Covariant Time-Frequency Distributions Based on Conjugate Operators

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Abstract— **We propose classes of quadratic time-frequency distributions that retain the inner structure of Cohen's class. Each of these classes is based on a pair of "conjugate" unitary operators producing time-frequency displacements. The classes satisfy covariance and marginal properties corresponding to these operators. For each class, we define a "central member" generalizing the Wigner distribution and the Q-distribution, and we specify a transformation by which the class can be derived from Cohen's class.**

I. Introduction

COHEN'S class with signal-independent kernels (Cohen's class hereafter) consists of all quadratic time-frequency representations (QTFR's) $T_x(t,f)$ that are *covariant* to time-frequency shifts $T_{\mathbf{S}_{\tau,\nu}x}(t,f) = T_x(t-\tau, f-\nu)$ [1]–[3]. Here, $x(t)$ is a signal with Fourier transform $X(f) = \int_t x(t)\, e^{-j2\pi ft}\, dt$, and $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ with the time-shift operator $(\mathbf{T}_\tau\, x)(t) = x(t-\tau)$ and the frequency-shift operator $(\mathbf{F}_\nu\, x)(t) = x(t)\, e^{j2\pi\nu t}$. The properties of the operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ entail a characteristic structure of Cohen's class. In this letter, this structure will be worked out in a generalized framework. We construct QTFR classes that are based on pairs of "conjugate" operators and that satisfy generalized covariance and marginal properties [4], [5]. Due to space limitations, we summarize our results without providing proofs. The concept of conjugate operators has been developed independently in [6] and [7].

II. Conjugate Operators

We consider two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. They are assumed to be *unitary* on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical *composition laws* $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}$ where $(\mathcal{G}, \bullet)$ is a commutative group [4], [8], [9]. The *eigenvalues* $\lambda^A_{\alpha,\tilde{\alpha}}$ and *eigenfunctions* $u^A_{\tilde{\alpha}}(t)$ of $\mathbf{A}_\alpha$ are defined by $\big(\mathbf{A}_\alpha\, u^A_{\tilde{\alpha}}\big)(t) = \lambda^A_{\alpha,\tilde{\alpha}}\, u^A_{\tilde{\alpha}}(t)$; they are indexed by a "dual parameter" $\tilde{\alpha}$. The **A**-*Fourier transform* (**A**-FT) [8] is defined as $X_A(\tilde{\alpha}) = (\mathcal{F}_A\, x)(\tilde{\alpha}) \triangleq \langle x, u^A_{\tilde{\alpha}}\rangle = \int_t x(t)\, u^{A*}_{\tilde{\alpha}}(t)\, dt$. Analogous definitions apply to $\lambda^B_{\beta,\tilde{\beta}}$, $u^B_{\tilde{\beta}}(t)$, and $X_B(\tilde{\beta}) = (\mathcal{F}_B\, x)(\tilde{\beta})$. We now assume that applying one operator to an eigenfunction of the other operator merely shifts the eigenfunction parameter [4], [5]:

Definition 1. Two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ as described above will be called *conjugate* if $\tilde{\alpha} \in \mathcal{G}$, $\tilde{\beta} \in \mathcal{G}$ and

$$\big(\mathbf{B}_\beta\, u^A_{\tilde{\alpha}}\big)(t) = u^A_{\tilde{\alpha}\bullet\beta}(t)\,, \qquad \big(\mathbf{A}_\alpha\, u^B_{\tilde{\beta}}\big)(t) = u^B_{\tilde{\beta}\bullet\alpha}(t)\,.$$

Two conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$ can be shown to satisfy the following remarkable properties [4]:

