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FRAME-THEORETIC ANALYSIS AND DESIGN OF OVERSAMPLED FILTER BANKS

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ABSTRACT

We provide a frame-theoretic analysis of oversampled and critically sampled, FIR and IIR, uniform filter banks (FBs). Our analysis is based on a relation between the polyphase matrices and the frame operator. For a given oversampled analysis FB, we parameterize all synthesis FBs providing perfect reconstruction, and we discuss the minimum norm synthesis FB and its approximative construction. We find conditions for a FB to provide a frame expansion. Paraunitary and biorthogonal FBs are shown to correspond to tight and exact frames, respectively. A new procedure for the design of paraunitary FBs is formulated. We show that the frame bounds are related with the eigenvalues of the polyphase matrices and the oversampling factor, and that they determine important numerical properties of the FB.

1. INTRODUCTION

Uniform filter banks (FBs)[1], i.e., filter banks with the same integer decimation ratio in each channel [1]-[4], correspond to a class of discrete-time signal expansions. The relation between discrete-time signal expansions and maximally decimated (or critically sampled) FBs has been studied in [5, 2, 6], and it has also been recognized that oversampled FBs [2, 4] correspond to redundant signal expansions [2], [7]-[10]. In [7]-[10] the use of the *theory of frames* [11] for the study of oversampled FBs has been proposed. In [12] a frame-theoretic analysis of continuous-time FBs is presented. In [7, 8] oversampled FIR FBs are studied using polynomial matrices [1]. A vector-filter framework for oversampled FIR FBs has been proposed in [10].

This paper presents a new frame-theoretic approach to oversampled and critically sampled FIR and IIR FBs. Our approach, which extends an analysis of continuous-time Weyl-Heisenberg frames proposed in [13], is based on an important relation between the FB's polyphase matrices and the frame operator. In Section 2, we review FBs, introduce the corresponding type of frames, and show that the FB's polyphase matrices provide matrix representations of the frame operator. Section 3 shows that the frame bounds determine important numerical properties of FBs, and that they are related to the eigenvalues of the polyphase matrices and the oversampling factor. In Section 4, we parameterize all synthesis FBs providing perfect reconstruction (PR) for a given oversampled analysis FB, and we discuss the minimum norm synthesis FB and its approximative construction. Conditions for a FB to provide a frame expansion are formulated, and the equivalence of critically sampled (biorthogonal) FBs and exact frames is discussed. In Section 5, we show that paraunitary FBs correspond to tight frames, and we propose a method for constructing paraunitary FBs from given nonparaunitary FBs.

[1]For the sake of brevity, we shall henceforth use the term *filter bank* (FB) instead of *uniform filter bank*.

2. FILTER BANKS AND FRAMES

We consider an N-channel FB with subsampling by the integer factor M in each channel, PR, and zero delay,[2] so that $\hat{x}[n] = x[n]$ where $x[n]$ and $\hat{x}[n]$ denote the input and reconstructed signal, respectively. The transfer functions of the analysis and synthesis filters are $H_k(z)$ and $F_k(z)$ $(0 \le k \le N-1)$, with corresponding impulse responses $h_k[n]$ and $f_k[n]$, respectively. The subband signals are

$$v_k[m] = \sum_{n=-\infty}^{\infty} x[n]\, h_k[mM-n]\,, \quad 0 \le k \le N-1\,, \tag{1}$$

and the reconstructed signal is

$$\hat{x}[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} v_k[m]\, f_k[n-mM]\,. \tag{2}$$

In the *oversampled* case $(N > M)$, the subband signals $v_k[m]$ are redundant since they contain more samples (per unit of time) than the input signal $x[n]$. Oversampled FBs offer more design freedom, improved numerical properties, and improved noise immunity as compared to critically sampled FBs. The increased design freedom corresponds to the fact that, for a given oversampled analysis FB, there exists a whole class of synthesis FBs providing PR (see Section 4). The improved noise immunity [11, 14, 15] allows a coarser quantization of the subband signals (see Section 3).

