Covariant Time-Frequency Distributions
Based on Conjugate Operators
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Abstract—We propose classes of quadratic time-frequency distributions that retain the inner structure of Cohen's class. Each of these classes is based on a pair of “conjugate” unitary operators producing time-frequency displacements. The classes satisfy covariance and marginal properties corresponding to these operators. For each class, we define a “central member” generalizing the Wigner distribution and the Q-distribution, and we specify a transformation by which the class can be derived from Cohen’s class.

I. INTRODUCTION

COHEN’S class with signal-independent kernels (Cohen’s class hereafter) consists of all quadratic time-frequency representations (QTFR’s) $T_{\alpha}(t, f)$ that are covariant to time-frequency shifts $T_{\alpha}(x, f) = T_{\alpha}(t - \tau, f - \nu)$ [1]-[3]. Here, $x(t)$ is a signal with Fourier transform $X(f) = \int_{\mathbb{R}} x(t) e^{-j2\pi ft} dt$, and $S_{\alpha,\nu} = F_x T_{\alpha}$ is the time-shift operator $(T_{\alpha} x)(t) = x(t - \tau)$ and the frequency-shift operator $(F_x x)(t) = x(t e^{j2\pi ft})$. The properties of the operators $T_{\alpha}$ and $F_x$ entail a characteristic structure of Cohen’s class. In this letter, this structure will be worked out in a generalized framework. We construct QTFR classes that are based on pairs of “conjugate” operators and that satisfy generalized covariance and marginal properties [4], [5]. Due to space limitations, we reserve our results without providing proofs. The concept of conjugate operators has been developed independently in [6] and [7].

II. CONJUGATE OPERATORS

We consider two operators $A_\alpha$ and $B_\beta$ indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. They are to be unitary on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical composition laws $A_{\alpha_2} A_{\alpha_1} = A_{\alpha_1 \circ \alpha_2}$ and $B_{\beta_2} B_{\beta_1} = B_{\beta_1 \circ \beta_2}$ where $(\mathcal{G}, \circ)$ is a commutative group [4], [8], [9]. The eigenvalues $\lambda_{\alpha,\beta}$ and eigenfunctions $u_{\alpha,\beta}(t)$ of $A_\alpha$ are defined by $(A_\alpha u_{\alpha,\beta}) (t) = \lambda_{\alpha,\beta} u_{\alpha,\beta}(t)$; they are indexed by a “dual parameter” $\alpha$. The A-Fourier transform (A-FT) [8] is defined as $X_{\alpha}(\hat{\alpha}) = (F_{A_\alpha} x)(\hat{\alpha}) \triangleq \int_{\mathbb{R}} x(t) u_{\alpha,\beta}^*(t) dt$. Analogous definitions apply to $\lambda_{\beta}, u_{\beta}(t)$, and $X_{\beta}(\hat{\beta}) = (F_{B_\beta} x)(\hat{\beta})$. We now assume that applying one operator to an eigenfunction of the other operator merely shifts the eigenfunction parameter [4], [5].

Definition 1. Two operators $A_\alpha$ and $B_\beta$ as described above will be called conjugate if $\alpha \in \mathcal{G}$, $\beta \in \mathcal{G}$ and

$$(B_\beta u_{\alpha,\beta}^*(t)) \triangleq \lambda_{\alpha,\beta} \triangleq (A_\alpha u_{\beta}^*(t)) \triangleq \lambda_{\beta}.$$

Two conjugate operators $A_\alpha, B_\beta$ can be shown to satisfy the following remarkable properties [4]:

1) Their eigenvalues can be written as $\lambda_{\alpha,\beta} = e^{\pm j2\pi \phi(\alpha) \phi(\beta)}$ and $\lambda_{\beta} = e^{\pm j2\pi \phi(\beta) \phi(\beta)} = (\lambda_{\beta})^*$. Here, $\phi(\alpha) \in \mathbb{R}$ maps $\mathcal{G}$ onto $\mathbb{R}$.+ in the sense that $\phi(\alpha_1 \cdot \alpha_2) = \phi(\alpha_1) + \phi(\alpha_2)$, $\phi(\alpha_0) = 0$, and $\phi(\beta_1) = -\phi(\beta)$ where $\alpha_0$ is the identity element in $\mathcal{G}$ and $\beta_1$ denotes the group-inverse of $\beta$. In the following, we shall simply write $\lambda_{\alpha,\beta} = \lambda_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}^* = \lambda_{\alpha,\beta}$.

