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COVARIANT (α, β), TIME-FREQUENCY, AND (a, b) REPRESENTATIONS*

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ABSTRACT—We present a general theory of bilinear/quadratic signal representations that are *covariant* to time-frequency (TF) displacement operators $\mathbf{D}_{\alpha,\beta}$. Covariant representations are developed in the (α, β) domain and in the TF domain. In the *conjugate case*, covariant representations can also be formulated in an eigenfunction parameter domain ((a, b) domain). The relations connecting the (α, β), TF, and (a, b) domains are epitomized by the *displacement function* and the *localization function*. For conjugate operators, (α, β) covariance and (a, b) covariance are equivalent.

1 INTRODUCTION AND OUTLINE

Important classes of bilinear/quadratic time-frequency representations (BTFRs) [1]-[9] can be defined by *covariance properties* with respect to families of unitary operators causing time-frequency (TF) displacements. This fact has motivated the development of a *covariance theory of TF analysis* [10]-[13], which is based on the concept of "TF displacement operators." In addition to providing a unified framework for important existing BTFR classes, the covariance theory also allows the systematic construction of new BTFR classes.

This paper presents an extended view of the covariance principle. After reviewing TF displacement operators (DOs) $\mathbf{D}_{\alpha,\beta}$, Section 2 characterizes the class of all bilinear "(α, β) representations" covariant to a given DO. Section 3 defines the *displacement function* of a DO and shows how it can be used to convert covariant (α, β) representations into covariant BTFRs. For the important "conjugate" case, Section 4 develops covariant "(a, b) representations" (where a and b are eigenfunction parameters), shows how these can be used for deriving covariant BTFRs, and argues that these BTFRs are identical to the covariant BTFRs from Section 3.

2 COVARIANT (α, β) REPRESENTATIONS

TF Displacement Operators. We consider a family of linear operators $\mathbf{D}_{\alpha,\beta} = \mathbf{D}_\theta$ defined on a linear space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, where $\mathcal{L}_2(\mathbb{R})$ is the space of finite-energy signals. The operator family $\{\mathbf{D}_\theta\}$ is indexed by the 2-D "displacement parameter" $\theta = (\alpha, \beta) \in \mathcal{D}$ with $\mathcal{D} \subseteq \mathbb{R}^2$. We assume that there exists an operation $\circ$ such that $(\mathcal{D}, \circ)$ forms a (possibly non-commutative) *group* [14]-[16] with identity element $\theta_0 \in \mathcal{D}$. We shall call $\{\mathbf{D}_\theta\}_{\theta\in\mathcal{D}}$ a *TF displacement operator* (DO) if $\mathbf{D}_\theta$ is *unitary* [17] for all $\theta \in \mathcal{D}$, and if it satisfies the *composition property* [10]-[13]

$$\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = e^{j\psi(\theta_1,\theta_2)}\,\mathbf{D}_{\theta_1\circ\theta_2} \tag{1}$$

where $\psi(\theta, \theta_0) = \psi(\theta_0, \theta) = 0$ *modulo* 2π. Thus, a displacement by θ_1 followed by a displacement by θ_2 is equivalent, up to a constant phase factor, to a displacement by $\theta_1 \circ \theta_2$. It follows that $\mathbf{D}_{\theta_0} = \mathbf{I}$ (the identity element θ_0 corresponds to *no* displacement) and $\mathbf{D}_\theta^{-1} = e^{-j\psi(\theta^{-1},\theta)}\,\mathbf{D}_{\theta^{-1}}$ (a TF displacement by θ can be undone, up to a constant phase factor, via a displacement by the group-inverse parameter θ^{-1}).