The *polyphase decomposition* of the analysis filters $H_k(z)$ reads $H_k(z) = \sum_{n=0}^{M-1} z^n\, E_{k,n}(z^M)$, $0 \le k \le N-1$, where

$$E_{k,n}(z) = \sum_{m=-\infty}^{\infty} h_k[mM-n]\, z^{-m}$$

with $0 \le k \le N-1$, $0 \le n \le M-1$ is the nth polyphase component of the kth analysis filter $H_k(z)$. The $N \times M$ analysis polyphase matrix $\mathbf{E}(z)$ is defined as $[\mathbf{E}(z)]_{k,n} = E_{k,n}(z)$. The synthesis filters can be similarly decomposed, $F_k(z) = \sum_{n=0}^{M-1} z^{-n}\, R_{k,n}(z^M)$, with the synthesis polyphase components

$$R_{k,n}(z) = \sum_{m=-\infty}^{\infty} f_k[mM+n]\, z^{-m}.$$

The $M \times N$ synthesis polyphase matrix $\mathbf{R}(z)$ is defined as $[\mathbf{R}(z)]_{k,n} = R_{n,k}(z)$.

FB analysis/synthesis can be interpreted as a signal expansion [2, 16, 5, 1, 17]. The subband signals in (1) can be written as inner products $v_k[m] = \langle x, h_{k,m}^{-*} \rangle =$

[2]We note that our theory can easily be extended to PR with nonzero delay.

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$\sum_{n=-\infty}^{\infty} x[n] \left(h_{k,m}^{-*}[n]\right)^*$ with $h_{k,m}^{-*}[n] = h_k^*[mM-n]$, where * stands for complex conjugation. Furthermore, with (2) and the PR property, we have

$$x[n] = \hat{x}[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left\langle x, h_{k,m}^{-*} \right\rangle f_{k,m}[n]$$

with $f_{k,m}[n] = f_k[n - mM]$. This shows that the FB corresponds to an expansion of the input signal $x[n]$ into the function set $\{f_{k,m}[n]\}$ $(0 \le k \le N-1, -\infty < m < \infty)$. In general the set $\{f_{k,m}[n]\}$ is not orthogonal, so that the expansion coefficients, i.e., the subband signal samples $v_k[m] = \left\langle x, h_{k,m}^{-*} \right\rangle$, are obtained by projecting the signal $x[n]$ onto a "dual" set of functions $\{h_{k,m}^{-*}[n]\}$. Critically sampled FBs provide orthogonal or biorthogonal signal expansions [16], whereas oversampled FBs correspond to redundant (overcomplete) expansions [2], [7]-[10], [17].

The *theory of frames* [11] is a powerful vehicle for the study of redundant signal expansions. The set $\{f_{k,m}[n]\}$ with $f_{k,m}[n] = f_k[n - mM]$ will be called a *uniform filter bank frame* (UFBF) for[3] $l^2(\mathbb{Z})$ if for all $x[n] \in l^2(\mathbb{Z})$

$$A\|x\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left|\left\langle x, f_{k,m} \right\rangle\right|^2 \le B\|x\|^2 \qquad (3)$$

with the *frame bounds* $A > 0$ and $B < \infty$. The frame bounds determine important numerical properties of the FB (see Sections 3 and 4). For synthesis filters $f_k[n]$ such that $\{f_{k,m}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, a particular analysis set (corresponding to expansion coefficients with minimum norm) is given by [11]

$$h_{k,m}^{-*}[n] = (S^{-1} f_{k,m})[n]. \qquad (4)$$

Here, S^{-1} is the inverse of the *frame operator* defined as $(Sx)[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left\langle x, f_{k,m} \right\rangle f_{k,m}[n]$. If the synthesis set $\{f_{k,m}[n]\}$ is a frame, then the analysis set $\{h_{k,m}^{-*}[n]\}$ defined by (4) is again a frame (the "dual" frame), with frame bounds $A' = 1/B$ and $B' = 1/A$. It can be shown [17] that the frame that is dual to a UFBF is itself a UFBF: If $\{f_{k,m}[n]\}$ is a UFBF with parameters M and N, then the dual frame $\{h_{k,m}^{-*}[n]\}$ is again a UFBF with the same parameters M and N, i.e.,

$$h_{k,m}^{-*}[n] = h_k^{-*}[n - mM], \qquad 0 \le k \le N-1,$$

where

$$h_k^{-*}[n] = h_k^*[-n] = (S^{-1} f_k)[n].$$

A frame is called *snug* if $B'/A' = B/A \approx 1$ and *tight* if $B'/A' = B/A = 1$. For a tight frame we have $S^{-1} = A' I$ (where I is the identity operator on $l^2(\mathbb{Z})$), and hence there is simply $f_k[n] = \frac{1}{A'} h_k^*[-n]$.

The following result [17] extends a similar result on continuous-time Weyl-Heisenberg frames [13].

