EXTENDING THE CHARACTERISTIC FUNCTION METHOD
FOR JOINT a-b AND TIME-FREQUENCY ANALYSIS*

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Abstract—We extend the characteristic function method (CFM) to more general groups, operators, and signal spaces. We show that the extended CFM can be applied to projected unitary operators as well as discrete-time/periodic signals.

1 INTRODUCTION AND OUTLINE

The characteristic function method (CFM) [1, 2] allows the construction of joint a-b energy distributions P_a(b) satisfying marginal properties

\[ \int_{-\infty}^{\infty} P_a(b) \, db = |\langle x, u_a \rangle|^2, \]

where \( v_a(t) \) and \( u_a(t) \) are (generalized) eigenfunctions of self-adjoint operators \( B \) and \( A \), respectively. \( B \) and \( A \) can be associated to \( A_0 = \text{e}^{i2\alpha a} \) and \( B_0 = \text{e}^{i2\beta a} \) which are unitary representations of the group \( \mathbb{R} \), respectively.

The eigenfunctions of \( B \) and \( A \) are those of \( A_0 \), \( B_0 \), respectively. A recent extension of the CFM allows \( \alpha, \beta \) to belong to more general LCA groups [9] \( (A, \bullet), (B, \ast) \), respectively, while making the following assumptions [10–13]:

1. \( A_0 \) and \( B_0 \) are unitary representations of the groups \( (A, \bullet) \) and \( (B, \ast) \), respectively, i.e., they are unitary operators satisfying \( A_0^2 = 1 \) and \( B_0^2 = 1 \).

2. \( (A, \bullet) \) and \( (B, \ast) \) are isomorphic to \( (\mathbb{R}, +) \), and thus also to each other. This implies (i) a correspondence between any extended a-b distribution \( \tilde{P}_a(b) \) and its deformation \( \tilde{P}_a(b) \) (i.e., the “extended” CFM is essentially equivalent to the conventional CFM), and (ii) \( A_0 = \text{e}^{i2\alpha a} \) and \( B_0 = \text{e}^{i2\beta a} \) which are self-adjoint with eigenfunctions equal to those of \( A_0 \), \( B_0 \), respectively.

3. The signal space is \( L^2(A, d\mu_A) \) or \( L^2(B, d\mu_B) \). Hence, all signals \( x(t) \) are defined for \( t \in (A, \bullet) \) or \( t \in (B, \ast) \).

4. The functions \( v_a(t) \), \( u_a(t) \) defining the marginals (cf. (1)) are eigenfunctions of \( A_0 \) and \( B_0 \), respectively.

This paper looks at the CFM from a new perspective and shows that all of the above assumptions 1-4 are unnecessary. This entails a real, essential extension of the CFM that admits much broader classes of operators, groups, and signal spaces. In particular, we shall show that our extended CFM can be applied to “projected” unitary operators and to discrete-time and periodic signals.

2 EXTENDED CFM FOR GROUP (\( \mathbb{R}, + \))

For the sake of simplicity, we shall first consider operators \( A_0 \) and \( B_0 \) indexed by \( \alpha, \beta \in (\mathbb{R}, +) \), i.e., \( \alpha, \beta \in (\mathbb{R}, +) \). Let \( A_0 : \mathbb{R} \rightarrow \mathbb{R} \) and \( B_0 : \mathbb{R} \rightarrow \mathbb{R} \) be two families of linear operators indexed by \( \alpha, \beta \). Let \( M_{\alpha,\beta} : \mathbb{R} \rightarrow \mathbb{R} \) be a linear operator satisfying \( M_{\alpha,\beta} = A_0 \), \( M_{\alpha,\beta} = B_0 \).