2) They commute up to a phase factor, $A_\alpha B_\beta = \lambda_{\alpha,\beta} B_\beta A_\alpha$.

3) Their eigenfunctions are related as $\langle u_{\alpha,\beta}^*, u_{\alpha,\beta}^* \rangle = \lambda_{\alpha,\beta} \langle u_{\alpha,\beta}^*, u_{\alpha,\beta}^* \rangle = \lambda_{\alpha,\beta} \langle u_{\alpha,\beta}^*, u_{\alpha,\beta}^* \rangle = \lambda_{\alpha,\beta} \langle u_{\alpha,\beta}^*, u_{\alpha,\beta}^* \rangle$.

4) The inner product of their kernels is $\int_{\mathbb{R}} A_\alpha(t, t') B_{\beta}(t', t) dt dt' = \delta(\phi(\alpha)) \delta(\phi(\beta))$ where $\delta(\cdot)$ denotes the Dirac delta function (cf. [10]).

5) The A-FT and B-FT satisfy $F_{A_\alpha B_\beta} x(\hat{\alpha}) = (F_{A_\alpha} x)(\hat{\alpha} \cdot \beta^{-1})$ and $F_{B_\beta A_\alpha} x(\hat{\beta}) = (F_{B_\beta} x)(\hat{\beta} \cdot \alpha^{-1})$, and they are related as $X_{\alpha}(\hat{\beta}) = \int_{\mathbb{R}} X_{\alpha}(\hat{\alpha}) \lambda_{\alpha,\beta} \delta(\phi(\beta))$.

We now compose two conjugate operators $A_\alpha, B_\beta$ as $D_\theta = B_\beta A_\alpha$, where $\theta = (\alpha, \beta) \in \mathbb{R}^2$ with $\mathbb{R}^2 \subseteq \mathbb{R} \times \mathbb{R}$. It is readily shown that $D_\theta$ is unitary on $\mathcal{X}$ and satisfies the composition property [4], [11]. $D_{\theta_2} D_{\theta_1} = \lambda_{\alpha_2,\alpha_1} D_{\beta_1,\beta_2}$, where $(\mathbb{R}^2, \circ)$ is the commutative group with group operation $\theta_1 \circ \theta_2 = (\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 \cdot \alpha_2, \beta_1 \cdot \beta_2)$, identity element $\theta_0 = (g_0, g_0)$, and invariable elements $\theta^{-1} = (\alpha^{-1}, \beta^{-1})$. Furthermore, $D_{\theta^{-1}} = \lambda_{\alpha,\beta} D_{\theta^{-1}}$ and $D_{\theta_0} = e$ where $\lambda_{\alpha,\beta}$ is the identity operator on $\mathcal{X}$.

Examples. The shift operators $T_{\alpha}$, $F_{\beta}$ underlying Cohen’s class are conjugate with $(\mathcal{G}, \bullet) = (\mathbb{R}, +)$ and $u_{\alpha,\beta}(t) = u_{\alpha,\beta}(t)$, eigenvalues $\lambda_{\alpha,\beta} = e^{\pm j2\pi \phi(\alpha) \phi(\beta)}$, $\lambda_{\alpha,\beta} = e^{\pm j2\pi \phi(\beta) \phi(\beta)}$, eigenfunctions $u_{\alpha,\beta}^*(t) = e^{\pm j2\pi \phi(\alpha) \phi(\beta)}$, and dual parameters $\hat{\alpha} = f$, $\hat{\beta} = f$. The operators are conjugate since $(F_{\beta} u_{\alpha,\beta}^*(t)) = u_{\alpha,\beta}^*(t)$ and $(T_{\alpha} u_{\alpha,\beta}^*(t)) = u_{\alpha,\beta}^*(t)$. The operators underlying the hyperbolic QTFR’s class [12] are conjugate as well, but the operators underlying the affine class and the power class [13],[15] are not conjugate.
III. Covariance and Marginal Properties