Theorem 1. Let $y[n] = (Sx)[n]$, where S is the frame operator corresponding to a UFBF. Then, the polyphase components $Y_n(z) = \sum_{m=-\infty}^{\infty} y[mM+n] z^{-m}$ of $Y(z)$ and the polyphase components $X_{n'}(z) = \sum_{m=-\infty}^{\infty} x[mM+n'] z^{-m}$ of $X(z)$ are related as

$$Y_n(z) = \sum_{n'=0}^{M-1} S_{n,n'}(z) X_{n'}(z), \qquad (5)$$

where[4] $S_{n,n'}(z) = \sum_{k=0}^{N-1} R_{k,n}(z) \tilde{R}_{k,n'}(z)$. Introducing the polyphase vectors $\mathbf{x}(z) = [X_0(z)\ X_1(z)\ \dots\ X_{M-1}(z)]^T$ and $\mathbf{y}(z) = [Y_0(z)\ Y_1(z)\ \dots\ Y_{M-1}(z)]^T$, (5) can be rewritten as[5]

$$\mathbf{y}(z) = \mathbf{S}(z)\,\mathbf{x}(z) \qquad \text{with} \quad \mathbf{S}(z) = \mathbf{R}(z)\,\tilde{\mathbf{R}}(z).$$

Thus, the frame operator S is expressed in the polyphase domain by the $M \times M$ *UFBF matrix* $\mathbf{S}(z) = \mathbf{R}(z)\tilde{\mathbf{R}}(z)$. Similarly, the inverse frame operator S^{-1} can be expressed in the polyphase domain by the $M \times M$ *inverse UFBF matrix* $\mathbf{S}^{-1}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$. Specializing to the unit circle ($z = e^{j2\pi\theta}$), it can be shown that $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$ and $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$ are *matrix representations* of the frame operator S and the inverse frame operator S^{-1}, respectively [17]. Most of our subsequent discussion of FBs will be based on these matrix representations.

An important consequence of Theorem 1 is the equality of the eigenvalues of the frame operator S and the eigenvalues $\lambda_n(\theta)$ of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$ [17]. Thus, the eigenanalysis of the frame operator (a matrix of infinite size) reduces to the eigenanalysis of an $M \times M$ matrix indexed by a parameter $\theta \in [0,1)$. Similarly, the eigenvalues of the inverse frame operator S^{-1} equal the eigenvalues of the inverse UFBF matrix $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$, which will be denoted $\lambda'_n(\theta)$. The matrices $\mathbf{S}(e^{j2\pi\theta})$ and $\mathbf{S}^{-1}(e^{j2\pi\theta})$ can be shown to be positive definite [17], hence their eigenvalues are positive.

3. FRAME BOUNDS

Important numerical properties of the UFBF $\{h_{k,m}^{-*}[n]\}$ and the associated FB are determined by its frame bounds A and B or, equivalently, $A' = 1/B$ and $B' = 1/A$ [11]. By analogy to (3), the subband signals $v_k[m] = \left\langle x, h_{k,m}^{-*} \right\rangle$ of a FB providing a UFBF expansion satisfy

$$A'\|x\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 \le B'\|x\|^2 \qquad (6)$$

with $0 < A' \le B' < \infty$. This generalizes the well-known energy conservation, $\sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 = \|x\|^2$, in orthogonal FBs [5]. Note that (6) shows that the subband signals $v_k[m]$ are in $l^2(\mathbb{Z})$ if the input signal $x[n]$ is in $l^2(\mathbb{Z})$.

Consider now subband signals $v_k[m]$ corresponding to input signal $x[n]$ and reconstructed signal $\hat{x}[n]$, and perturbed subband signals $v'_k[m] = v_k[m] + \Delta v_k[m]$ corresponding to input signal $x'[n] = x[n] + \Delta x[n]$ and reconstructed signal $\hat{x}'[n] = \hat{x}[n] + \Delta\hat{x}[n]$. Using the PR property, $\hat{x}[n] = x[n]$ and $\hat{x}'[n] = x'[n]$, it follows from (6) that the reconstruction error $\Delta\hat{x}[n] = \hat{x}'[n] - \hat{x}[n] = x'[n] - x[n]$ is bounded as

$$A'\,\|\Delta\hat{x}\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\Delta v_k[m]|^2 \le B'\,\|\Delta\hat{x}\|^2.$$

With $\|\Delta v\|^2 = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\Delta v_k[m]|^2$ denoting the total energy of the subband signal perturbations $\Delta v_k[m]$, this can be written as

[3] Here $l^2(\mathbb{Z})$ denotes the space of square-summable functions $x[n]$, i.e., $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$.