**Theorem 1.** Let \( X \) be a signal (Hilbert) space with some inner product \( \langle \cdot, \cdot \rangle_X \). Let \( A_0 : \mathbb{R} \rightarrow \mathbb{R} \) and \( B_0 : \mathbb{R} \rightarrow \mathbb{R} \) be two families of linear operators indexed by \( \alpha, \beta \in (\mathbb{R}, +) \). Let \( M_{\alpha,\beta} : \mathbb{R} \rightarrow \mathbb{R} \) be a linear operator satisfying

\[
M_{\alpha,\beta} = A_0, \quad M_{\alpha,\beta} = B_0.
\]

Let \( \phi(\alpha, \beta) \) be a complex-valued function satisfying

\[
\phi(\alpha, 0) = \phi(0, \beta) = 1.
\]

Finally, let \( v_a(t) \) and \( v_b(t) \) be two families of functions indexed by \( a, b \in (\mathbb{R}, +) \). Then, the a-b representation

\[
P_a(b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\alpha, \beta) \langle M_{\alpha,\beta}^{-1} x(t), x(t) \rangle_X \text{e}^{i2\alpha a} \text{d}\beta \text{d}a
\]

satisfies the marginal property

\[
\int_{-\infty}^{\infty} P_a(b) \text{d}b = |\langle x, v_a \rangle_X|^2, \quad \forall b \in \mathbb{R}
\]

if and only if \( A_0 \) is related to \( v_a(t) \) as

\[
A_0 = \int_{-\infty}^{\infty} (v_b \circ v_a) \text{e}^{i2\alpha b} \text{d}b.
\]

Similarly, \( P_b(a) \) satisfies the marginal property

\[
P_b(a) = \int_{-\infty}^{\infty} \langle x(t), u_b(t) \rangle_X \text{d}a,
\]

if and only if \( B_0 \) is related to \( u_b(t) \) as

\[
B_0 = \int_{-\infty}^{\infty} (u_a \circ u_b) \text{e}^{i2\beta a} \text{d}a.
\]

**Proof.** We shall prove (5); the proof of (7) is analogous. With (4), we have (here, all integrals are from \(-\infty\) to \(\infty\))

\[
M_{\alpha,\beta} = A_0, \quad M_{\alpha,\beta} = B_0.
\]

Here \( v_b \circ v_a \) is the linear operator defined by \( (v_b \circ v_a) x(t) = \langle x, v_b \rangle_X v_a(t) \).
\[
\int_a^b P_x(a, b) \, da = \\
= \int_a^b \phi(a, b) \langle M_{a, \alpha} x, x \rangle \left[ \int_a^b e^{-j2\alpha x^2} \, da \right] e^{-j2\alpha x} \, dx \\
= \int_a^b \phi(a, b) \langle M_{a, \alpha} x, x \rangle e^{-j2\alpha x} \, dx \\
= \int_a^b \langle A_{\alpha} x, x \rangle e^{-j2\alpha x} \, dx = \left( \int_a^b \langle A_{\alpha} e^{-j2\alpha x} \, da \right) (x, x) \\
= (v_0 \otimes v_0) (x, x) = (x, x) B_\alpha, 
\]
so that (5) is proved. Note that we have used (3), (2), and (1) in its proof, the central condition in (6) or (8) might be suspected to imply all or some of these assumptions. We now dispel this notion by means of a counterexample.

**Projected unitary operators.** Let \( A : X \to X \)' and \( B : X' \to X' \) with \( \alpha, \beta \in (R, +) \) be unitary representations of \((R, +)\) on a Hilbert space \( X \). Let \( v_0(t) \), \( u_0(t) \) be the eigenvectors of \( A \), \( B \), respectively. We make the usual assumption \([10, 15]\) that \( v_0(t) \) and \( u_0(t) \) are complete and orthogonal function sets inducing spectral decompositions
\[
A' = \int_{-\infty}^{\infty} \langle v_0(t), v_0(t) \rangle e^{j2\alpha x} \, dx, \quad B' = \int_{-\infty}^{\infty} \langle u_0(t), u_0(t) \rangle e^{j2\alpha x} \, dx.
\]
Now, let \( X \subset X' \) be some proper subspace of \( X' \), orthogonal projection operator \( P_X \), and define the "projected operators" \( A = P_X A P_X \) and \( B = P_X B P_X \) and the projected eigenvectors \( v_0(t) = (P_X v_0(t)) (0) \) and \( u_0(t) = (P_X u_0(t)) (0) \). With (9), we obtain
\[
A = \int_{-\infty}^{\infty} \langle P_X v_0(t), P_X v_0(t) \rangle e^{j2\alpha x} \, dx = \int_{-\infty}^{\infty} \langle v_0(t), v_0(t) \rangle e^{j2\alpha x} \, dx,
\]
where we have used the identity \( P_X (v_0(t) \otimes u_0(t)) P_X = (P_X v_0(t) \otimes (P_X u_0(t)) = v_0(t) \otimes u_0(t) \). Similarly, we can show that
\[
B = \int_{-\infty}^{\infty} \langle u_0(t), u_0(t) \rangle e^{j2\alpha x} \, dx \quad \text{(11)}
\]
While the operators \( A_\alpha \) and \( B_\beta \) are defined on all of \( X' \), they map \( X \to X \); hence we may redefine them as \( A_\alpha : X \to X \) and \( B_\beta : X \to X \). Since (10) and (11) are still valid, the conditions (6) and (8) of Theorem 1 are satisfied. Thus, any \( a, b \)-representation constructed as in (4) will satisfy the marginal properties (5), (7). We emphasize that \( v_0(t) \), \( u_0(t) \) are not the eigenvectors, and consequently (10) and (11) are not the spectral decompositions, of \( A_\alpha \), \( B_\beta \), respectively (the projected eigenvectors of an operator are not the eigenvectors of the projected operator).