Example. The TF shift operator $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu\mathbf{T}_\tau$, where $(\mathbf{T}_\tau x)(t) = x(t-\tau)$ and $(\mathbf{F}_\nu x)(t) = x(t)\,e^{j2\pi\nu t}$, is a DO with $\theta = (\tau, \nu)$, $\mathcal{D} = \mathbb{R}^2$, $(\tau_1, \nu_1) \circ (\tau_2, \nu_2) = (\tau_1+\tau_2,\ \nu_1+\nu_2)$, $\theta_0 = (0, 0)$, $\theta^{-1} = (-\tau, -\nu)$, and $\psi(\theta_1, \theta_2) = -2\pi\nu_1\tau_2$. Thus, (1) reads $\mathbf{S}_{\tau_2,\nu_2}\mathbf{S}_{\tau_1,\nu_1} = e^{-j2\pi\nu_1\tau_2}\,\mathbf{S}_{\tau_1+\tau_2,\,\nu_1+\nu_2}$.

Covariant Bilinear (α, β) Representations. Although we are primarily interested in covariant BTFRs, we first consider bilinear (α, β) representations (or θ representations).

Definition 1 [13]. A bilinear θ representation $B_{x,y}(\theta)$ will be called *covariant to a DO* $\mathbf{D}_\theta$ if

$$B_{\mathbf{D}_{\tilde{\theta}}x,\mathbf{D}_{\tilde{\theta}}y}(\theta) = B_{x,y}\big(\theta \circ \tilde{\theta}^{-1}\big)\,. \tag{2}$$

Theorem 1 [10]-[13]. A bilinear θ representation is covariant to a DO $\mathbf{D}_\theta$ if and only if it can be written as[1]

$$B_{x,y}(\theta) = \int_{t_1}\int_{t_2} x(t_1)\,y^*(t_2)\,\big(\mathbf{D}_\theta^{\otimes} h\big)^*(t_1, t_2)\,dt_1 dt_2\,, \tag{3}$$

where $h(t_1, t_2)$ is an arbitrary 2-D function independent of $x(t)$, $y(t)$ and $\mathbf{D}_\theta^{\otimes}$ is the outer product of $\mathbf{D}_\theta$ by itself[2].

We see that the class of all bilinear θ representations covariant to a DO is parameterized by a 2-D "kernel" $h(t_1, t_2)$. Eq. (3) can equivalently be expressed as the bilinear form $B_{x,y}(\theta) = \langle x, \mathbf{H}_\theta^D y\rangle = \int_{t_1}\int_{t_2} x(t_1)\,y^*(t_2)\,h_\theta^{D*}(t_1, t_2)\,dt_1 dt_2$ with $\mathbf{H}_\theta^D = \mathbf{D}_\theta \mathbf{H}\,\mathbf{D}_\theta^{-1}$, where $\mathbf{H}$ is the linear operator whose kernel is $h(t_1, t_2)$ in (3).

Example. For the TF shift operator $\mathbf{S}_{\tau,\nu}$, the covariance (2) is $B_{\mathbf{S}_{\tilde{\tau},\tilde{\nu}}x,\mathbf{S}_{\tilde{\tau},\tilde{\nu}}y}(\tau, \nu) = B_{x,y}(\tau - \tilde{\tau}, \nu - \tilde{\nu})$, and, with (3), the class of all covariant bilinear (τ, ν) representations is $B_{x,y}(\tau, \nu) = \int_{t_1}\int_{t_2} x(t_1)\,y^*(t_2)\ h^*(t_1 - \tau, t_2 - \tau)\,e^{-j2\pi\nu(t_1-t_2)}\,dt_1 dt_2$, which is Cohen's class [1]-[4].

3 COVARIANT TF REPRESENTATIONS

We shall now show how to convert covariant bilinear θ representations into covariant BTFRs.

Displacement Function. If $x(\cdot)$ is localized, in the TF plane, about some (fixed) reference TF point $z_0 = (t_0, f_0)$,

[1] Integrals are over the support of the integrand.

[2] $\mathbf{D}_\theta^{\otimes}$ acts on a 2-D function $h(t_1, t_2)$ as $\big(\mathbf{D}_\theta^{\otimes} h\big)(t_1, t_2) = \int_{t_1'}\int_{t_2'} D_\theta(t_1, t_1')\,D_\theta^*(t_2, t_2')\,h(t_1', t_2')\,dt_1' dt_2'$, where $D_\theta(t, t')$ is the kernel of $\mathbf{D}_\theta$, i.e., $(\mathbf{D}_\theta\, x)(t) = \int_{t'} D_\theta(t, t')\,x(t')\,dt'$. For example, $(\mathbf{S}_{\tau,\nu}^{\otimes}\, h)(t_1, t_2) = h(t_1 - \tau, t_2 - \tau)\ e^{j2\pi\nu(t_1-t_2)}$.