Let \( \nu^A(t) \) and \( \tau^B(f) \) denote the instantaneous frequency and group delay of the eigenfunctions \( u^A(t) \) and \( u^B(t) \), respectively. For any \( \tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}) \in \mathbb{R}^2 \), the corresponding functions \( \nu^{A}_{\tilde{\theta}}(t) \) and \( \tau^{B}_{\tilde{\theta}}(f) \) are assumed to be invertible and to intersect in a unique time-frequency (TF) point \( z = (t, f) \). Hence, \( z = \tilde{l}(\tilde{\theta}) \) where \( \tilde{l}(\tilde{\theta}) \) will be called the localization function (LF) of the operator \( D_{\theta} \) [4], [5]. The LF is constructed by solving the system of equations \( \nu^{A}_{\tilde{\theta}}(t) = f \), \( \tau^{B}_{\tilde{\theta}}(f) = t \) for \( (t, f) = z \) [4], [10], [16]. It is assumed to be invertible, i.e., \( z = \tilde{l}(\tilde{\theta}) \) \( \Leftrightarrow \tilde{\theta} = l^{-1}(z) \).

The LF describes the TF displacements caused by \( D_{\theta} \). If a signal \( x(t) \) is localized about a TF point \( z = (t, f) \), then \( (D_{\theta}x)(t) \) will be localized about a new TF point \( z' = (t', f') \). Since \( z \) is the intersection of \( u^A(t) \) and \( u^B(t) \) with \( (\tilde{\alpha}, \tilde{\beta}) = \tilde{\theta} = l^{-1}(z) \), \( z' \) will be the intersection of \( (D_{\theta}u^A)(t) \) and \( (D_{\theta}u^B)(t) \). Due to the conjugateness of \( A_{\alpha}, B_{\beta}, (D_{\theta}u^A)(t) = \lambda^{*}_{\alpha,\beta}u^A(t) \) and \( (D_{\theta}u^B)(t) = \lambda^{*}_{\beta,\alpha}u^B(t) \).

Hence, \( z' = l((\tilde{\alpha} \cdot \tilde{\beta}, \tilde{\beta} \cdot \alpha) = l(\theta \cdot \theta^T) = l(l^{-1}(z) \cdot \theta^T) \) with \( \theta^T = (\beta^{-1}, \alpha^{-1}) \). This motivates the following definition [4], [5]:

**Definition 2.** A QTFR \( T_x(z) = T_x(t, f) \) is called covariant to \( D_{\theta} \) if

\[
T_{D_{\theta}x}(z) = T_x(l(l^{-1}(z) \cdot \theta^{-T}))
\]

(1)

with \( \theta^{-T} = (\theta^{-1})^T = (\beta^{-1}, \alpha^{-1}) \).

The class of all QTFR’s covariant to \( D_{\theta} \) is characterized as follows (cf. [4], [11]):

**Theorem 1.** A QTFR \( T_x(z) = T_x(t, f) \) is covariant to an operator \( D_{\theta} \) if and only if

\[
T_x(z) = \langle x, \mathbf{H}_x^D \rangle \\
= \int_{t_1} \int_{t_2} x(t_1) x^*(t_2) h^D_x(t_1, t_2) dt_1 dt_2
\]

(2)

with \( \mathbf{H}_x^D = D_{\theta}^{-1}(z_0) \mathbf{H}_{\mathbf{x}^*} D_{\theta}(z_0)^T \), i.e., \( h^D_x(t_1, t_2) = \int_{t_1} \int_{t_2} D_{\theta}(z_0)^T(t_1, t_1') h(t_1', t_2') D_{\theta}(z_0)(t_2', t_2) dt_1' dt_2' \). Here, \( \mathbf{H} \) is an arbitrary linear operator with kernel \( h(t_1, t_2) \), assumed independent of \( x(t) \), and \( D_{\theta}(t_2, t_1) \) and \( D_{\theta}^{-1}(t_1, t_2) \) are the kernels of \( D_{\theta} \) and \( D_{\theta}^{-1} \), respectively.

For given operator \( D_{\theta} \), (2) defines a class of QTFR’s parameterized by the 2-D kernel \( h(t_1, t_2) \) of the operator \( H \). This class consists of all QTFR’s satisfying the covariance (1).

For \( D_{\theta} = S_{\tau}, T_{\tau} \), (1) becomes the TF shift covariance \( T_{S_{\tau}}(s, f) = T_{\tau}(s - \tau, f - \varphi) \) and (2) becomes Cohen’s class where \( h^D_x(t_1, t_2) = h^D_x(t_1, t_2) = h(t_1 - t, t_2 - t) e^{j2\pi f(t_2 - t)} \).