1) Their eigenvalues can be written as $\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ and $\lambda^B_{\beta,\tilde{\beta}} = e^{\mp j2\pi\,\mu(\beta)\,\mu(\tilde{\beta})} = \big(\lambda^A_{\beta,\tilde{\beta}}\big)^*$. Here, $\mu(g) \in \mathbb{R}$ maps $(\mathcal{G},\bullet)$ onto $(\mathbb{R},+)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$ where g_0 is the identity element in $\mathcal{G}$ and g^{-1} denotes the group-inverse of g. In the following, we shall simply write $\lambda^A_{\alpha,\beta} = \lambda_{\alpha,\beta}$ and $\lambda^B_{\alpha,\beta} = \lambda^*_{\alpha,\beta}$.
2) They commute up to a phase factor, $\mathbf{A}_\alpha\mathbf{B}_\beta = \lambda_{\alpha,\beta}\,\mathbf{B}_\beta\mathbf{A}_\alpha$.
3) Their eigenfunctions are related as $\langle u^B_{\tilde{\beta}}, u^A_{\tilde{\alpha}}\rangle = \lambda_{\tilde{\alpha},\tilde{\beta}}$, $\int_\mathcal{G} u^B_{\tilde{\beta}}(t)\,\lambda^*_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta}) = u^A_{\tilde{\alpha}}(t)$, and $\int_\mathcal{G} u^A_{\tilde{\alpha}}(t)\,\lambda_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha}) = u^B_{\tilde{\beta}}(t)$, where $d\mu(g) \triangleq |\mu'(g)|\, dg$.
4) The inner product of their kernels is $\int_t \int_{t'} A_\alpha(t,t')\, B^*_\beta(t,t')\, dt\, dt' = \delta\big(\mu(\alpha)\big)\ \ \delta\big(\mu(\beta)\big)$ where $\delta(\cdot)$ denotes the Dirac delta function (cf. [10]).
5) The **A**-FT and **B**-FT satisfy $\big(\mathcal{F}_A\mathbf{B}_\beta\, x\big)(\tilde{\alpha}) = (\mathcal{F}_A\, x)(\tilde{\alpha}\bullet\beta^{-1})$ and $\big(\mathcal{F}_B\mathbf{A}_\alpha\, x\big)(\tilde{\beta}) = (\mathcal{F}_B\, x)(\tilde{\beta}\bullet\alpha^{-1})$, and they are related as $X_B(\tilde{\beta}) = \int_\mathcal{G} X_A(\tilde{\alpha})\,\lambda^*_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha})$ and $X_A(\tilde{\alpha}) = \int_\mathcal{G} X_B(\tilde{\beta})\,\lambda_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta})$ (cf. [6], [7]).

We now compose two conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$ as $\mathbf{D}_\theta \triangleq \mathbf{B}_\beta\mathbf{A}_\alpha$ where $\theta = (\alpha,\beta) \in \mathcal{G}^2$ with $\mathcal{G}^2 = \mathcal{G}\times\mathcal{G}$. It is readily shown that $\mathbf{D}_\theta$ is unitary on $\mathcal{X}$ and satisfies the *composition property* [4], [11] $\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = \lambda_{\alpha_2,\beta_1}\,\mathbf{D}_{\theta_1\circ\theta_2}$ where $(\mathcal{G}^2,\circ)$ is the commutative group with group operation $\theta_1\circ\theta_2 = (\alpha_1,\beta_1)\circ(\alpha_2,\beta_2) = (\alpha_1\bullet\alpha_2, \beta_1\bullet\beta_2)$, identity element $\theta_0 = (g_0, g_0)$, and inverse elements $\theta^{-1} = (\alpha^{-1},\beta^{-1})$. Furthermore, $\mathbf{D}^{-1}_\theta = \lambda_{\alpha,\beta}\,\mathbf{D}_{\theta^{-1}}$ and $\mathbf{D}_{\theta_0} = \mathbf{I}$ where $\mathbf{I}$ is the identity operator on $\mathcal{X}$.

Examples. The shift operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ underlying Cohen's class are conjugate with $(\mathcal{G},\bullet) = (\mathbb{R},+)$, $\mu(g) = g$, eigenvalues $\lambda^T_{\tau,f} = e^{-j2\pi\tau f}$, $\lambda^F_{\nu,t} = e^{j2\pi\nu t}$, eigenfunctions $u^T_f(t) = e^{j2\pi ft}$, $u^F_t(t') = \delta(t'-t)$, and dual parameters $\tilde{\tau} = f$, $\tilde{\nu} = t$. The operators are conjugate since $\big(\mathbf{F}_\nu\, u^T_f\big)(t) = u^T_{f+\nu}(t)$ and $\big(\mathbf{T}_\tau\, u^F_t\big)(t') = u^F_{t+\tau}(t')$. The operators underlying the *hyperbolic* QTFR's class [12] are conjugate as well, but the operators underlying the *affine class* and the *power classes* [13]-[15] are *not* conjugate.