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then $(\mathbf{D}_\theta\, x)(\cdot)$ will be localized about some other TF point $z=(t,f)$ that depends on θ, i.e., $z=d(\theta)$ where $d(\cdot)$ is the *displacement function* (DF) of the DO $\mathbf{D}_\theta$ [10]-[13]. In order to construct the DF, we define the family of signals $x_\theta(t) \triangleq D_\theta(t,\hat{t})$ where $D_\theta(t,t')$ is the kernel of $\mathbf{D}_\theta$ and $\hat{t}$ is some *fixed* time. Similarly, we define the signal family $\{y_\theta(t)\}_{\theta\in\mathcal{D}}$ in the frequency domain as $Y_\theta(f) \triangleq \tilde{D}_\theta(f,\hat{f})$ where $\tilde{D}_\theta(f,f') = \int_t \int_{t'} D_\theta(t,t')\, e^{-j2\pi(ft-f't')}\, dt\, dt'$ is the DO's frequency-domain kernel[3] and $\hat{f}$ is some *fixed* frequency. Eq. (1) then implies $\big(\mathbf{D}_{\theta_2} x_{\theta_1}\big)(t) = e^{j\psi(\theta_1,\theta_2)}\, x_{\theta_1\circ\theta_2}(t)$ and $\big(\mathbf{D}_{\theta_2} y_{\theta_1}\big)(t) = e^{j\psi(\theta_1,\theta_2)}\, y_{\theta_1\circ\theta_2}(t)$, i.e., application of $\mathbf{D}_\theta$ to a member of the family $\{x_\theta(t)\}_{\theta\in\mathcal{D}}$ or $\{y_\theta(t)\}_{\theta\in\mathcal{D}}$ yields another member of the respective family (up to a constant phase factor). In the following, we shall use the group delay of $x_\theta(t)$, $\tau_\theta(f) = -\frac{1}{2\pi}\frac{d}{df}\arg\{X_\theta(f)\}$, and the instantaneous frequency of $y_\theta(t)$, $\nu_\theta(t) = \frac{1}{2\pi}\frac{d}{dt}\arg\{y_\theta(t)\}$.

Based on the "invariant signal families" $\{x_\theta(t)\}_{\theta\in\mathcal{D}}$ and $\{y_\theta(t)\}_{\theta\in\mathcal{D}}$, the DF is now constructed as follows (see *Fig. 1*). Let $z_0=(t_0,f_0)$ denote the intersection of the group delay $\tau_{\theta_0}(f)$ of $x_{\theta_0}(t)$ and the instantaneous frequency $\nu_{\theta_0}(t)$ of $y_{\theta_0}(t)$, i.e., there is $\nu_{\theta_0}(t_0)=f_0$ and $\tau_{\theta_0}(f_0)=t_0$. Thus, $z_0=(t_0,f_0)$ is the intersection of the effective TF loci of $x_{\theta_0}(t)$ and $y_{\theta_0}(t)$. We shall consider z_0 as a *fixed* reference TF point[4]. We now apply the DO to $x_{\theta_0}(t)$ and $y_{\theta_0}(t)$ to obtain $\big(\mathbf{D}_\theta\, x_{\theta_0}\big)(t) = e^{j\psi(\theta_0,\theta)}\, x_{\theta_0\circ\theta}(t) = x_\theta(t)$ and $\big(\mathbf{D}_\theta\, y_{\theta_0}\big)(t) = e^{j\psi(\theta_0,\theta)}\, y_{\theta_0\circ\theta}(t) = y_\theta(t)$ (where $\psi(\theta_0,\theta)=0$ has been used), with group delay $\tau_\theta(f)$ and instantaneous frequency $\nu_\theta(t)$, respectively. The intersection $z=(t,f)$ of $\tau_\theta(f)$ and $\nu_\theta(t)$, defined by the system of equations

$$\nu_\theta(t) = f\,, \qquad \tau_\theta(f) = t\,,$$

depends on θ, i.e., $z=d(\theta)$. This defines the DF provided that the above system of equations has a unique solution $z=(t,f)$ for any $\theta\in\mathcal{D}$.

