[2] The subscripts Q, $Q^{(\alpha)}$, etc. will be used to label the kernels of the QTFR's $Q_X(t,f)$, $Q_X^{(\alpha)}(t,f)$, etc., whereas the subscript T denotes the kernel of a general HC QTFR $T_X(t,f)$. For $\kappa = 1$, $P_X^{(\kappa)}(t,f)$ simplifies to $P_X(t,f)$.

TABLE III
PROPERTY CONSTRAINTS FORMULATED WITH THE KERNELS $A_T(\beta)$, $B_T(\beta)$ OF THE LOCALIZED-KERNEL HC, THE KERNEL $B_T(\beta)$ OF THE AFFINE HC, AND THE KERNEL $s_T(\eta)$ (WITH FOURIER TRANSFORM $S_T(\delta)$) OF THE POWER-WARP HC. THE PROPERTIES ARE DEFINED IN TABLE II OF PART I [1]. THE RELATIONSHIPS OF $A_T(\beta)$, $B_T(\beta)$, AND $S_T(\delta)$ WITH THE HC KERNEL $\Phi_T(b,\beta)$ ARE PROVIDED IN THE LAST COLUMN OF TABLE I. FOR THE AFFINE HC, $A_T(\beta) = \ln(\frac{\sin\mathrm{h}(\beta/2)}{(\beta/2)}) = A_P(\beta)$. THE CONSTRAINTS FOR THE POWER HC'S ARE DISCUSSED IN SECTION II-C. WE NOTE THAT $A'_T(0) = \frac{d}{d\beta}A_T(\beta)|_{\beta=0}$ ETC.

QTFR PROPERTY	LOCALIZED-KERNEL HC KERNEL CONSTRAINT	AFFINE HC KERNEL CONSTRAINT	POWER-WARP HC KERNEL CONSTRAINT
P_1: hyp. time-shift covar.	always satisfied	always satisfied	always satisfied
P_2: scale covariance	always satisfied	always satisfied	always satisfied
P_3: real-valuedness	$A_T(-\beta) = A_T(\beta)$, $B_T^*(-\beta) = B_T(\beta)$	$B_T^*(-\beta) = B_T(\beta)$	$S_T(\gamma) \in \mathrm{R}$
P_4: energy distribution	$B_T(0) = 1$	$B_T(0) = 1$	$S_T(0) = 1$
P_5: frequency marginal	$A_T(0) = 0$, $B_T(0) = 1$	$B_T(0) = 1$	$S_T(0) = 1$
P_6: Moyal's formula/unitarity	$\lvert B_T(\beta)\rvert = 1$	$\lvert B_T(\beta)\rvert = 1$	$\lvert S_T(\gamma)\rvert = 1$
P_7: group delay	$A_T(0) = 0$, $B_T(0) = 1$, $A'_T(0) = B'_T(0) = 0$	$B_T(0) = 1$, $B'_T(0) = 0$	$S_T(0) = 1$, $S'_T(0) = 0$
P_8: finite frequency support	$B_T(\beta)=0$ for all β for which $\lvert A_T(\beta)\rvert > \lvert\frac{\beta}{2}\rvert$	always satisfied	$s_T(\eta) = 0$ for $\lvert\eta\rvert > \frac{1}{2}$
P_9: Dirac freq. localization	$A_T(0) = 0$, $B_T(0) = 1$	$B_T(0) = 1$	$S_T(0) = 1$
P_{10}: hyperbolic marginal	$B_T(\beta) = 1$	$B_T(\beta) = 1$	$S_T(0) = 1$
P_{11}: hyperbolic localization	$B_T(\beta) = 1$	$B_T(\beta) = 1$	$S_T(0) = 1$
P_{12}: finite hyperbolic support	$\int_{-\infty}^{\infty} B_T(\beta)\, e^{j2\pi(c\beta+\zeta A_T(\beta))} d\beta = 0$ for $\lvert\frac{c}{\zeta}\rvert > \frac{1}{2}$	$\int_{-\infty}^{\infty} B_T(\beta)\, e^{j2\pi(c\beta+\zeta A_P(\beta))} d\beta = 0$ for $\lvert\frac{c}{\zeta}\rvert > \frac{1}{2}$	$s_T(\eta) = 0$ for $\lvert\eta\rvert > \frac{1}{2}$
P_{13}: hyperbolic moments	$B_T(\beta) = 1$	$B_T(\beta) = 1$	$S_T(0) = 1$
P_{14}: power-warp covariance	$A_T(\beta) = \alpha\beta$, $B_T(\beta) = 1$	not satisfied	always satisfied
P_{15}: time-shift covariance	$A_T(\beta) = \ln(\frac{\sinh(\beta/2)}{\beta/2})$	always satisfied	not satisfied
P_{16}: hyperbolic axis reversal	$A_T(-\beta) = -A_T(\beta)$, $B_T(-\beta) = B_T(\beta)$	not satisfied	always satisfied
P_{17}: hyp. freq. moments	$A_T(0) = 0$, $B_T(0) = 1$	$B_T(0) = 1$	$S_T(0) = 1$
P_{18}: hyp. instantaneous frequency	$A_T(\beta) = 0$, $B_T(\beta) = 1$ (only Q-distribution!)	not satisfied	$S_T(0) = 1$, $S'_T(0) = 0$
P_{19}: hyperbolic convolution	$B_T(\beta) = 1$	$B_T(\beta) = 1$	$S_T(\gamma) = e^{c\gamma}$
P_{20}: hyp. multiplication	$A_T(\beta) = \alpha\beta$, $B_T(\beta) = e^{\alpha'\beta}$	not satisfied	$S_T(\gamma) = e^{c\gamma}$
P_{21}: convolution	$B_T(\beta) = e^{A_T(\beta)}$	$B_T(\beta) = \frac{\sinh(\beta/2)}{\beta/2}$	$S_T(\gamma) = 1$ (only Q-distribution!)
P_{22}: hyp. chirp localization	$A_T(\beta) = 0$, $B_T(\beta) = 1$ (only Q-distribution!)	not satisfied	$S_T(\gamma) = 1$ (only Q-distribution!)

The *unitary* P_0-distribution in (6) is a special case, as discussed below. The Q-distribution is not a member of the affine HC since it is not time-shift covariant.

Following [8], we shall now derive a condition for the time-shift covariance (12) within the HC. Inserting (1) into (12), it is easily shown that (12) is satisfied by an HC QTFR T if and only if

$$\Phi_T(b,\beta)\exp\{-j2\pi c[e^{-b}\,(e^{\beta/2} - e^{-\beta/2}) - \beta]\} = \Phi_T(b,\beta) \quad \forall c, b, \beta.$$

This will be true if and only if the argument of the exponential in the left-hand side is zero, i.e., $e^{-b}\,(e^{\beta/2} - e^{-\beta/2}) - \beta = 0$, which yields the condition $b = \ln(\frac{\sinh(\beta/2)}{\beta/2})$, expressing an explicit dependence of b on β. The kernel $\Phi_T(b,\beta)$ must be such that $b = \ln(\frac{\sinh(\beta/2)}{\beta/2})$ is enforced, which leads to

$$\Phi_T(b,\beta) = B_T(\beta)\,\delta(b - A_P(\beta)) \tag{13}$$

with

$$A_P(\beta) = \ln\left(\frac{\sinh(\beta/2)}{\beta/2}\right) \quad \text{and} \quad B_T(\beta) \text{ arbitrary.}$$