The function sets \( v_0(t) \) and \( u_0(t) \) are complete in \( X \), i.e.
\[
\int_{-\infty}^{\infty} \langle \sum_{n=0}^{\infty} (a_n \otimes u_n) \rangle \, dx = \int_{-\infty}^{\infty} (u_n \otimes u_n) \, dx = 1_X \quad (\text{the identity operator on } X)
\]
but they are not orthogonal, i.e., \( \langle v_0(t), v_0(t) \rangle \neq \delta (b - b') \) and \( \langle u_0(t), u_0(t) \rangle \neq \delta (a - a') \). Furthermore, \( A_\alpha \), \( B_\beta \) are not unitary. They satisfy \( A_0 = B_0 = 1_X \) but not the usual composition properties, i.e., \( A_\alpha A_\alpha, A_\alpha \neq A_{\alpha+\alpha_0} \) and \( B_\beta B_\beta \neq B_{\beta+\beta_0} \). They cannot be written as exponentiated self-adjoint operators \( B \) and \( A \). Finally, \( X \) is generally different from \( L^2 (B, d\mu_B) = L^2 (B, d\mu_B) = L^2 (X, d\mu_B) \). Hence, assumptions 1.3, and 4 are indeed violated.

**3 Extended CFM for General Groups.**

We now consider the general case \( \alpha \in (A \bullet) \) and \( \beta \in (B \bullet) \), where \( (A \bullet) \) and \( (B \bullet) \) are arbitrary LCA groups that are not assumed to be isomorphic to each other or to \((R, +)\), i.e., assumption 2 is not made.

Let \( \lambda^A_{\alpha, \beta} \) and \( \lambda^B_{\beta, \gamma} \) denote the group characters of \( (A \bullet) \) and \( (B \bullet) \), respectively \([9]\), where \( \langle \hat{A} \bullet, \hat{A} \rangle \) and \( (\hat{B} \bullet, \hat{B} \bullet) \) are the dual groups of \( (A \bullet) \) and \( (B \bullet) \), respectively \([9] \). Our central result is formulated as follows.