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III. Covariance And Marginal Properties

Let $\nu^A_{\tilde\alpha}(t)$ and $\tau^B_{\tilde\beta}(f)$ denote the instantaneous frequency and group delay of the eigenfunctions $u^A_{\tilde\alpha}(t)$ and $u^B_{\tilde\beta}(t)$, respectively. For any $\tilde\theta = (\tilde\alpha, \tilde\beta) \in \mathcal{G}^2$, the corresponding functions $\nu^A_{\tilde\alpha}(t)$ and $\tau^B_{\tilde\beta}(f)$ are assumed[1] to intersect in a unique time-frequency (TF) point $z = (t, f)$. Hence, $z = l(\tilde\theta)$ where $l(\tilde\theta)$ will be called the *localization function* (LF) of the operator $\mathbf{D}_\theta$ [4], [5]. The LF is constructed by solving the system of equations $\nu^A_{\tilde\alpha}(t) = f$, $\tau^B_{\tilde\beta}(f) = t$ for $(t, f) = z$ [4], [10], [16]. It is assumed to be invertible, i.e., $z = l(\tilde\theta) \Leftrightarrow \tilde\theta = l^{-1}(z)$.

The LF describes the *TF displacements* caused by $\mathbf{D}_\theta$. If a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(\mathbf{D}_\theta x)(t)$ will be localized about a new TF point $z' = (t', f')$. Since z is the intersection[2] of $u^A_{\tilde\alpha}(t)$ and $u^B_{\tilde\beta}(t)$ with $(\tilde\alpha, \tilde\beta) = \tilde\theta = l^{-1}(z)$, z' will be the intersection of $(\mathbf{D}_\theta u^A_{\tilde\alpha})(t)$ and $(\mathbf{D}_\theta u^B_{\tilde\beta})(t)$. Due to the conjugateness of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, $(\mathbf{D}_\theta u^A_{\tilde\alpha})(t) = \lambda_{\alpha,\tilde\alpha}\, u^A_{\tilde\alpha \bullet \beta}(t)$ and $(\mathbf{D}_\theta u^B_{\tilde\beta})(t) = \lambda^*_{\beta, \tilde\beta \bullet \alpha}\, u^B_{\tilde\beta \bullet \alpha}(t)$. Hence, $z' = l(\tilde\alpha \bullet \beta, \tilde\beta \bullet \alpha) = l(\tilde\theta \circ \theta^T) = l(l^{-1}(z) \circ \theta^T)$ with $\theta^T = (\beta, \alpha)$. This motivates the following definition [4], [5]:

Definition 2. A QTFR $T_x(z) = T_x(t, f)$ will be called *covariant to* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = T_x\Big(l\big(l^{-1}(z) \circ \theta^{-T}\big)\Big) \tag{1}$$

with $\theta^{-T} = (\theta^{-1})^T = (\beta^{-1}, \alpha^{-1})$.

The class of all QTFR's covariant to $\mathbf{D}_\theta$ is characterized as follows (cf. [4], [11]):

Theorem 1. A QTFR $T_x(z) = T_x(t, f)$ is covariant to an operator $\mathbf{D}_\theta$ if and only if

$$\begin{aligned} T_x(z) &= \langle x, \mathbf{H}^D_z x \rangle \\ &= \int_{t_1}\int_{t_2} x(t_1)\, x^*(t_2)\, h^{D*}_z(t_1, t_2)\, dt_1 dt_2 \end{aligned} \tag{2}$$

with $\mathbf{H}^D_z = \mathbf{D}_{[l^{-1}(z)]^T}\, \mathbf{H}\, \mathbf{D}^{-1}_{[l^{-1}(z)]^T}$, i.e. $h^D_z(t_1, t_2) = \int_{t'_1}\int_{t'_2} D_{[l^{-1}(z)]^T}(t_1, t'_1)\ h(t'_1, t'_2)\ D^{-1}_{[l^{-1}(z)]^T}(t'_2, t_2)\ dt'_1 dt'_2$. Here, $\mathbf{H}$ is an arbitrary linear operator with kernel $h(t_1, t_2)$, assumed independent of $x(t)$, and $D_\theta(t_1, t_2)$ and $D^{-1}_\theta(t_1, t_2)$ are the kernels of $\mathbf{D}_\theta$ and $\mathbf{D}^{-1}_\theta$, respectively.

For given operator $\mathbf{D}_\theta$, (2) defines a class of QTFR's parameterized by the 2-D kernel $h(t_1, t_2)$ of the operator $\mathbf{H}$. This class consists of *all* QTFR's satisfying the covariance (1). For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, (1) becomes the TF shift covariance $T_{\mathbf{S}_{\tau,\nu} x}(t, f) = T_x(t - \tau, f - \nu)$, and (2) becomes Cohen's class where $h^D_z(t_1, t_2) = h^S_z(t_1, t_2) = h(t_1 - t, t_2 - t)\, e^{j2\pi f(t_1 - t_2)}$.