This form of $\Phi_T(b,\beta)$ is necessary and sufficient for the time-shift covariance (12), and thus characterizes the affine HC (cf. Table I). Comparing (13) with (7), we see that *the affine HC is a subclass of the localized-kernel HC,* with kernel $A_T(\beta) = A_P(\beta) = \ln(\frac{\sinh(\beta/2)}{\beta/2})$, which is the "$A$" kernel of the unitary P_0-distribution, whereas $B_T(\beta)$ is arbitrary. Combining (8) and (13), we can write any affine HC member as

$$T_X(t,f) = f\int_{-\infty}^{\infty} X(f\,G_P(\beta)\,e^{\beta/2})X^*(fG_P(\beta)e^{-\beta/2}) \times G_P(\beta)B_T(\beta)e^{j2\pi tf\beta}\,d\beta \tag{14}$$

with $G_P(\beta) = \frac{\beta/2}{\sinh(\beta/2)} = e^{-A_P(\beta)}$ and $B_T(\beta)$ arbitrary.[3]

Since the kernel $A_T(\beta)$ is fixed, only $B_T(\beta)$ remains to be chosen. Hence, members of the affine HC are characterized in terms of the single 1-D kernel $B_T(\beta)$. The kernel constraints for desirable QTFR properties can now be reformulated in terms of $B_T(\beta)$, as listed in the third column of Table III.

A prominent member of the affine HC is the unitary P_0-distribution $P_X(t,f)$ in (6), for which $B_T(\beta) = B_P(\beta) = 1$. From Table III, it follows that the unitary P_0-distribution is the *only* QTFR of the affine HC that satisfies the hyperbolic marginal property P_{10} (or the hyperbolic localization property P_{11}). Thus, this property uniquely specifies the unitary P_0-distribution inside the affine HC. Furthermore, using (14) we can show that *any member of the affine HC can be derived from the unitary P_0-distribution $P_X(t,f)$ through a scaled convolution with respect to the time variable*

$$T_X(t,f) = f\int_{-\infty}^{\infty} b_T(f(t-t'))\,P_X(t',f)\,dt' \tag{15}$$

where $b_T(c)$ is the inverse Fourier transform of the kernel $B_T(\beta)$. (This result is important since it permits the implementation of affine HC QTFR's using existing algorithms for the unitary P_0-distribution [10].) For these reasons, the unitary P_0-distribution can be considered to be the "central" QTFR of the affine HC.

In [7], [9], [11], and [12], the *localized-kernel subclass of the affine QTFR class* is defined as the subclass of all affine QTFR's $T_X^{(A)}(t,f)$ that can be written in the form

$$T_X^{(A)}(t,f) = f\int_{-\infty}^{\infty} X\left(f\left(F_T(\beta)+\frac{\beta}{2}\right)\right) \times X^*\left(f\left(F_T(\beta)-\frac{\beta}{2}\right)\right)G_T(\beta)e^{j2\pi tf\beta}\,d\beta \tag{16}$$

where $F_T(\beta)$ and $G_T(\beta)$ are two 1-D kernel functions. It is easily checked that any QTFR of the affine HC in (14) can be written in the form (16), with

$$F_T(\beta) = F_P(\beta) = \frac{\beta}{2}\coth\frac{\beta}{2},$$
$$G_T(\beta) = \frac{\beta/2}{\sinh(\beta/2)}B_T(\beta) = G_P(\beta)B_T(\beta). \tag{17}$$

This shows that *the affine HC is also a subclass of the localized-kernel affine subclass*, with kernel $F_T(\beta) = F_P(\beta) = \frac{\beta}{2}\coth\frac{\beta}{2}$ equal to the kernel of the unitary P_0-distribution in the framework of the localized-kernel affine subclass [11]. The second kernel $G_T(\beta)$ is arbitrary but related to the corresponding HC kernel $B_T(\beta)$ according to (17). For the unitary P_0-distribution, $B_P(\beta) = 1$, and hence, $G_P(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$.

[3] Equation (14) is Equation (38) in [8] for the class of P_0-distributions, with $\lambda_0(\beta)$ and $\mu(\beta)$ used in [8] related to $G_P(\beta)$ and $B_T(\beta)$ as $\lambda_k(\beta) = \lambda_0(\beta) = G_P(\beta)\,e^{\beta/2}$ and $\mu(\beta) = G_P(\beta)\,B_T(\beta)$. The unitary P_0-distribution is obtained when $\mu(\beta) = G_P(\beta)$ [8].

C. The Power HC Subclasses

The affine HC concept can be extended by considering a generalization of the affine class. The *power class* PC_κ associated with the power parameter $\kappa \in \mathbb{R}$, $\kappa \neq 0$ has been introduced in [29] and [30] as the class of all QTFR's T satisfying the scale covariance property and the "power time-shift covariance property"

$$T_{\mathcal{D}_c^{(\kappa)}X}(t,f) = T_X(t-c\,\tau_\kappa(f),f) \tag{18}$$

with

$$\left(\mathcal{D}_c^{(\kappa)}X\right)(f) = e^{-j2\pi c\,\xi_\kappa(f)}X(f).$$

Here, $\xi_\kappa(f) = (\frac{f}{f_r})^\kappa$ and $\tau_\kappa(f) = \frac{d}{df}\,\xi_\kappa(f) = \frac{\kappa}{f_r}(\frac{f}{f_r})^{\kappa-1}$ with $f>0$ and $\kappa \neq 0$ (cf. Tables I and II). The operator $\mathcal{D}_c^{(\kappa)}$ is an allpass filter with power-law group delay $c\,\tau_\kappa(f)$. The affine class is equivalent to the power class when the power parameter $\kappa = 1$ and is, thus, a special case of the family PC_κ of power classes.

For a given power κ, we define the *power subclass* of the HC (which is abbreviated power-HC_κ) to consist of all HC QTFR's satisfying the power time-shift covariance (18) [22], [29], [30]; this is an important property when analyzing signals propagating through dispersive systems. The power-HC_κ is a subclass of both the HC and the power class PC_κ; in fact, it forms the intersection of the two classes. The power-HC_κ with $\kappa = 1$ equals the affine HC. It can be shown that *any QTFR of the power-HC_κ is a member of the localized-kernel HC* in (8), with kernels

$$A_T(\beta) = A_{P^{(\kappa)}}(\beta) = \frac{1}{\kappa}\ln\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right)$$

and

$$B_T(\beta) \text{ arbitrary}$$

where $A_{P^{(\kappa)}}(\beta)$ is the "A" kernel of the κth power unitary P_0-distribution (cf. (11)). Thus, the power-HC_κ is a subclass of the localized-kernel HC parameterized in terms of the 1-D kernel $B_T(\beta)$. With (8), it follows that any QTFR of the power-HC_κ can be written as a generalization of (14)

$$T_X(t,f) = f\int_{-\infty}^{\infty} X(f\,G_{P^{(\kappa)}}(\beta)\,e^{\beta/2})\,X^*(f\,G_{P^{(\kappa)}}(\beta)\,e^{-\beta/2}) \times G_{P^{(\kappa)}}(\beta)\,B_T(\beta)e^{j2\pi tf\beta}\,d\beta$$

with $G_{P^{(\kappa)}}(\beta) = (\frac{\kappa\beta/2}{\sinh(\kappa\beta/2)})^{1/\kappa} = e^{-A_{P^{(\kappa)}}(\beta)}$ and $B_T(\beta)$ arbitrary. Note that (14) is obtained when $\kappa = 1$.