[4] $\tilde{R}_{k,n}(z) = R_{k,n}^*(1/z^*)$ is the paraconjugate of $R_{k,n}(z)$.

[5] $\tilde{\mathbf{R}}(z) = \mathbf{R}^H(1/z^*)$ where ^{H} denotes conjugate transposition.

$$A \leq \frac{\|\Delta\hat{x}\|^2}{\|\Delta v\|^2} \leq B, \qquad (7)$$

where $A = 1/B'$ and $B = 1/A'$. Thus, for given subband perturbation energy $\|\Delta v\|^2$, the frame bounds A and B provide lower and upper bounds on the resulting reconstruction error energy $\|\Delta\hat{x}\|^2$. The reconstruction error energy is minimized by making A as small as possible and B as close to A as possible. Note that $A \approx B$ implies $A' \approx B'$. Thus it is desirable to have $A' \approx B'$, i.e., a snug frame.

The frame bounds can be related to the *oversampling factor* N/M. It can be shown [17, 10] that $A' \leq \frac{1}{M}\sum_{k=0}^{N-1}\|h_k\|^2 \leq B'$. Normalizing the $h_k[n]$ such that $\|h_k\|^2 = 1$ for $0 \leq k \leq N-1$, we get $\sum_{k=0}^{N-1}\|h_k\|^2 = N$ and thus $A' \leq N/M \leq B'$ or equivalently

$$A \leq \frac{1}{N/M} \leq B.$$

We see that A will be smaller for larger N/M, i.e., more oversampling. In particular, in the case of a tight frame (corresponding to a paraunitary FB, see Section 5), we have

$$A = B = \frac{1}{N/M}.$$

Inserting this in (7) yields

$$\frac{\|\Delta\hat{x}\|^2}{\|\Delta v\|^2} = \frac{1}{N/M}.$$

Thus, the energy of the reconstruction error for given subband perturbation energy $\|\Delta v\|^2$ is here inversely proportional to the oversampling factor. The improved noise immunity of oversampled FBs allows a coarser quantization of the subband signals [15]. A similar result exists for oversampled A/D conversions [18].

Since the frame bounds characterize important numerical properties of a FB as demonstrated above, their calculation is of importance. It can be shown [17] that the (tightest possible) frame bounds A' and B' of a FB providing a UFBF expansion are given by the essential infimum and supremum [11], respectively, of the eigenvalues $\lambda'_n(\theta)$ of the inverse UFBF matrix $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$,

$$\begin{aligned} A' &= \operatorname{ess\,inf}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \lambda'_n(\theta) \\ B' &= \operatorname{ess\,sup}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \lambda'_n(\theta). \end{aligned}$$

4. PERFECT RECONSTRUCTION AND FRAME PROPERTIES

The next theorem [17] states a PR condition for oversampled FBs, as well as a general expression of the synthesis FB providing PR for given analysis FB.

Theorem 2. An oversampled or critically sampled FB satisfies the PR condition $\hat{x}[n] = x[n]$ if and only if

$$\mathbf{R}(z)\,\mathbf{E}(z) = \mathbf{I}_M, \qquad (8)$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. This condition has solutions $\mathbf{R}(z)$ for given $\mathbf{E}(z)$ if and only if $\mathbf{E}(z)$ has full rank, $\text{rank}\{\mathbf{E}(z)\} = M$, almost everywhere. Any solution of (8) can be written as

$$\mathbf{R}(z) = \hat{\mathbf{R}}(z) + \hat{\mathbf{R}}(z)\mathbf{W}(z)\left[\mathbf{I}_N - \mathbf{E}(z)\,\hat{\mathbf{R}}(z)\right], \qquad (9)$$

where $\hat{\mathbf{R}}(z)$, the *para-pseudo-inverse* of $\mathbf{E}(z)$, is a particular solution of (8) defined as

$$\hat{\mathbf{R}}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z), \qquad (10)$$

and $\mathbf{W}(z)$ is an $N \times N$ matrix with arbitrary elements $W_{k,l}(z)$ satisfying $|W_{k,l}(e^{j2\pi\theta})| < \infty$.