[1] In certain cases, this assumption holds if one uses the group delay of $u^A_{\tilde\alpha}(t)$ and the instantaneous frequency of $u^B_{\tilde\beta}(t)$; here, an analogous theory can be formulated.

[2] z is the intersection of $u^A_{\tilde\alpha}(t)$ and $u^B_{\tilde\beta}(t)$ in the sense that $u^A_{\tilde\alpha}(t)$ and $u^B_{\tilde\beta}(t)$ are concentrated, in the TF plane, along $\nu^A_{\tilde\alpha}(t)$ and $\tau^B_{\tilde\beta}(f)$, respectively, and z is the intersection of $\nu^A_{\tilde\alpha}(t)$ and $\tau^B_{\tilde\beta}(f)$.

Besides the covariance property (1), the *marginal properties* [4], [8], [17]

$$\begin{aligned} &\int_{\mathcal{G}} T_x\big(l(\tilde\theta)\big)\, d\mu(\tilde\beta) = |X_A(\tilde\alpha)|^2, \\ &\int_{\mathcal{G}} T_x\big(l(\tilde\theta)\big)\, d\mu(\tilde\alpha) = |X_B(\tilde\beta)|^2 \end{aligned} \tag{3}$$

are of importance. A class of QTFR's satisfying (3) is

$$\bar{T}_x(z) = \iint_{\mathcal{G}^2} \Psi(\theta)\, A^D_x(\theta)\, \Lambda\big(l^{-1}(z), \theta\big)\, d\mu^2(\theta) \tag{4}$$

where $\Lambda(\tilde\theta, \theta) \triangleq \lambda_{\alpha,\tilde\alpha}\, \lambda^*_{\beta,\tilde\beta}$, $A^D_x(\theta) \triangleq \langle \mathbf{D}_{\theta^{-1/2}}\, x, \mathbf{D}_{\theta^{1/2}}\, x \rangle = \lambda^{-1/2}_{\alpha,\beta} \langle x, \mathbf{D}_\theta\, x \rangle$ (the "characteristic function"[3]), $d\mu^2(\theta) \triangleq d\mu(\alpha)\, d\mu(\beta)$, and $\Psi(\theta) = \Psi(\alpha, \beta)$ is a kernel (assumed independent of $x(t)$) satisfying $\Psi(\alpha, g_0) = \Psi(g_0, \beta) = 1$ [4], [8], [17]. In the case of the conjugate operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, the marginal properties (3) become $\int_t T_x(t, f)\, dt = |X(f)|^2$ and $\int_f T_x(t, f)\, df = |x(t)|^2$, $A^D_x(\theta) = A^S_x(\tau, \nu)$ becomes the symmetric ambiguity function [3], and the QTFR class (4) becomes Cohen's class.

So far, we have formulated the QTFR class $\mathcal{T} = \{T_x(z)\}$ in (2) comprising all QTFR's satisfying the covariance property (1), and the QTFR class $\bar{\mathcal{T}} = \{\bar{T}_x(z)\}$ in (4) related to the marginal properties (3). These classes are equivalent in the conjugate case [4], [5]:

Theorem 2. For conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$, there is $\mathcal{T} = \bar{\mathcal{T}}$ or equivalently $T_x(z) \equiv \bar{T}_x(z)$ where the kernel $h(t_1, t_2)$ of $T_x(z)$ and the kernel $\Psi(\theta)$ of $\bar{T}_x(z)$ are related as $h(t_1, t_2) = \iint_{\mathcal{G}^2} \Psi^*(\theta)\, D_\theta(t_1, t_2)\, \lambda^{1/2}_{\alpha,\beta}\, d\mu^2(\theta)$.

Hence, in the conjugate case considered, the "covariance approach" and the "characteristic function approach" to the construction of QTFR classes are fully equivalent.