**Theorem 2.** Let \( X \) be a signal (Hilbert) space with some inner product \( \langle \cdot , \cdot \rangle \). Let \( A : X \to X \) and \( B : X \to X \) be two families of linear operators indexed by \( \alpha \in (A \bullet) \) and \( \beta \in (B \bullet) \). Let \( M_{\alpha, \beta} : X \to X \) be a linear operator satisfying
\[
M_{\alpha, \beta} = A, \quad M_{\alpha, 0} = B,
\]
with \( \alpha, 0 \) the identity elements of \( (A \bullet), (B \bullet) \), respectively. Let \( \phi(\alpha, \beta) \) be a complex-valued function satisfying
\[
\phi(\alpha, \beta) = \phi(\alpha, 0) = \phi(0, \beta) = 1.
\]
Finally, let \( v_0(t) \) and \( u_0(t) \) be two families of functions indexed by \( \alpha \in (B \bullet) \), \( b \in (\hat{A} \bullet) \). Then, the \( a, b \)-representation
\[
P_x(a, b) = \int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \lambda^A_{\alpha, \beta} \, d\mu_B (b) \, d\mu_A (a)
\]
for \( \alpha \in (B \bullet), b \in (\hat{A} \bullet) \) satisfies the marginal property
\[
P_{x, b} (a) = \int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \, d\mu_B (b) \times \forall \alpha \in (B \bullet)
\]
if and only if \( A_\alpha \) is related to \( v_0(t) \) as
\[
A_\alpha = \int_B \langle v_0(t), u_0(t) \rangle \lambda^A_{\alpha, \beta} \, d\mu_H (b).
\]
Similarly, \( P_x(a, b) \) satisfies the marginal property
\[
P_{x, a} (b) = \int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \, d\mu_B (b) \times \forall a \in (B \bullet)
\]
if and only if \( B_\beta \) is related to \( u_0(t) \) as
\[
B_\beta = \int_B \langle u_0(t), v_0(t) \rangle \lambda^B_{\alpha, \beta} \, d\mu_B (b).
\]

**Proof.** With (14), the left-hand side of (15) is
\[
\int_B P_x(a, b) \, d\mu_B (a) = \int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \, d\mu_B (b) \times \forall a \in (B \bullet)
\]

It follows from the theory of the group Fourier transform\(^8\) that \( \int_B \lambda^B_{\alpha, \beta} \, d\mu_B (a) = \delta_{\alpha, 0} \delta_{\beta, 0} \) satisfies
\[
\int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \, d\mu_B (b) = \int_B \phi(\alpha, \beta) (M_{\alpha, \beta} x, x) \lambda^B_{\alpha, \beta} \, d\mu_B (b)
\]
where \( \delta (\alpha) \in L^2 (B, d\mu_B) \).
so that (15) is proved. Here we have used (13), (12), and\(\int A(x) \lambda_{\alpha \beta} \, d\mu_\alpha(x) = 0\) by inversion of (16), cf. Footnote 8. From the above, it is evident that (16) is also necessary for (15). The proof of (17) is analogous. □

We emphasize that none of the assumptions 1–4 have been used in Theorem 2 or its proof. The following example shows that some of these assumptions may in fact be violated.

**Projected unitary operators.** Let \(A_\alpha : X \to X'\) and \(B_\beta : X' \to X''\) be unitary representations of \((A, \bullet)\) and \((B, *\)), respectively. Let \(v_\alpha(t), u_\beta(t)\) be the eigenfunctions of \(A_\alpha\) and \(B_\beta\), respectively, assumed to be complete and orthogonal function sets inducing spectral decompositions

\[
A_\alpha = \int \langle v_\alpha(t) \mid v_\alpha\rangle A_\alpha \, d\mu_\alpha(t), \quad B_\beta = \int \langle u_\beta(t) \mid u_\beta\rangle B_\beta \, d\mu_\beta(t),
\]

where the eigenvalues \(\lambda_{\alpha \beta}^A\) and \(\lambda_{\beta \alpha}^B\) are the characters of \((A, \bullet)\) and \((B, *\)), respectively. Let \(X \subset X'\) and define \(A_{\alpha} = P_X A_\alpha P_X, \quad B_{\beta} = P_{X'} B_{\beta} P_{X'}\) and \(v_{\alpha}(t) = (P_X v_{\alpha}) (t), \quad u_{\beta}(t) = (P_{X'} u_{\beta}) (t)\). With (19) we obtain (cf. the derivation in Section 2)

\[
A_\alpha = \int (v_{\alpha}(t) \otimes v_{\alpha}(t)) A_{\alpha} \, d\mu_\alpha(t), \quad B_{\beta} = \int (u_{\beta}(t) \otimes u_{\beta}(t)) B_{\beta} \, d\mu_\beta(t).
\]