According to its construction above, the DF $z=d(\theta)$ yields the TF point $z=(t,f)$ that is obtained when displacing the initial reference TF point $z_0=(t_0,f_0)$ by the displacement parameter θ (see Fig. 1). Note that $d(\theta_0)=z_0$. Let $\mathcal{Z}\subseteq\mathbb{R}^2$ denote the range of the DF, i.e., the set of all TF points $z=d(\theta)$ with $\theta\in\mathcal{D}$. We assume that the DF is a one-to-one mapping from $\mathcal{D}$ onto $\mathcal{Z}$, i.e., the inverse DF $d^{-1}(\cdot)$ exists.

Extended Displacement Function. The DF also describes the displacement of arbitrary TF points. Indeed, the TF point, denoted $z_2=(t_2,f_2)$, that is obtained by displacing a given TF point $z_1=(t_1,f_1)\in\mathcal{Z}$ by θ is easily seen to be given by $z_2 = d\big(d^{-1}(z_1)\circ\theta\big)$. The function

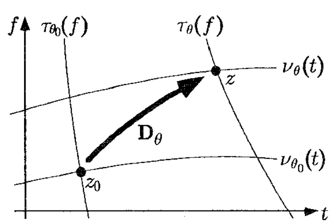

Fig. 1. Construction of the displacement function.

[3]The Fourier transform of $(\mathbf{D}_\theta x)(t)$ is $\int_{f'} \tilde{D}_\theta(f,f')\, X(f')\, df'$ where $X(f)=\int_t x(t)\, e^{-j2\pi ft}\, dt$ is the Fourier transform of $x(t)$.

[4]Note that $z_0=(t_0,f_0)$ depends on $\hat{t}$ and $\hat{f}$ used for the definition of $\{x_\theta(t)\}$ and $\{y_\theta(t)\}$, respectively.

$$e(z,\theta) \triangleq d\big(d^{-1}(z)\circ\theta\big)$$

will be called the *extended displacement function* of the DO $\mathbf{D}_\theta$. It can be shown that $e(z_0,\theta)=d(\theta)$, $e(z,\theta_0)=z$, $e\big(e(z,\theta_1),\theta_2\big) = e(z,\theta_1\circ\theta_2)$, $d^{-1}\big(e(z,\theta)\big) = d^{-1}(z)\circ\theta$, $z'=e(z,\theta) \Leftrightarrow z=e(z',\theta^{-1})$ or equivalently $e\big(e(z,\theta),\theta^{-1}\big)=z$. We assume (or conjecture) that for any $\theta\in\mathcal{D}$, the "TF coordinate transform" $z\to z'=e(z,\theta)$ is a one-to-one, differentiable, and area-preserving mapping from $\mathcal{Z}$ onto $\mathcal{Z}$.

Example. The invariant signals of $\mathbf{S}_{\tau,\nu}$ are obtained as $x_{\tau,\nu}(t)=\delta(t-\tau)\,e^{j2\pi\nu t}$, with group delay $\tau_{\tau,\nu}(f)=\tau$, and $Y_{\tau,\nu}(f)=\delta(f-\nu)$, with instantaneous frequency $\nu_{\tau,\nu}(t)=\nu$ (we have set $\hat{t}=0$ and $\hat{f}=0$). It follows that the DF of $\mathbf{S}_{\tau,\nu}$ is simply $d(\tau,\nu)=(\tau,\nu)$, with range $\mathcal{Z}=\mathbb{R}^2$ (the entire TF plane). The extended DF of $\mathbf{S}_{\tau,\nu}$ is $e(t,f;\tau,\nu)=(t+\tau,f+\nu)$.

Covariant BTFRs. Using the extended DF, we are now able to define a covariance property in the TF domain.