With $\Psi(\theta) \equiv 1$, the "central member" $W^D_x(z) \triangleq \iint_{\mathcal{G}^2} A^D_x(\theta)\ \Lambda\big(l^{-1}(z), \theta\big)\ d\mu^2(\theta)$ of the QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ is obtained [5], [18]. It can be expressed as

$$\begin{aligned} W^D_x(z) &= \int_{\mathcal{G}} X_A(\tilde\alpha \bullet \beta^{1/2})\, X^*_A(\tilde\alpha \bullet \beta^{-1/2})\, \lambda^*_{\beta,\tilde\beta}\, d\mu(\beta) \\ &= \int_{\mathcal{G}} X_B(\tilde\beta \bullet \alpha^{1/2})\, X^*_B(\tilde\beta \bullet \alpha^{-1/2})\, \lambda_{\alpha,\tilde\alpha}\, d\mu(\alpha) \end{aligned}$$

where$(\tilde\alpha, \tilde\beta) = l^{-1}(z)$. Any QTFR $T_x(z)$ of $\mathcal{T} = \bar{\mathcal{T}}$ can be derived from $W^D_x(z)$ as

$$T_x(z) = \iint_{\mathcal{G}^2} W^D_x\big(l(\tilde\theta)\big)\ \psi\Big(l^{-1}(z) \circ \tilde\theta^{-1}\Big)\ d\mu^2(\tilde\theta)$$

where $\psi(\tilde\theta) = \iint_{\mathcal{G}^2} \Psi(\theta)\, \Lambda(\tilde\theta, \theta)\, d\mu^2(\theta)$ [5]. In the special cases of Cohen's class and the hyperbolic class, the central member becomes the Wigner distribution and the Q-distribution, respectively [3], [12].

[3] We note that $\theta^{1/2}$ is defined by $\theta^{1/2} \circ \theta^{1/2} = \theta$, and that $\lambda^{-1/2}_{\alpha,\beta} = (e^{\pm j2\pi\, \mu(\alpha)\, \mu(\beta)})^{-1/2} = e^{\mp j\pi\, \mu(\alpha)\, \mu(\beta)}$.

IV. Transformation of Operators and QTFR Classes

The QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ can be constructed using a transformation approach, a fact linking our theory to the "warping" theory in [10], [16]. Let $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ be conjugate operators on a signal space $\mathcal{X}$, with group $(\mathcal{G}, \bullet)$, and consider the operators $\mathbf{C}_\gamma \triangleq \mathbf{V}\,\mathbf{A}_{s(\gamma)}\mathbf{V}^{-1}$ and $\mathbf{D}_\delta \triangleq \mathbf{V}\,\mathbf{B}_{s(\delta)}\mathbf{V}^{-1}$. Here, $\mathbf{V}$ is an isometric isomorphism mapping $\mathcal{X}$ onto some other space $\mathcal{Y}$, and $s(\cdot)$ is a one-to-one function mapping some commutative group $(\mathcal{H}, *)$ onto $(\mathcal{G}, \bullet)$, such that $s(h_1 * h_2) = s(h_1) \bullet s(h_2)$ for all $h_1, h_2 \in \mathcal{H}$. Assuming suitable choice of the dual parameters $\tilde{\gamma}$ and $\tilde{\delta}$, the eigenvalues/functions of $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are $\lambda^C_{\gamma,\tilde{\gamma}} = \lambda^A_{s(\gamma),s(\tilde{\gamma})}$, $u^C_{\tilde{\gamma}}(t) = \left(\mathbf{V}u^A_{s(\tilde{\gamma})}\right)(t)$ and $\lambda^D_{\delta,\tilde{\delta}} = \lambda^B_{s(\delta),s(\tilde{\delta})}$, $u^D_{\tilde{\delta}}(t) = \left(\mathbf{V}u^B_{s(\tilde{\delta})}\right)(t)$, respectively, and $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are *conjugate* operators on $\mathcal{Y}$, with group $(\mathcal{H}, *)$. Thus, isometric isomorphisms $\mathbf{V}$ and one-to-one group transformations $s(\cdot)$ preserve the conjugateness property of two operators. The following theorem [5] states that *any* QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ corresponding to conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from Cohen's class using a transformation. Similar results have been derived independently in [6], [7].