Since power-HC_κ QTFR's have $A_T(\beta)$ fixed, they are characterized by the single 1–D kernel $B_T(\beta)$. The kernel constraints on $B_T(\beta)$ for desirable QTFR properties equal those for the affine HC in the third column of Table III, except for the following:

i) In P_{12}, $A_P(\beta)$ is replaced by $A_{P^{(\kappa)}}(\beta)$.
ii) P_{15} is replaced by the power time-shift covariance property (18).

iii) The constraint for P_{21} becomes $B_T(\beta) = (\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2})^{1/\kappa}$.

A "central" QTFR in the power-HC$_\kappa$ is the *power unitary P_0-distribution* $P_X^{(\kappa)}(t,f)$ defined by $B_T(\beta) = B_{P^{(\kappa)}}(\beta) = 1$ [22]. Inside the power-HC$_\kappa$, the power unitary P_0-distribution is uniquely specified by the hyperbolic marginal property P_{10} (or the hyperbolic localization property P_{11}). It can be shown that *any member of the power-HC$_\kappa$ can be derived from the power unitary P_0-distribution $P_X^{(\kappa)}(t,f)$ through a scaled convolution with respect to the time variable*

$$T_X(t,f) = \frac{f}{|\kappa|}\int_{-\infty}^{\infty} b_T\left(\frac{f}{\kappa}(t-t')\right) P_X^{(\kappa)}(t',f)\,dt' \quad (19)$$

where $b_T(c)$ is the inverse Fourier transform of $B_T(\beta)$. Note that this relation generalizes (15).

D. The Power-Warp HC Subclass

An important subclass of Cohen's class is the *shift-scale covariant subclass* [32], [5], [20], [21], [33], which consists of all Cohen's class QTFR's satisfying the scale covariance property. Prominent members of this class are the generalized Wigner distribution [32] and the Choi–Williams distribution [33], [49].

We define the *power-warp subclass* of the HC (power-warp HC) as the HC subclass whose members satisfy the power-warp covariance property introduced in Section IV-D of Part I [1]

$$T_{\mathcal{P}_a X}(t,f) = T_X\left(\frac{at}{\left(\frac{f}{f_r}\right)^{\frac{1}{a}-1}}, f_r\left(\frac{f}{f_r}\right)^{\frac{1}{a}}\right)$$

with

$$(\mathcal{P}_a X)(f) \triangleq \sqrt{\frac{1}{a}\left(\frac{f}{f_r}\right)^{\frac{1}{a}-1}}\, X\left(f_r\left(\frac{f}{f_r}\right)^{\frac{1}{a}}\right), \quad a>0.$$

Since this property corresponds to the scale covariance property in Cohen's class, the power-warp HC is the HC counterpart of the shift-scale covariant subclass of Cohen's class. The operator $\mathcal{P}_a$ produces a power-law frequency warping $f \to f_r(\frac{f}{f_r})^{1/a}$; the factor $\sqrt{\frac{1}{a}(\frac{f}{f_r})^{1/a-1}}$ assures the unitarity of $\mathcal{P}_a$ [23], [29]. With the kernel relations in Equations (20) and (21) of Part I [1], the theory of the shift-scale covariant subclass of Cohen's class [32], [20], [21] can be reformulated for its HC counterpart: the power-warp HC. In particular, the 2-D kernels of a QTFR T of the power-warp HC are parameterized in terms of a 1-D "kernel" $s_T(\eta)$ or its Fourier transform $S_T(\gamma) = \int_{-\infty}^{\infty} s_T(\eta)\,e^{-j2\pi\gamma\eta}\,d\eta$ (cf. Table I)

$$\Phi_T(b,\beta) = \frac{1}{|\beta|}s_T\left(-\frac{b}{\beta}\right), \quad \Psi_T(\zeta,\beta) = S_T(\zeta\beta). \quad (20)$$

Note that $\Psi_T(\zeta,\beta)$ is a "product kernel" that depends only on the product of ζ and β. From (20), it follows that any QTFR of the power-warp HC can be written as

$$T_X(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_X\left(\ln\frac{f}{f_r}+\eta\beta, \beta\right) s_T(\eta) e^{j2\pi tf\beta}\,d\eta\,d\beta$$
$$= f\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X\big(fe^{(\frac{1}{2}+\eta)\beta}\big)X^*\big(fe^{-(\frac{1}{2}-\eta)\beta}\big) \times s_T(\eta)e^{\eta\beta}\,e^{j2\pi tf\beta}\,d\eta\,d\beta. \quad (21)$$

The kernel constraints associated with desirable QTFR properties can be reformulated in terms of the power-warp HC kernels $s_T(\eta)$ or $S_T(\gamma)$, as summarized in the fourth column of Table III.

A prominent member of the power-warp HC is the *generalized Q-distribution*

$$Q_X^{(\alpha)}(t,f) = f\int_{-\infty}^{\infty} X\big(fe^{(\frac{1}{2}-\alpha)\beta}\big)\,X^*\big(fe^{-(\frac{1}{2}+\alpha)\beta}\big)\, e^{-\alpha\beta}e^{j2\pi tf\beta}d\beta \quad (22)$$

which has been introduced in Section V-B of Part I [1] as the HC counterpart of the generalized Wigner distribution of Cohen's class. The Q-distribution in (5) is the special case when $\alpha = 0$. The kernels of the generalized Q-distribution are $s_{Q^{(\alpha)}}(\eta) = \delta(\eta+\alpha)$ and $S_{Q^{(\alpha)}}(\gamma) = e^{j2\pi\alpha\gamma}$. With (21), it follows that *any QTFR of the power-warp HC is a superposition of generalized Q-distributions*

$$T_X(t,f) = \int_{-\infty}^{\infty} s_T(-\eta)Q_X^{(\eta)}(t,f)d\eta.$$

Hence, the generalized Q-distribution can be considered to be the basic QTFR type of the power-warp HC.

Another QTFR of the power-warp HC is the "hyperbolic Choi–Williams distribution" (the HC counterpart of the Choi–Williams distribution of Cohen's class [49]) whose kernels are given by $s_{\mathrm{HCWD}}(\eta) = \sqrt{\frac{\sigma}{4\pi}}\exp(-\frac{\sigma}{4}\eta^2)$ and $S_{\mathrm{HCWD}}(\gamma) = \exp(-\frac{(2\pi\gamma)^2}{\sigma})$, where $\sigma > 0$. More generally, the HC counterparts of the "reduced interference distributions" introduced in [33] are power-warp HC members.

III. Regularity

Besides the discussion of four subclasses of the HC, the second contribution of this paper is a study of the fundamental QTFR properties of regularity and unitarity [14], [21] within the HC. These properties are important for various methods of (statistical) TF signal processing [4], [34]–[43]. The *regularity* property considered in this section expresses the QTFR's "invertibility" in the sense that no essential information about the signal is lost in the signal's QTFR. Regularity has several further important implications.