1In certain cases, this assumption holds if one uses the group delay of \( u^A(t) \) and the instantaneous frequency of \( u^B(t) \); here, an analogous theory can be formulated.

2\( z \) is the intersection of \( u^A(t) \) and \( u^B(t) \) in the sense that \( u^A(t) \) and \( u^B(t) \) are concentrated in the TF plane, along \( \nu^A(t) \) and \( \tau^B(f) \), respectively, and \( z \) is the intersection of \( \nu^{A}_{\tilde{\theta}}(t) \) and \( \tau^{B}_{\tilde{\theta}}(f) \).

Besides the covariance property (1), the marginal properties [4], [8], [17]

\[
\int_{\mathbb{R}^2} T_x(l(\tilde{\theta})) d\mu(\tilde{\beta}) = |X_{A}(\tilde{\alpha})|^2
\]

\[
\int_{\mathbb{R}^2} T_x(l(\tilde{\theta})) d\mu(\tilde{\alpha}) = |X_{B}(\tilde{\beta})|^2
\]

(3)

are of importance. A class of QTFR’s satisfying (3) is

\[
T_x(z) = \int_{\mathbb{R}^2} \Psi(\theta) A^D_{\theta}(\theta) \Lambda(l^{-1}(z), \theta) d\mu^2(\theta)
\]

(4)

where \( \Lambda(\tilde{\alpha}, \tilde{\beta}) = \lambda_{\alpha,\beta} \lambda_{\beta,\alpha}^* A_{\alpha}^D(\theta) = \lambda_{\alpha,\beta}^{-1/2} (x, D_{\theta} x) \) (the “characteristic function”), \( d\mu^2(\theta) = d\mu(\alpha) d\mu(\beta) \), and \( \Psi(\theta) = \Psi(\alpha, \beta) \) is a kernel (assumed independent of \( x(t) \)) satisfying \( \Psi(\alpha, g_0) = \Psi(g_0, \beta) = 1 \) [4], [8], [17]. In the case of the conjugate operators \( T_{\tau}, F_{\nu} \), the marginal properties (3) become \( T_{S_{\tau}}(s, f) d\mu = |X_{\tau}(f)|^2 \) and \( \int_{\mathbb{R}^2} T_{S_{\tau}}(s, f) df = |X_{\tau}(f)|^2 \) [5], [18]. It can be expressed as

\[
W^D_x(z) = \int_{\mathbb{R}^2} X_{\tau}(\tilde{\alpha} \cdot \beta, \beta) X_{\tau}^* (\tilde{\alpha} \cdot \beta^{-1/2}) \lambda_{\beta, \beta}^* d\mu(\beta)
\]

\[
= \int_{\mathbb{R}^2} X_{\tau}(\tilde{\beta} \cdot \alpha, \alpha) X_{\tau}^* (\tilde{\beta} \cdot \alpha^{-1/2}) \lambda_{\alpha, \alpha} d\mu(\alpha)
\]

where \( (\tilde{\alpha}, \tilde{\beta}) = l^{-1}(z) \). Any QTFR \( T_x(z) \) of \( T = T_{\tau} \) can be derived from \( W^D_x(z) \) as

\[
T_x(z) = \int_{\mathbb{R}^2} W^D_x(l(\tilde{\theta})) \psi(l^{-1}(z) \cdot \theta^{-1}) d\mu^2(\tilde{\theta})
\]

where \( \psi(\tilde{\theta}) = \int_{\mathbb{R}^2} \Psi(\theta) \Lambda(\tilde{\theta}, \theta) d\mu^2(\theta) \) [5]. In the special cases of Cohen’s class and the hyperbolic class, the central member becomes the Wigner distribution and the Q-distribution, respectively [3], [12].