Theorem 3: Let $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ be conjugate with group $(\mathcal{G}, \bullet)$ corresponding to function $\mu(\cdot)$, so that $\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{-j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ ($-$ sign), then $\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{T}_{t_r\mu(\alpha)}\mathbf{V}^{-1}$ and $\mathbf{B}_\beta = \mathbf{V}\,\mathbf{F}_{\mu(\beta)/t_r}\mathbf{V}^{-1}$, where $t_r > 0$ is an arbitrary reference time constant, and $(\mathbf{V}_x^{-1})(t) = \frac{1}{\sqrt{t_r}}X_B(\mu^{-1}(\frac{t}{t_r}))$ with $\mu^{-1}(\cdot)$ denoting the function inverse to $\mu(\cdot)$. Furthermore, any QTFR $T_x(z) = T_x(t,f)$ of the QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ associated to $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from a corresponding QTFR $C_x(t,f)$ of Cohen's class as

$$T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\beta}), \frac{\mu(\tilde{\alpha})}{t_r}\right)\Bigg|_{\tilde{\theta}=l^{-1}(z)}$$

where $l^{-1}(\cdot)$ is the inverse LF of $\mathbf{D}_\theta = \mathbf{B}_\beta\mathbf{A}_\alpha$. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ (+ sign), then the above relations have to be replaced by $\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{F}_{\mu(\alpha)/t_r}\mathbf{V}^{-1}$ and $\mathbf{B}_\beta = \mathbf{V}\,\mathbf{T}_{t_r\mu(\beta)}\mathbf{V}^{-1}$, $(\mathbf{V}_x^{-1})(t) = \frac{1}{\sqrt{t_r}}X_A(\mu^{-1}(\frac{t}{t_r}))$, and $T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\alpha}), \frac{\mu(\tilde{\beta})}{t_r}\right)\Big|_{\tilde{\theta}=l^{-1}(z)}$.

References

[1] L. Cohen, *Time-Frequency Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[2] P. Flandrin, *Temps-fréquence.* Paris: Hermès, 1993.

[3] F. Hlawatsch, "Duality and classification of bilinear time-frequency signal representations," *IEEE Trans. Signal Processing*, vol. 39, no. 7, pp. 1564–1574, July 1991.

[4] F. Hlawatsch and H. Bölcskei, "Displacement-covariant time-frequency energy distributions," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 2, pp. 1025–1028.

[5] ——, "Time-frequency distributions based on conjugate operators," in *Proc. IEEE UK Sympos. Applications of Time-Frequency and Time-Scale Methods*, Univ. of Warwick, Coventry, UK, Aug. 1995, pp. 187–193a.

[6] A. M. Sayeed and D. L. Jones, "On the equivalence of generalized joint signal representations," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 3, pp. 1533–1536.

[7] ——, "Integral transforms covariant to unitary operators and their implications for joint signal representations," submitted to *IEEE Trans. Signal Processing*, Sept. 1994.

[8] R. G. Baraniuk, "Beyond time-frequency analysis: Energy densities in one and many dimensions," *Proc. IEEE ICASSP-94*, Adelaide, Australia, Apr. 1994, vol. 3, pp. 357–360.

[9] W. Rudin, *Fourier Analysis on Groups.* New York: Wiley, 1967.

[10] R. G. Baraniuk and D. L. Jones, "Unitary equivalence: A new twist on signal processing," *IEEE Trans. Signal Processing*, vol. 43, no. 10, pp. 2269–2282, Oct. 1995.

[11] F. Hlawatsch and H. Bölcskei, "Unified theory of displacement-covariant time-frequency analysis," in *Proc. IEEE Int. Sympos. Time-Frequency Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 524–527.

[12] A. Papandreou, F. Hlawatsch, and G. F. Boudreaux-Bartels, "The hyperbolic class of quadratic time-frequency representations, Part I," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3425–3444, Dec. 1993.

[13] J. Bertrand and P. Bertrand, "Affine time-frequency distributions," in *Time-Frequency Signal Analysis—Methods and Applications*, ed. B. Boashash, Longman-Cheshire, Melbourne, 1992, pp. 118–140.

[14] O. Rioul and P. Flandrin, "Time-scale energy distributions: A general class extending wavelet transforms," *IEEE Trans. Signal Processing*, vol. 40, no. 7, pp. 1746–1757, July 1992.

[15] A. Papandreou, F. Hlawatsch, and G. F. Boudreaux-Bartels, "A unified framework for the scale covariant affine, hyperbolic, and power class quadratic time-frequency representations using generalized time shifts," in *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 2, pp. 1017–1020.

[16] R. G. Baraniuk, "Warped perspectives in time-frequency analysis," in *Proc. IEEE Int. Sympos. Time-Frequency Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 528–531.

[17] L. Cohen, "The scale representation," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3275–3292, Dec. 1993.

[18] F. Hlawatsch, T. Twaroch, and H. Bölcskei, "Wigner-type a-b and time-frequency analysis based on conjugate operators," to appear in *Proc. ICASSP-96*, Atlanta, GA, May 1996.