A. Inverse Kernels

The expressions (1) and (2) state that an HC QTFR $T_X(t,f)$ can be derived from $V_X(b,\beta)$ and $B_X(\zeta,\beta)$ by characteristic linear transformations. An HC QTFR $T_X(t,f)$ is said to be *regular* if these linear transformations are invertible. Inversion of (1) and (2) can be shown to result in

$$V_X(b,\beta) = \int_{-\infty}^{\infty}\int_0^{\infty} \Phi_T^{-1}\left(b-\ln\frac{f}{f_r}, \beta\right)T_X(t,f)e^{-j2\pi\beta tf}\,dt\,df \quad (23)$$

$$B_X(\zeta,\beta) = \Psi_T^{-1}(\zeta,\beta)\int_{-\infty}^{\infty}\int_0^{\infty} T_X(t,f)e^{-j2\pi(\beta tf-\zeta\ln\frac{f}{f_r})}dt\,df$$

where the "inverse kernels" $\Phi_T^{-1}(b,\beta)$ and $\Psi_T^{-1}(\zeta,\beta)$ are related as $\Psi_T^{-1}(\zeta,\beta) = \int_{-\infty}^{\infty} \Phi_T^{-1}(b,\beta)\, e^{j2\pi\zeta b}\, db$. Inverse kernels and "direct" kernels are related as

$$\int_{-\infty}^{\infty} \Phi_T^{-1}(b-b',\beta)\Phi_T(b',\beta)\, db' = \delta(b), \qquad \Psi_T^{-1}(\zeta,\beta)\Psi_T(\zeta,\beta) = 1. \tag{24}$$

The last relation implies $\Psi_T^{-1}(\zeta,\beta) = 1/\Psi_T(\zeta,\beta)$; therefore, an HC QTFR is regular if and only if $\Psi_T(\zeta,\beta)$ is essentially nonzero. In general, such a simple expression does not exist for the other inverse kernels, except for unitary QTFR's (see Section IV). It is easily shown that an HC QTFR is regular if and only if the Cohen's class QTFR corresponding to it, via the constant-Q warping mapping [1], is regular. The (generalized) Q-distribution, unitary P_0-distribution, and power unitary P_0-distribution are all regular.

Example: The inverse kernels of the Q-distribution $Q_X(t,f)$ are $\Phi_Q^{-1}(b,\beta) = \Phi_Q(b,\beta) = \delta(b)$ and $\Psi_Q^{-1}(\zeta,\beta) = \Psi_Q(\zeta,\beta) = 1$. Specializing (23) to the Q-distribution by inserting $\Phi_Q^{-1}(b,\beta) = \delta(b)$ yields the relation $V_X(b,\beta) = \int_{-\infty}^{\infty} Q_X(\frac{c}{f_r e^b}, f_r e^b)\, e^{-j2\pi\beta c}\, dc$, which can be reformulated as

$$X(f_1)\, X^*(f_2) = \int_{-\infty}^{\infty} Q_X(t, \sqrt{f_1 f_2}) e^{-j2\pi\sqrt{f_1 f_2}(\ln\frac{f_1}{f_2})\, t} dt.$$

B. Implications of Regularity

The regularity of an HC QTFR has far-reaching implications, which are summarized in the following.

Implication 1: From a regular HC QTFR, the signal can be recovered up to a constant phase. In order to recover the signal, we first calculate $V_X(b,\beta)$ via (23). Then, it follows from (3) that the signal, up to an unknown constant phase $\hat{\phi}$, can be derived from $V_X(b,\beta)$ as

$$X(f) = \frac{V_X\left(\ln\frac{\sqrt{f\hat{f}}}{f_r}, \ln\frac{f}{\hat{f}}\right)}{\sqrt{f V_X\left(\ln\frac{\hat{f}}{f_r}, 0\right)}} e^{j\hat{\phi}}.$$

The frequency $\hat{f}$ can be chosen arbitrarily apart from the requirement that $V_X(\ln\frac{\hat{f}}{f_r}, 0)$ be nonzero.

Implication 2: From a regular HC QTFR, any quadratic signal representation can be derived by a linear transformation. Any quadratic signal representation of an analytic signal can be written as [21]

$$\tilde{T}_X(\Theta) = \int_0^{\infty}\int_0^{\infty} K_{\tilde{T}}(\Theta;\, f_1, f_2)\, X(f_1)\, X^*(f_2) df_1\, df_2. \tag{25}$$

Here, Θ is a parameter vector such as (t,f) in the case of a QTFR, and $K_{\tilde{T}}(\Theta;\, f_1, f_2)$ is a kernel characterizing $\tilde{T}$. Expressing $X(f_1)\, X^*(f_2)$ in terms of $V_X(b,\beta)$ (see (3)) and inserting (23), it is seen that $\tilde{T}_X(\Theta)$ can be derived from a *regular* HC QTFR $T_X(t,f)$ via a linear transformation

$$\tilde{T}_X(\Theta) = \int_{-\infty}^{\infty}\int_0^{\infty} L_{\tilde{T}T}(\Theta;\, t', f')\, T_X(t',f') dt'\, df' \tag{26}$$

where the kernel $L_{\tilde{T}T}(\Theta; t', f')$ is constructed as

$$L_{\tilde{T}T}(\Theta;\, t', f') = \int_0^{\infty}\int_{-\infty}^{\infty} K_{\tilde{T}}(\Theta;\, fe^{\beta/2}, fe^{-\beta/2}) \times \Phi_T^{-1}\left(\ln\frac{f}{f'}, \beta\right) e^{-j2\pi t' f' \beta}\, df\, d\beta. \tag{27}$$

If the quadratic signal representation $\tilde{T}_X(\Theta)$ is itself an HC QTFR, then (26) simplifies to

$$\tilde{T}_X(t,f) = \int_{-\infty}^{\infty}\int_0^{\infty} \lambda_{\tilde{T}T}\left(tf - t'f', \ln\frac{f}{f'}\right) T_X(t',f') dt'\, df' \tag{28}$$

with

$$\lambda_{\tilde{T}T}(c,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{\tilde{T}}(b-b',\beta)\Phi_T^{-1}(b',\beta) e^{j2\pi c\beta} db'\, d\beta. \tag{29}$$

An important degenerate special case of a quadratic signal representation (25) is a *quadratic form*

$$\Lambda_X = \int_0^{\infty}\int_0^{\infty} X^*(f_1) D(f_1, f_2)\, X(f_2) df_1\, df_2 \tag{30}$$

for which $K_\Lambda(\Theta;\, f_1, f_2) = D(f_2, f_1)$. Specializing (26) and (27), the quadratic form can be expressed as a weighted TF integral of a regular HC QTFR $T_X(t,f)$

$$\Lambda_X = \int_{-\infty}^{\infty}\int_0^{\infty} L_{\Lambda T}(t,f) T_X(t,f)\, dt\, df \tag{31}$$

where

$$L_{\Lambda T}(t,f) = \int_0^{\infty}\int_{-\infty}^{\infty} D(f'e^{-\beta/2}, f'e^{\beta/2})\Phi_T^{-1}\left(\ln\frac{f'}{f}, \beta\right) \times e^{-j2\pi t f \beta}\, df'\, d\beta.$$

Quadratic forms are important as they occur in many applications, especially in statistical signal processing. For example, the optimum detection statistic for the "Gauss-Gauss" detection problem involves a quadratic form [50]. The expression (31) can here be interpreted as a TF correlator using a TF reference (weighting) function $L_{\Lambda T}(t,f)$ that incorporates the statistical *a priori* information [34]. Although this TF formulation does not yield an efficient implementation, it often facilitates the detector's interpretation [34]–[37].