Since \(A_\alpha\) and \(B_\beta\) map \(X\) into \(X'\), we will redefine them as \(A_\alpha : X \to X'\) and \(B_\beta : X' \to X''\). With (20) the conditions (16), (18) are satisfied, and thus any a-b representation constructed as in (14) satisfies the marginal properties (15), (17). Again \(v_{\alpha}(t), u_{\beta}(t)\) are the eigenfunctions, and the expressions (20) are not the spectral decompositions, of \(A_{\alpha}\) and \(B_{\beta}\). The function sets \(v_{\alpha}(t), u_{\beta}(t)\) are complete in \(X\) (i.e., \([x, v_{\alpha}(t)]^2\) and \([x, u_{\beta}(t)]^2\) are valid 1-D energy distributions) but not orthogonal. Auto- \(A_\alpha\) and \(B_\beta\) are unitary and cannot be written as exponentiated self-adjoint operators. There is \(A_{\alpha} = B_{\beta} = 1_X\) but \(A_{\alpha} B_{\beta} \neq A_{\alpha} A_{\alpha} * A_{\alpha} A_{\alpha} * A_{\alpha} A_{\alpha} \neq A_{\alpha} * B_{\beta} * A_{\alpha} * B_{\beta} * A_{\alpha} * B_{\beta} \). \(X\) is generally different from \(L^2(A, d\mu_\alpha)\) and \(L^2(B, d\mu_\beta)\). Hence, assumptions 1–4 are violated.

### 4 EXAMPLE: SCALING AND TIME SHIFT OPERATORS ON \(L^2([0, T], dt)\)

In the following example, assumptions 1, 3, and 4 are violated.

Consider the scaling and time-shift operators

\[
A_\alpha' : X_1 \to X_1', \quad A_\alpha' (x) (t) = \frac{1}{\sqrt{\lambda}} x \left( \frac{t}{\lambda} \right), \quad \lambda \in (R^+, \\
B_\beta' : X_2 \to X_2', \quad B_\beta' (x) (t) = x (t - \beta), \quad \beta \in (R^+, \\
\text{where} \ X_1 = L^2(R^+, dt) \text{ and} \ X_2 = L^2(R, dt). \ A_\alpha' \text{ and} \ B_\beta' \text{ are unitary representations of the LCA groups} (A, \bullet) = (R^+, \cdot) \text{ and}(B, * \text{)} = (R^+, +) \text{ (the group of positive} \ \alpha \text{ with multiplicity as group operation and invariant measure} \ d\mu_\alpha (\alpha) = \frac{d\alpha}{\lambda} ) \text{ and} \ (B, * \text{)} = (R^+, +) \text{, respectively.} \ A_\alpha' \text{ and} \ B_\beta' \text{ allow (spectral) decompositions} \ (19) \text{ with eigenvalues} \ (= \text{group characters}) \lambda_{\alpha \beta} = e^{i 2\pi \lambda \beta / \lambda} \text{ and} \lambda_{\beta \alpha} = e^{- i 2\pi \beta \alpha / \lambda} \text{ and eigenfunctions}
\]

\[
v_\alpha(t) = \frac{1}{\sqrt{\lambda}} e^{- i 2\pi \alpha \lambda t / \lambda} \text{ for} \ t > 0 \text{ (} 0 > \alpha \text{ is an arbitrary reference time)} \text{ and} \ u_\beta(t) = e^{i 2\pi \beta t}, \text{ where} \ b \in (A, \bullet) = (R^+, +) \text{ and} \ a \in (B, * \text{)} = (R^+, +).
\]