In the oversampled case $N > M$, the nonuniqueness of the synthesis FB for given analysis FB as expressed by (9) entails a freedom of design that does not exist in the case of critical sampling (for critical sampling, $N = M$, (9) reduces to the *unique* solution $\mathbf{R}(z) = \hat{\mathbf{R}}(z) = \mathbf{E}^{-1}(z)$). The expression (9) for $\mathbf{R}(z)$ is a canonical parameterization in terms of the N^2 complex numbers $W_{k,l}(z)$ that can be chosen arbitrarily [19, 20]. It can be shown [17, 8] that the particular synthesis polyphase matrix given by the para-pseudo-inverse $\hat{\mathbf{R}}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$ corresponds to the synthesis FB provided by frame theory according to $f_k[n] = (Sh_k^{-*})[n]$, or in other words, $\{h_{k,m}^{-*}[n]\}$ is the UFBF that is dual to $\{f_{k,m}[n]\}$. It is shown in [17] that this frame-theoretic solution minimizes $\sum_{k=0}^{N-1}\|f_k\|^2$ among all possible solutions. We shall hereafter restrict our attention to this *minimum norm synthesis FB*, i.e., to the particular synthesis polyphase matrix $\mathbf{R}(z) = \hat{\mathbf{R}}(z) = \left[\tilde{\mathbf{E}}(z)\,\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$ (denoted simply $\mathbf{R}(z)$ in the following).

We now formulate conditions for a FB to provide a UFBF expansion. In addition to PR, the frame property guarantees a certain degree of numerical stability (see Section 3).

Theorem 3 [17]. An oversampled or critically sampled FB with BIBO stable[6] analysis filters $h_k[n]$ provides a UFBF expansion in $l^2(\mathbb{Z})$, i.e., the analysis set $\{h_{k,m}^{-*}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, if and only if the analysis polyphase matrix $\mathbf{E}(z)$ has full rank on the unit circle, i.e.,

$$\text{rank}\{\mathbf{E}(e^{j2\pi\theta})\} = M \qquad \text{for} \quad 0 \leq \theta < 1.$$

We note that Theorem 3 holds for both FIR and IIR FBs. For the FIR case, where stability is inherently guaranteed, the full rank condition has been found independently in [8]. In the following, a FB providing a UFBF expansion in $l^2(\mathbb{Z})$ will be called a *frame FB* (FFB). It can also be shown [17] that an FB is an FFB if and only if the eigenvalues $\lambda'_n(\theta)$ of the inverse UFBF matrix $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$ satisfy the conditions $\operatorname{ess\,inf}_{\theta \in [0,1), n=0,1,\ldots,M-1} \lambda'_n(\theta) > 0$ and $\operatorname{ess\,sup}_{\theta \in [0,1), n=0,1,\ldots,M-1} \lambda'_n(\theta) < \infty$.

In the case of an FFB, it follows from (10) that calculation of the minimum norm synthesis FB requires the inversion of the matrix $\tilde{\mathbf{E}}(z)\mathbf{E}(z)$. However, by analogy to the approximation of dual frames described in [11], a simple approximation to the synthesis FB can be based on a series expansion of $\mathbf{S}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}$ which results in

$$\mathbf{R}(z) = \frac{2}{A'+B'}\sum_{n=0}^{\infty}\left[\mathbf{I}_M - \frac{2}{A'+B'}\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^n\tilde{\mathbf{E}}(z).$$

We shall restrict our attention to the first-order approximation of $\mathbf{R}(z)$ obtained by truncating this series at $n = 0$,

$$\mathbf{R}^{(0)}(z) = \frac{2}{A'+B'}\tilde{\mathbf{E}}(z),$$

or equivalently

$$f_k^{(0)}[n] = \frac{2}{A'+B'}h_k^*[-n].$$

With $\hat{x}^{(0)}[n]$ denoting the signal reconstructed using this "first-order synthesis FB," the resulting reconstruction error can be bounded in terms of the frame bounds as [11]

[6]BIBO stability means $h_k[n] \in l^1(\mathbb{Z})$, i.e., $\sum_{n=-\infty}^{\infty}|h_k[n]| < \infty$ for $0 \leq k \leq N-1$.

$$\|\hat{x}^{(0)} - x\| \leq \frac{B'/A' - 1}{A'/B' + 1} \|x\|.$$

Note that the reconstruction error is small for $B'/A' \approx 1$, i.e., when the underlying UFBF is snug. Here, the first-order impulse responses $f_k^{(0)}[n]$ are good approximations to the minimum norm, PR impulse responses $f_k[n]$. In the tight case where $B'/A' = 1$, we have $\|\hat{x}^{(0)} - x\| = 0$ and the approximation is exact, $f_k^{(0)}[n] = f_k[n] = \frac{1}{A'} h_k^*[-n]$.