Example: Since the Q-distribution is regular, any quadratic signal representation can be derived from the Q-distribution via a linear transformation. Specializing (27) and (29) to the Q-distribution by inserting $\Phi_Q^{-1}(b,\beta) = \delta(b)$, we obtain, respectively, the transformation kernels

$$L_{\tilde{T}Q}(\Theta;\, t', f') = f' \int_{-\infty}^{\infty} K_{\tilde{T}}(\Theta;\, f'e^{\beta/2}, f'e^{-\beta/2}) \times e^{-j2\pi t' f' \beta}\, d\beta,$$

$$\lambda_{\tilde{T}Q}(c,b) = \int_{-\infty}^{\infty} \Phi_{\tilde{T}}(b,\beta)\, e^{j2\pi c\beta}\, d\beta.$$

By way of example, the following relations show how the hyperbolic ambiguity function $B_X(\zeta,\beta)$ in (4), the unitary P_0-distribution $P_X(t,f)$ in (6), the generalized Q-distribution $Q_X^{(\alpha)}(t,f)$ in (22), the hyperbologram $Y_X(t,f) = \frac{f_r}{f}\left|\int_0^\infty X(f')\Gamma^*(\frac{f_r}{f}f') e^{j2\pi tf\ln(f'/f_r)}df'\right|^2$ [1], and the wideband ambiguity function $\chi_X(\tau,\alpha) = \int_0^\infty X(\sqrt{\alpha}f)X^*(\frac{f}{\sqrt{\alpha}})e^{j2\pi\tau f}df$ [1] can be derived from the Q-distribution:

$$B_X(\zeta,\beta) = \int_{-\infty}^{\infty}\int_0^\infty Q_X(t,f)e^{-j2\pi\left(\beta tf-\zeta\ln\frac{f}{f_r}\right)}dt\,df,$$

$$P_X(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_X\left(\frac{c}{fG_P(\beta)}, fG_P(\beta)\right) \times e^{j2\pi(tf-c)\beta}dc\,d\beta$$

with

$$G_P(\beta) = \frac{\beta/2}{\sinh(\beta/2)},$$

$$Q_X^{(\alpha)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_X\left(\frac{c}{fe^{-\alpha\beta}}, fe^{-\alpha\beta}\right) \times e^{j2\pi(tf-c)\beta}dc\,d\beta,$$

$$Y_X(t,f) = \int_{-\infty}^{\infty}\int_0^\infty Q_\Gamma\left(\frac{1}{f_r}\frac{f}{f'}(t'f'-tf),\, f_r\frac{f'}{f}\right) \times Q_X(t',f')\,dt'\,df',$$

$$\chi_X(\tau,\alpha) = \int_{-\infty}^{\infty}\int_0^\infty Q_X(t,f)e^{-j2\pi((\ln\alpha)tf+\tau f)}\,dt\,df.$$

Furthermore, any quadratic form (30) can be expressed as a weighted integral of the Q-distribution $\Lambda_X = \int_{-\infty}^{\infty}\int_0^\infty L_{\Lambda T}(t,f)Q_X(t,f)dt\,df$. The TF weighting function is given by

$$L_{\Lambda Q}(t,f) = f\int_{-\infty}^{\infty} D(fe^{-\beta/2}, fe^{\beta/2})e^{-j2\pi tf\beta}\,d\beta$$

where $D(f_1,f_2)$ is given in (30). We note that $L_{\Lambda Q}(t,f)$ is the HC counterpart of the *Weyl symbol* [51]–[53].

Implication 3: For a regular HC QTFR, a linear transformation of the signal results in a linear transformation of the QTFR. If T is a regular HC QTFR, then the QTFR $T_{\tilde{X}}(t,f)$ of a linearly transformed signal $\tilde{X}(f) = \int_0^\infty H(f,f')X(f')\,df'$ is a linearly transformed QTFR of the original signal

$$T_{\tilde{X}}(t,f) = \int_{-\infty}^{\infty}\int_0^\infty L_{HT}(t,f;t',f')T_X(t',f')\,dt'\,df'$$

where the kernel $L_{HT}(t,f;t',f')$ is constructed as [21]

$$\begin{aligned} L_{HT}&(t,f;t',f') \\ &= \int_0^\infty\int_{-\infty}^{\infty}\int_0^\infty\int_{-\infty}^{\infty} H(f_1e^{\beta_1/2},\, f_2\,e^{\beta_2/2}) \\ &\quad\times H^*(f_1e^{-\beta_1/2},\, f_2\,e^{-\beta_2/2}) \\ &\quad\times \Phi_T\left(\ln\frac{f}{f_1},\beta_1\right)\Phi_T^{-1}\left(\ln\frac{f_2}{f'},\beta_2\right) \\ &\quad\times e^{j2\pi(tf\beta_1-t'f'\beta_2)}\,df_1\,d\beta_1\,df_2\,d\beta_2. \end{aligned} \tag{32}$$

This is a mathematical framework for "covariance properties."

Example: For the Q-distribution (i.e., $T=Q$), the kernel (32) becomes

$$\begin{aligned} L_{HQ}&(t,f;t',f') \\ &= ff'\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(fe^{\beta/2},\, f'e^{\beta'/2}) \\ &\quad\times H^*(fe^{-\beta/2}, f'e^{-\beta'/2})e^{j2\pi(tf\beta-t'f'\beta')}d\beta\,d\beta'. \end{aligned}$$

For example, with $H(f,f') = G(f)\delta(f'-f)$, the following covariance property of the Q-distribution is obtained: $\tilde{X}(f) = G(f)\,X(f) \Rightarrow Q_{\tilde{X}}(t,f) = \int_{-\infty}^{\infty} Q_G(t-t',f)\,Q_X(t',f)\,dt'$. Further special cases involving the Q-distribution are the covariance properties P_1, P_2, P_{14}, P_{16}, P_{19}, and P_{20} in Table II of Part I [1].

Implication 4: In the case of a regular QTFR, a signal basis induces a QTFR basis. If the signals $Y_k(f)$ with $k=1,\ldots,\infty$ are linearly independent and complete in the linear space of all square-integrable (finite-energy), analytic signals, and if T is a *regular* HC QTFR, then the signals' auto- and cross-QTFR's $T_{Y_k,Y_l}(t,f)$ with $k=1,\ldots,\infty$ and $l=1,\ldots,\infty$ are linearly independent and complete in $L_2(\mathbb{R}\times\mathbb{R}_+)$, which is the space of all square-integrable 2-D functions $F(t,f)$ defined for $f>0$ [21]. Hence, any function $F(t,f)\in L_2(\mathbb{R}\times\mathbb{R}_+)$ can be expanded into the basis $\{T_{Y_k,Y_l}(t,f)\}$, and the expansion coefficients are uniquely determined. Such a "TF expansion" is useful for TF optimization problems [38]–[42], [44]–[46], [54].

C. Regularity in the HC Subclasses

The calculus of inverse kernels is greatly simplified in the various subclasses of the HC.