3We note that \( \theta^{1/2} \) is defined by \( \theta^{1/2} = g^{1/2} \circ g^{-1/2} = \theta, \) and that \( \lambda_{\alpha,\beta}^{1/2} = (\cos \frac{\pi}{2} \mu(\alpha) \mu(\beta) - \sin \frac{\pi}{2} \mu(\alpha) \mu(\beta)) e^{j\pi \frac{1}{2} \mu(\alpha) \mu(\beta)} \).
IV. TRANSFORMATION OF OPERATORS AND QTFR CLASSES

The QTFR class \( T = \hat{T} \) can be constructed using a transformation approach, a fact linking our theory to the “warping” theory in \([10, 16]\). Let \( A_\alpha \) and \( B_\beta \) be conjugate operators on a signal space \( \mathcal{X} \), with group \((G, \bullet)\), and consider the operators \( C_\gamma \triangleq V A_\gamma V^{-1} \) and \( D_\delta \triangleq V B_\delta V^{-1} \). Here, \( V \) is an isometric isomorphism mapping the operators \( C, \) and \( D \), onto some other space \( \mathcal{Y} \), and \( s(\cdot) \) is a one-to-one function mapping some commutative group \((\mathcal{H}, \ast)\) onto \((G, \bullet)\), such that \( s(h_1 \ast h_2) = s(h_1) \bullet s(h_2) \) for all \( h_1, h_2 \in \mathcal{H} \). Assuming suitable choice of the dual parameters \( \hat{\gamma} \) and \( \hat{\delta} \), the eigenvalues/functions of \( C_\gamma \) and \( D_\delta \) are \( \lambda_{\gamma, \hat{\gamma}} = \lambda_{\gamma(\hat{\gamma})} \) and \( u_{\gamma, \hat{\gamma}}(t) = \left( V u_{\gamma(\hat{\gamma})}^A(t) \right)^{\hat{\gamma}} \), respectively, and \( C_\gamma \) and \( D_\delta \) are conjugate operators on \( \mathcal{Y} \), with group \((\mathcal{H}, \ast)\). Thus, isometric isomorphisms \( V \) and one-to-one group transformations \( s(\cdot) \) preserve the conjugateness property of two operators. The following theorem \([5]\) states that any QTFR class \( T = \hat{T} \) corresponding to conjugate operators \( A_\alpha, B_\beta \) can be derived from Cohen’s class using a transformation. Similar results have been derived independently in \([6, 7]\).

Theorem 3: Let \( A_\alpha \), \( B_\beta \) be conjugate with group \((G, \bullet)\) corresponding to function \( \mu(\cdot) \), so that \( \lambda_{\alpha, \hat{\alpha}} = e^{-j2\pi \varphi(\mu(\cdot))} \). If \( \lambda_{\alpha, \hat{\alpha}} = e^{-j2\pi \varphi(\mu(\cdot))} \) (sign), then \( A_\alpha = V T_{\varphi(\mu(\cdot))} V^{-1} \) and \( B_\beta = V T_{\varphi(\mu(\cdot))} V^{-1} \), where \( \varphi(\cdot) > 0 \) is an arbitrary reference time constant, and \( \left( T_{\varphi(\cdot)} \right)(t) = \frac{1}{\sqrt{\varphi}} X_B(\mu(\varphi^{-1}(\frac{t}{\varphi}))) \) with \( \mu^{-1}(\cdot) \) denoting the function inverse to \( \mu(\cdot) \). Furthermore, any QTFR \( T_{\varphi}(x) = T_{\varphi}(t, f) \) of the QTFR class \( T = \hat{T} \) associated to \( A_\alpha, B_\beta \) can be derived from a corresponding QTFR \( C_{\varphi}(t, f) \) of Cohen’s class as

\[
T_{\varphi}(x) = C_{\varphi^{-1}} \left( \frac{f}{\sqrt{\varphi}}, \frac{\mu(\varphi^{-1}(\frac{x}{\varphi}))}{\sqrt{\varphi}} \right) \bigg|_{\varphi=\varphi^{-1}(x)}
\]

where \( \varphi^{-1}(\cdot) \) is the inverse LF of \( D_\delta = B_\beta A_\alpha \). The above relations have to be replaced by \( A_\alpha = V F_{\varphi(\mu(\cdot))} V^{-1} \) and \( B_\beta = V F_{\varphi(\mu(\cdot))} V^{-1} \), \( \left( T_{\varphi^{-1}(\cdot)} \right)(t) = \sqrt{\varphi} X_A(\mu^{-1}(\frac{t}{\varphi})) \), and

\[
T_{\varphi}(x) = C_{\varphi^{-1}} \left( \frac{f}{\sqrt{\varphi}}, \frac{\mu(\varphi^{-1}(\frac{x}{\varphi}))}{\sqrt{\varphi}} \right) \bigg|_{\varphi=\varphi^{-1}(x)}
\]

REFERENCES