Assume now that we wish to operate on \(X = L^2([0, T], dt)\), the space of square-integrable signals \(x(t)\) defined for \(t \in [0, T]\), with inner product \((x, y)_X = \int_0^T x(t) y(t)^* dt\). Extending signals as \(x(t) = 0\) outside the respective time interval, we have \(X \subset X_1 \subset X_2\). Unfortunately, \(A_\alpha'\) and \(B_\beta'\) cannot be defined on \(X\) since they may map signals \(x(t) \in X\) onto signals outside \(X\). Hence we use \(A_\alpha = P_X A_\alpha' P_X, \ B_\beta = P_{X'} B_\beta' P_{X'}\), which can be considered to be defined on \(X\), i.e., \(A_\alpha : X \to X\) and \(B_\beta : X' \to X\). Here, \((P_X x)(t) = x(t) I(t)\) where \(I(t) = 1\) for \(t \in [0, T]\) and 0 otherwise. \(A_\alpha\) and \(B_\beta\) allow the decompositions (not spectral decompositions) (20) with \(\lambda_{\alpha \beta} = e^{i 2\pi \lambda \beta / \lambda} \text{ and} \lambda_{\beta \alpha} = e^{- i 2\pi \beta \alpha / \lambda} \text{ as before and} v_\alpha(t) = (P_X v_\alpha) (t) = \frac{1}{\sqrt{\lambda}} e^{- i 2\pi \alpha \lambda t / \lambda} \text{ and} u_\beta(t) = (P_{X'} u_\beta) (t) = e^{i 2\pi \beta t}. \text{ Hence, conditions (16), (17), (18) are satisfied and joint energy distributions can be constructed according to (14), which yields}

\[
P_a (a, b) = \int_0^\infty \int_0^\infty \langle (M_\alpha x, x)_X \rangle e^{2i \pi (\alpha - b) t / \lambda} \, d\mu_\alpha (\alpha) \, dB (\beta)
\]

for \(a, b \in \mathbb{R}\), where \(\langle \phi, \phi \rangle X = \int_0^T \langle (P_X \phi, P_X \phi) \rangle_X \, dt\) and \((P_X \phi) (t) = \frac{1}{\sqrt{\lambda}} e^{- i 2\pi \alpha \lambda t / \lambda}\).

### 5 EXTENDED CFM FOR DISCRETE-TIME AND/OR PERIODIC SIGNALS

The removal of assumption 2 in Section 3 allows the extended CFM to be applied to discrete-time and/or periodic signals. (This application has independently been considered in [14].) We shall discuss simple specific examples based on suitably defined time and frequency shift operators.

**Discrete-time signals.** Let \(X = \mathbb{Z}\), the space of square-integrable discrete-time signals \(x(n) \in \mathbb{Z}\) with inner product \((x, y)_X = \sum_{n=\infty}^\infty x(n) y(n)^*\), and consider the time shift and frequency shift operators

\[
A_\alpha : X \to X, \quad (A_\alpha x) (n) = x(n - \alpha), \quad \alpha \in (\mathbb{Z}, +) \\
B_\beta : X \to X, \quad (B_\beta x) (n) = e^{i 2\pi \beta n} x(n), \quad \beta \in (\mathbb{R}^{mod 1}, +)
\]

\(A_\alpha\) and \(B_\beta\) are unitary representations of the LCA groups \((A, \bullet) = (\mathbb{Z}, +)\) and \((B, * \text{)} = (\mathbb{R}^{mod 1}, +)\) (i.e., the interval \([0, 1]\) with group operation \(+ \text{mod 1}\)). These groups are not isomorphic to \((\mathbb{R}, +)\), i.e., assumption 2 is violated. They are dual, i.e., the dual group of \((\mathbb{Z}, +)\) is \((\mathbb{R}^{mod 1}, +)\) and vice versa. \(A_\alpha\) and \(B_\beta\) allow (the spectral) decompositions

\[
A_\alpha = \int_0^1 \langle v_{\alpha} \mid v_{\alpha} \rangle A_\alpha \, dv_{\alpha}, \quad B_\beta = \int_0^1 \langle u_{\beta} \mid u_{\beta} \rangle B_\beta \, du_{\beta}
\]

9A similar problem occurs in [16], where the time shift operator is considered on \(L^2([0, T], dt)\). Our approach using projected operators explains why the results in [16] are nevertheless valid.
where $\phi(t)$ must satisfy $\phi(t) = \phi(t)$. Setting $A_s = M_a$, we obtain $M_{a+M_s} = 1$. Substituting $x(n) = \sum_{k=-\infty}^{\infty} x(n-k)$ and $P(a,b) = \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} P(a,b) = 1$. Finally, let $X = \{0, 1\}$ and $Y = \{0, 1\}$. We use the discrete cyclic time and frequency shift operators 

$P(a,b) = 1 \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} P(a,b) = 1$. Setting $A_s = M_a$, we obtain $M_{a+M_s} = 1$.