1) Localized-Kernel HC: The inverse kernels of a regular localized-kernel HC QTFR can be written as

$$\begin{aligned} \Phi_T^{-1}(b,\beta) &= \frac{1}{B_T(\beta)}\delta(b+A_T(\beta)), \\ \Psi_T^{-1}(\zeta,\beta) &= \frac{1}{B_T(\beta)}e^{-j2\pi\zeta A_T(\beta)} \end{aligned} \tag{33}$$

where $A_T(\beta)$ and $B_T(\beta)$ were defined in (7). Hence, T is regular if and only if its kernel $B_T(\beta)$ is essentially nonzero. Note that this condition does not constrain the kernel $A_T(\beta)$. If T and $\tilde{T}$ are members of the localized-kernel HC, and if T is regular, then the conversion kernel $\lambda_{\tilde{T}T}(c,b)$ in (28), (29) is given by

$$\lambda_{\tilde{T}T}(c,b) = \int_{-\infty}^{\infty}\frac{B_{\tilde{T}}(\beta)}{B_T(\beta)}\,\delta(b-[A_{\tilde{T}}(\beta)-A_T(\beta)])\,e^{j2\pi c\beta}d\beta.$$

If $A_{\tilde{T}}(\beta) = A_T(\beta)$, which is satisfied within the affine HC or the power HC, then $\lambda_{\tilde{T}T}(c,b) = \omega_{\tilde{T}T}(c)\delta(b)$ with $\omega_{\tilde{T}T}(c) = \int_{-\infty}^{\infty}\frac{B_{\tilde{T}}(\beta)}{B_T(\beta)}e^{j2\pi c\beta}\,d\beta$, and (28) simplifies to the convolution (extending (15) and (19))

$$\tilde{T}_X(t,f) = f\int_{-\infty}^{\infty}\omega_{\tilde{T}T}(f(t-t'))T_X(t',f)\,dt'. \tag{34}$$

2) Affine HC and Power HC's: Since the affine HC is a subclass of the localized-kernel HC, it follows from (33) that the inverse kernels of a regular QTFR of the affine HC can be written as

$$\begin{aligned} \Phi_T^{-1}(b,\beta) &= \frac{1}{B_T(\beta)}\delta(b+A_P(\beta)), \\ \Psi_T^{-1}(\zeta,\beta) &= \frac{1}{B_T(\beta)}e^{-j2\pi\zeta A_P(\beta)} \end{aligned}$$

where $A_P(\beta) = \ln(\frac{\sinh(\beta/2)}{\beta/2})$. Hence, T is regular if and only if $B_T(\beta)$ is nonzero. From a regular affine HC QTFR

T, any other affine HC QTFR $\tilde{T}$ can be derived via the convolution in (34) (generalizing (15)). This discussion can be generalized to all power HC's (simply replace $A_P(\beta)$ by $A_{P^{(\kappa)}}(\beta) = \frac{1}{\kappa}\ln(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2})$).

3) Power-Warp HC: The inverse kernels of a regular QTFR of the power-warp HC can be written as

$$\Phi_T^{-1}(b,\beta) = \frac{1}{|\beta|} s_T^{-1}\left(-\frac{b}{\beta}\right), \quad \Psi_T^{-1}(\zeta,\beta) = S_T^{-1}(\zeta\beta)$$

where the inverse 1-D kernels $s_T^{-1}(\eta)$ and $S_T^{-1}(\gamma)$ are a Fourier transform pair and are related to the power-warp HC kernels in (20) according to $\int_{-\infty}^{\infty} s_T^{-1}(\eta-\eta')\, s_T(\eta')\, d\eta' = \delta(\eta)$ and $S_T^{-1}(\gamma)\, S_T(\gamma) = 1$. From $S_T^{-1}(\gamma) = 1/S_T(\gamma)$, it follows that T is regular if and only if $S_T(\gamma)$ is nonzero. If T and $\tilde{T}$ are members of the power-warp HC, and if T is regular, then the conversion kernel $\lambda_{\tilde{T}T}(c,b)$ in (28) and (29) is given by

$$\lambda_{\tilde{T}T}(c,b) = \int_{-\infty}^{\infty} \mu_{\tilde{T}T}(\eta)\, \frac{1}{|\eta|} e^{-j2\pi\frac{cb}{\eta}}\, d\eta$$

with

$$\mu_{\tilde{T}T}(\eta) = \int_{-\infty}^{\infty} s_{\tilde{T}}(\eta-\eta') s_T^{-1}(\eta')\, d\eta'.$$

IV. Unitarity

The QTFR property of unitarity [21], or validity of Moyal's formula, expresses a preservation of inner products and norms. A unitary QTFR is also regular, with the inverse kernels being essentially equal to the "direct" kernels. The unitarity property has some further important implications.

A. Moyal's Formula, Kernel Constraints, and Inverse Kernels

An HC QTFR is *unitary* if the linear transformations underlying the general expressions (1) and (2) preserve inner products, i.e., if[4] $\langle T_X, T_Y\rangle = \langle V_X, V_Y\rangle$ and $\langle T_X, T_Y\rangle = \langle B_X, B_Y\rangle$, where $V_X(b,\beta)$ and $B_X(\zeta,\beta)$ are defined in (3) and (4), respectively. These equations are equivalent and can be rewritten as

$$\langle T_X, T_Y\rangle = |\langle X, Y\rangle|^2$$

which is known as *Moyal's formula* [55]. Thus, a QTFR is unitary if and only if it satisfies Moyal's formula. An important special case of Moyal's formula is $\|T_X\| = \|X\|^2$. It can be shown that an HC QTFR is unitary if and only if the corresponding Cohen's class QTFR is unitary. Examples of unitary HC QTFR's are the (generalized) Q-distribution, the unitary P_0-distribution, and the power unitary P_0-distribution.

The following equivalent kernel constraints can be shown to be necessary and sufficient for an HC QTFR to be unitary [14], [21]:

$$\int_{-\infty}^{\infty} \Phi_T^*(b'-b,\beta)\Phi_T(b',\beta)db' = \delta(b), \qquad \Psi_T^*(\zeta,\beta)\,\Psi_T(\zeta,\beta) = 1. \tag{35}$$

[4]The inner product is defined as $\langle X_1, X_2\rangle = \int_0^{\infty} X_1(f)\, X_2^*(f)\, df$ for analytic signals and $\langle T_1, T_2\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} T_1(t,f)\, T_2^*(t,f)\, dt\, df$ for QTFR's. The associated norms $\|X\|$ and $\|T\|$ are defined by $\|X\|^2 = \langle X, X\rangle = \int_0^{\infty} |X(f)|^2\, df$ (the energy of $X(f)$) and $\|T\|^2 = \langle T, T\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} |T(t,f)|^2\, dt\, df$.

The last relation shows that $\Psi_T(\zeta,\beta)$ must be unimodular, $|\Psi_T(\zeta,\beta)| = 1$. Comparing (35) with (24), it follows that *a unitary HC QTFR is regular*, with inverse kernels $\Phi_T^{-1}(b,\beta) = \Phi_T^*(-b,\beta)$ and $\Psi_T^{-1}(\zeta,\beta) = \Psi_T^*(\zeta,\beta)$. Hence, the inverse kernels of a unitary QTFR are easily calculated from the direct kernels.

B. Implications of Unitarity

Since a unitary QTFR is regular, all results discussed in Section III-B apply. Moreover, unitarity has some important implications of its own, three of which are summarized in the following.