It is well known that in a *critically sampled* FB biorthogonality of the analysis and synthesis filters and their shifted versions is equivalent to PR [16, 5, 1]. The following theorem [17] states that critically sampled FFBs correspond to *exact* UFBFs. Exact frames consist of linearly independent frame functions; they are minimal in that removal of an arbitrary frame function from the set $\{f_{k,m}[n]\}$ leaves an incomplete set [11]. Furthermore, the frame functions here satisfy the biorthogonality relation $\langle f_{k,m}, h_{k',m'}^{-*} \rangle = \delta_{k,k'}\, \delta_{m,m'}$ and, in particular, $\langle f_k, h_{k'}^{-*} \rangle = \delta_{k,k'}$ [11].

Theorem 4. An FFB provides an *exact* UFBF expansion if and only if it is critically sampled.

The linear independence of the set $\{f_{k,m}[n]\}$ for a critically sampled FB has also been observed in [5], and the biorthogonality has been reported in [16]. Note that Theorem 4 implies that an oversampled UFBF cannot be exact and, hence, the corresponding FB cannot be biorthogonal.

5. PARAUNITARY FILTER BANKS AND TIGHT FRAMES

The analysis UFBF $\{h_{k,m}^{-*}[n]\}$ is *tight* if $A' = B'$. From the theory of frames, we know that here $\boldsymbol{S}^{-1} = A'\boldsymbol{I}$ [11], and hence the minimum norm synthesis FB is $f_k[n] = \frac{1}{A'} h_k^*[-n]$. This is precisely the relation between the synthesis and analysis filters in a paraunitary FB [1]. In fact we can formulate the following theorem [17].

Theorem 5. An FFB (oversampled or critically sampled) provides a *tight* UFBF expansion in $l^2(\mathbb{Z})$ if and only if it is *paraunitary*, i.e.,

$$\mathbf{S}^{-1}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z) \equiv A'\,\mathbf{I}_M.$$

The frame bound is $A' = S_{n,n}^{-1}(z) = \sum_{k=0}^{N-1} \tilde{E}_{k,n}(z)\, E_{k,n}(z)$.

The equivalence of tight Weyl-Heisenberg frames (a subclass of UFBFs) and paraunitary DFT FBs has been noted in [6] and independently in [7, 8]. For oversampled FIR FBs, a result similar to Theorem 5 has been stated independently in [8]. Paraunitary FBs are also known as orthogonal FBs. However, a paraunitary FFB is orthogonal only if it is critically sampled: an oversampled paraunitary FB corresponds to a UFBF that is tight but not orthogonal.

The following theorem [17] describes a method for the derivation of a paraunitary FFB from a given nonparaunitary FFB. This method is an adaptation of a procedure for the design of tight frames described in [11].

Theorem 6. Consider an FFB with polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$, and let $\mathbf{P}(z)$ be an invertible, parahermitian,[7] $M \times M$ matrix $\mathbf{P}(z)$ such that $\mathbf{P}^2(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$. Then the FFB with analysis polyphase matrix

$$\mathbf{E}_p(z) = \mathbf{E}(z)\,\mathbf{P}^{-1}(z)$$

is paraunitary with frame bound $A' = 1$, i.e., $\mathbf{S}_p^{-1}(z) = \tilde{\mathbf{E}}_p(z)\,\mathbf{E}_p(z) \equiv \mathbf{I}_M$. If, moreover, the original FFB is biorthogonal, then the FFB with analysis polyphase matrix $\mathbf{E}_p(z)$ is orthogonal.

[7] A matrix $\mathbf{P}(z)$ is said to be parahermitian if $\tilde{\mathbf{P}}(z) = \mathbf{P}(z)$.

6. CONCLUSION

We have shown that the theory of frames is a powerful vehicle for the analysis and design of oversampled uniform filter banks (FBs). The frame-theoretic analysis of FBs was seen to be consistent with many results previously derived in the FB literature, and it also provided a number of new results and insights. In particular, we showed that most of the results formulated in [7, 8, 10] for the FIR case also hold in the IIR case. We discussed the relevance of the frame bounds and the benefits of oversampling. We furthermore presented a general expression of the PR synthesis FB for given oversampled analysis FB, and we discussed the special synthesis FB provided by frame theory.

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