Implication 1: The squared magnitude of any quadratic signal representation can be written as a quadratic form involving an arbitrary unitary HC QTFR. For the general quadratic signal representation $\tilde{T}_X(\Theta)$ in (25), $|\tilde{T}_X(\Theta)|^2$ can be written as a quadratic form

$$|\tilde{T}_X(\Theta)|^2 = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} T_X^*(t,f) L_{\tilde{T}T}^{(\Theta)}(t,f;t',f') \times T_X(t',f')dt\, df\, dt'\, df' \tag{36}$$

where T is an arbitrary *unitary* HC QTFR, and $L_{\tilde{T}T}^{(\Theta)}(t,f;t',f')$ is given by [21]

$$\begin{aligned} L_{\tilde{T}T}^{(\Theta)}&(t,f;t',f') \\ &= \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} K_{\tilde{T}}(\Theta;\, f_2\, e^{\beta_2/2}, f_1 e^{\beta_1/2}) \\ &\quad \times K_{\tilde{T}}^*(\Theta;\, f_2\, e^{-\beta_2/2}, f_1 e^{-\beta_1/2}) \\ &\quad \times \Phi_T\left(\ln\frac{f}{f_1},\beta_1\right)\Phi_T^*\left(\ln\frac{f'}{f_2},\beta_2\right) \\ &\quad \times e^{j2\pi(tf\beta_1 - t'f'\beta_2)}\, df_1\, d\beta_1\, df_2\, d\beta_2. \end{aligned} \tag{37}$$

Example: Since the Q-distribution $Q_X(t,f)$ is unitary, the squared magnitude of any quadratic signal representation can be written as a quadratic form of $Q_X(t,f)$. Specializing (37) to $T = Q$ yields the kernel

$$\begin{aligned} L_{\tilde{T}Q}^{(\Theta)}&(t,f;t',f') \\ &= ff' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\tilde{T}}(\Theta;\, f' e^{\beta'/2}, fe^{\beta/2}) \\ &\quad \times K_{\tilde{T}}^*(\Theta;\, f' e^{-\beta'/2}, fe^{-\beta/2}) e^{j2\pi(tf\beta - t'f'\beta')} d\beta\, d\beta'. \end{aligned}$$

Two special cases of (36) involving the Q-distribution are the relations

$$\begin{aligned} [Q_X(t,f)]^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_X\left(\frac{tf+\zeta/2}{fe^{\beta/2}}, fe^{\beta/2}\right) \\ &\quad \times Q_X\left(\frac{tf-\zeta/2}{fe^{-\beta/2}}, fe^{-\beta/2}\right) d\zeta\, d\beta \\ |B_X(\zeta,\beta)|^2 &= \int_{-\infty}^{\infty}\int_0^{\infty} Q_X\left(\frac{tf+\zeta/2}{fe^{\beta/2}}, fe^{\beta/2}\right) \\ &\quad \times Q_X\left(\frac{tf-\zeta/2}{fe^{-\beta/2}}, fe^{-\beta/2}\right) dt\, df. \end{aligned}$$

The first relation (derived independently in [12]) is the HC counterpart of the *interference formula* of the Wigner distribution [56].

Implication 2: In the case of a unitary HC QTFR, an orthonormal signal basis induces an orthonormal QTFR basis. If the signals $Y_k(f)$ with $k = 1, \ldots, \infty$ are orthonormal and complete in the space of finite-energy analytic signals, and if T is *unitary,* then the auto- and cross-QTFR's $T_{Y_k,Y_l}(t,f)$ with $k = 1, \ldots, \infty$ and $l = 1, \ldots, \infty$ are orthonormal and complete in $L_2(\mathbb{R} \times \mathbb{R}_+)$ [21]. Thus, unitarity assures that the orthonormality of the basis signals $Y_k(f)$ carries over to the 2-D basis functions $T_{Y_k,Y_l}(t,f)$. This will not be the case if the QTFR T is merely regular (cf. Section III-B).

Implication 3: For a unitary QTFR, there exist standard solutions to the problem of least-squares signal synthesis. The least-squares signal synthesis problem [38], [39], [44]–[46], [54] is the calculation of a signal $X_{\rm opt}(f)$ whose QTFR is closest to a given square-integrable TF function ("model") $M(t,f)$,

$$X_{\rm opt}(f) \triangleq \arg\min_X \|M - T_X\|.$$

Signal synthesis methods are useful for the isolation of signal components [39], [41], [45]–[47], [57].

If T is a *unitary* HC QTFR, then the signal synthesis result $X_{\rm opt}(f)$ can be derived as follows [38] (see [38], [39], [44], [54] for alternative formulations using an orthonormal signal basis):

- The transformation (23) is applied to $M(t,f)$, $\tilde{V}(b,\beta) = \int_{-\infty}^{\infty}\int_0^{\infty} \Phi_T^*(\ln\frac{f}{f_r} - b, \beta) M(t,f)\, e^{-j2\pi\beta t f}\, dt\, df$.
- The function $\tilde{V}(b,\beta)$ is converted as $J(f_1,f_2) = \frac{1}{\sqrt{f_1 f_2}} \tilde{V}(\ln\frac{\sqrt{f_1 f_2}}{f_r}, \ln\frac{f_1}{f_2})$.
- The Hermitian part of $J(f_1,f_2)$ is formed, $J_h(f_1,f_2) = \frac{1}{2}[J(f_1,f_2) + J^*(f_2,f_1)]$.
- The largest eigenvalue λ_1 and the corresponding eigenfunction $E_1(f)$ of $J_h(f_1,f_2)$ are calculated.[5]
- If $\lambda_1 > 0$, the signal synthesis result is given by $X_{\rm opt}(f) = \sqrt{\lambda_1}\; e^{j\hat{\phi}}\; E_1(f)$, where $\hat{\phi}$ is an arbitrary constant phase [48]. If $\lambda_1 \le 0$, the signal synthesis result is $X_{\rm opt}(f) = 0$.

Example: The signal synthesis problem $X_{\rm opt}(f) = \arg\min_X \|M - Q_X\|$ for the Q-distribution can be solved using the method summarized above. The kernel $J(f_1,f_2)$ involved in this method is given by

$$J(f_1,f_2) = \int_{-\infty}^{\infty} M(t, \sqrt{f_1 f_2})\, e^{-j2\pi\sqrt{f_1 f_2}(\ln\frac{f_1}{f_2})t} dt.$$

C. Unitarity in the HC Subclasses

A QTFR of the localized-kernel HC, affine HC, or power-HC$_\kappa$ is unitary if and only if the kernel $B_T(\beta)$ in (7) is unimodular (see P_6 in Table III)

$$|B_T(\beta)| = 1$$

which implies $\frac{1}{B_T(\beta)} = B_T^*(\beta)$. Note that the condition for unitarity does not constrain the kernel $A_T(\beta)$.

A QTFR of the power-warp HC is unitary if and only if the kernel $S_T(\gamma)$ in (20) is unimodular

$$|S_T(\gamma)| = 1.$$

For a unitary power-warp HC QTFR, the inverse kernels are simply $s_T^{-1}(\eta) = s_T^*(-\eta)$ and $S_T^{-1}(\gamma) = S_T^*(\gamma)$.

V. Conclusion

The hyperbolic class (HC) comprises quadratic time-frequency representations (QTFR's) with constant-Q characteristic. In this paper, we have defined and discussed several important subclasses of the HC in which the description of an HC QTFR in terms of 2-D kernel functions is simplified. Furthermore, we have studied the properties of regularity and unitarity in the HC and its subclasses.

Motivated by the localized-kernel affine subclass, we have introduced the *localized-kernel subclass* of the HC via a parameterization of the 2-D kernels in terms of two 1-D functions. The localized-kernel structure was shown to be related to a time-frequency concentration property.

The *affine subclass* of the HC is the intersection of the HC with the affine QTFR class. It is part of the localized-kernel HC as well as part of the localized-kernel affine class. Any QTFR of the affine HC can be derived from the unitary P_0-distribution through a 1-D convolution. In a similar manner, the *power subclasses* of the HC have been defined as the intersections of the HC and the power QTFR classes.

A further HC subclass is the *power-warp subclass*, which consists of all HC QTFR's satisfying the power-warp covariance property. The power-warp HC is the HC counterpart of the shift-scale covariant subclass of Cohen's class, and its members are superpositions of generalized Q-distributions.

We have furthermore discussed two fundamental QTFR properties and their implications. The first property—*regularity*—expresses, basically, the QTFR's "invertibility." The regularity property has several further important implications. In particular, any quadratic signal representation can be derived from a regular QTFR, and linear signal transformations are equivalent to linear QTFR transformations.

A more restrictive property than regularity is a QTFR's *unitarity* (validity of Moyal's formula), which allows a simple calculation of the inverse QTFR kernels. The squared magnitude of any quadratic signal representation can be expressed in terms of a unitary QTFR. In addition, there exist standard solutions to the signal synthesis problem in the case of a unitary QTFR. Specialization of general results from the regularity/unitarity theory yielded interesting new relations involving the Q-distribution.

Appendix
Proof of Theorem 1

Inserting (1) into the left-hand side of the concentration property (9), performing a Fourier transform with respect to t and an inverse Fourier transform with respect to c on both sides of the resulting equation, and simplifying yields the equation (equivalent to (9))

$$\int_{-\infty}^{\infty} \Phi_T(b,\beta) R_{f,\beta}(b) \delta(\Xi_{f,\beta}(b) - \gamma) db = r^2(f)\delta(\gamma - f\xi'(f)\beta) \tag{38}$$

[5] The eigenvalues λ_k and eigenfunctions $E_k(f)$ of $J_h(f_1,f_2)$ are defined by $\int_0^{\infty} J_h(f_1,f_2)\, E_k(f_2)\, df_2 = \lambda_k\, E_k(f_1)$.

where $R_{f,\beta}(b)$ and $\Xi_{f,\beta}(b)$ have been defined in Theorem 1. If (as assumed in Theorem 1) $\Xi_{f,\beta}(b)$ is one-to-one and differentiable for all fixed f, β, then the left-hand side of (38) can be written as

$$\int_{-\infty}^{\infty} \frac{\Phi_T(b,\beta)\, R_{f,\beta}(b)}{|\Xi'_{f,\beta}(b)|} \delta\big(b - \Xi^{-1}_{f,\beta}(\gamma)\big)\, db = \frac{\Phi_T\big(\Xi^{-1}_{f,\beta}(\gamma), \beta\big) R_{f,\beta}\big(\Xi^{-1}_{f,\beta}(\gamma)\big)}{\big|\Xi'_{f,\beta}\big(\Xi^{-1}_{f,\beta}(\gamma)\big)\big|}$$

where $\Xi^{-1}_{f,\beta}(\cdot)$ is the function inverse to $\Xi_{f,\beta}(\cdot)$. Substituting $\Xi^{-1}_{f,\beta}(\gamma) = b$, (38) then becomes

$$\frac{\Phi_T(b,\beta)\, R_{f,\beta}(b)}{|\Xi'_{f,\beta}(b)|} = r^2(f)\, \delta(\Xi_{f,\beta}(b) - f\xi'(f)\beta)$$

or equivalently

$$\Phi_T(b,\beta) = \frac{r^2(f)}{R_{f,\beta}\big(\Xi^{-1}_{f,\beta}(f\,\xi'(f)\,\beta)\big)} \delta\big(b - \Xi^{-1}_{f,\beta}(f\,\xi'(f)\,\beta)\big). \tag{39}$$

Obviously, the right-hand side of (39) must not depend on f. This requires, first of all, that $\Xi^{-1}_{f,\beta}(f\xi'(f)\beta)$ is independent of f, so that $\Xi^{-1}_{f,\beta}(f\,\xi'(f)\,\beta) = A_T(\beta)$ with some real-valued function $A_T(\beta)$. Thus, there must exist a function $A_T(\beta)$ such that $\Xi^{-1}_{f,\beta}(f\,\xi'(f)\,\beta) = A_T(\beta)$ or, equivalently, $f\xi'(f)\beta = \Xi_{f,\beta}(A_T(\beta))$ for all $f, \beta \in \mathbb{R}$, which is Condition 1 in Theorem 1. If this first condition is satisfied, (39) simplifies to

$$\Phi_T(b,\beta) = \frac{r^2(f)}{R_{f,\beta}(A_T(\beta))} \delta(b - A_T(\beta))$$

and we have now to make sure that $\frac{r^2(f)}{R_{f,\beta}(A_T(\beta))}$ is independent of f, which is Condition 2. If this second condition is satisfied too, then we can write $\frac{r^2(f)}{R_{f,\beta}(A_T(\beta))} = B_T(\beta)$ with some function $B_T(\beta) \geq 0$, and we finally obtain $\Phi_T(b,\beta) = B_T(\beta)\delta\big(b - A_T(\beta)\big)$, which is Condition 3.

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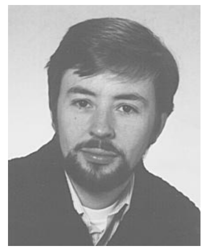

Franz Hlawatsch (M'88) received the Diplom-Ingenieur, Dr. techn. and Univ.-Dozent degrees in electrical engineering/signal processing from the Vienna University of Technology, Vienna, Austria, in 1983, 1988, and 1996, respectively.

Since 1983, he has been with the Department of Communications and Radio-Frequency Engineering, Vienna University of Technology. During 1991–1992, he spent a sabbatical year with the Department of Electrical Engineering, University of Rhode Island, Kingston. His research interests are mainly in signal processing with emphasis on time-frequency methods.

Antonia Papandreou-Suppappola (M'95) received the B.S., M.S., and Ph.D. degrees in electrical engineering from the University of Rhode Island, Kingston, in 1989, 1991, and 1995, respectively.

She is currently a Research Assistant Professor of Electrical Engineering at the University of Rhode Island. Her research interests are in the areas of digital signal processing with emphasis on time-frequency analysis, detection, and estimation.

Dr. Papandreou-Suppappola won a Fulbright Scholarship for her undergraduate studies and a University of Rhode Island Graduate Fellowship for two consecutive academic years. She is a member of the honor societies Eta Kappa Nu, Golden Key, Phi Eta Sigma, Phi Kappa Phi, and Tau Beta Pi.

G. Faye Boudreaux-Bartels (SM'96) received the B.S. degree (summa cum laude) in computer science from the University of Southwestern Louisiana, Lafayette, in 1974 and the M.S. and Ph.D. degrees in electrical engineering from Rice University, Houston, TX, in 1980 and 1989, respectively.

She was a Research Mathematician for Shell Development Company, Houston, TX, from 1971 to 1977 and a Fulbright Scholar at École Nationale Supérieure des Télécommunications, Paris, France from 1981 to 1982. Since 1984, she has been teaching at the University of Rhode Island (URI), Kingston, where she is currently Professor of Electrical Engineering, Associate Dean of the Graduate School, and Director of the URI Women in Engineering Program. Her research and teaching interests are in the areas of digital signal processing, communications, and numerical methods.

Dr. Boudreaux-Bartels has published over 60 papers, one of which won the "1988 Senior Paper Award" given by the IEEE Signal Processing Society (SPS). She has also received two URI Faculty Excellence Awards and has served as an associate editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING and the IEEE SIGNAL PROCESSING LETTERS. She has also served as Vice-Chair of the Digital Signal Processing Technical Committee and was an elected member of the IEEE SPS Advisory Committee.