Definition 2 [10]-[13]. A BTFR $B_{x,y}(z)=B_{x,y}(t,f)$ will be called *covariant to a DO* $\mathbf{D}_\theta$ if

$$B_{\mathbf{D}_\theta x,\mathbf{D}_\theta y}(z) = B_{x,y}\big(e(z,\theta^{-1})\big)\,. \tag{4}$$

This is identical to the θ representation covariance (2) up to a mapping $z=d(\theta)$; hence the class of all covariant BTFRs is obtained by remapping the class of covariant bilinear θ representations $B'_{x,y}(\theta)$ in (3) as

$$B_{x,y}(z) = B'_{x,y}(\theta)\big|_{\theta=d^{-1}(z)}\,. \tag{5}$$

This yields the following corollary to Theorem 1:

Corollary 1 [10]-[13]. A BTFR is covariant to a DO $\mathbf{D}_\theta$ if and only if it can be written as

$$B_{x,y}(z) = \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\, \big(\mathbf{D}^{\otimes}_{d^{-1}(z)} h\big)^*(t_1,t_2)\, dt_1 dt_2\,, \tag{6}$$

where $h(t_1,t_2)$ is some 2-D function independent of $x(t)$, $y(t)$.

The class of all BTFRs covariant to a DO is seen to be parameterized by a 2-D kernel $h(t_1,t_2)$. The covariant BTFRs (6) can also be expressed as the bilinear form $B_{x,y}(z) = \big\langle x,\, \mathbf{H}^{D}_{d^{-1}(z)} y\big\rangle$ with $\mathbf{H}^{D}_{\theta} = \mathbf{D}_\theta \mathbf{H}\, \mathbf{D}_\theta^{-1}$, where $\mathbf{H}$ is the linear operator whose kernel is $h(t_1,t_2)$ in (6).

Example. For $\mathbf{S}_{\tau,\nu}$, (4) is the TF shift covariance $B_{\mathbf{S}_{\tau,\nu}x,\mathbf{S}_{\tau,\nu}y}(t,f) = B_{x,y}(t-\tau,f-\nu)$, and the covariant BTFR class (6) is $B_{x,y}(t,f) = \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\ h^*(t_1-t,t_2-t)\, e^{-j2\pi f(t_1-t_2)}\, dt_1 dt_2$, which is Cohen's class.

4 THE CONJUGATE CASE

In the "conjugate" or "dual" case [11], [18]-[22], there exists an alternative approach to the definition of covariant BTFRs.

The Separable Case. Under certain conditions regarding the group operation $\circ$ [11], a DO $\mathbf{D}_\theta = \mathbf{D}_{\alpha,\beta}$ is *separable*,

$$\mathbf{D}_\theta = \mathbf{B}_\beta \mathbf{A}_\alpha\,,$$

where the *partial DOs* (PDOs) $\{\mathbf{A}_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{\mathbf{B}_\beta\}_{\beta\in\mathcal{B}}$ are families of unitary, linear operators indexed by the 1-D displacement parameters $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$, respectively (note that $\mathcal{D}=\mathcal{A}\times\mathcal{B}$). The PDOs satisfy the composition properties $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1\bullet\alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 * \beta_2}$, where $(\mathcal{A},\bullet)$

and $(\mathcal{B},*)$ are commutative groups with identity elements α_0 and β_0, respectively (note that $\theta_0=(\alpha_0,\beta_0)$).

Localization Function. The (generalized) eigenvalues $\lambda^{A}_{\alpha,b}$ and eigenfunctions $u^{A}_{b}(t)$ of $\mathbf{A}_\alpha$ are defined by the eigenequation $(\mathbf{A}_\alpha u^{A}_{b})(t)=\lambda^{A}_{\alpha,b}\,u^{A}_{b}(t)$; they are indexed by a "dual" parameter b which belongs to the commutative *dual group* $(\tilde{\mathcal{A}},\tilde{\bullet})$ of $(\mathcal{A},\bullet)$ [23, 15, 16, 24]. A similar definition applies to the eigenvalues $\lambda^{B}_{\beta,a}$ and eigenfunctions $u^{B}_{a}(t)$ of $\mathbf{B}_\beta$, where a belongs to the dual group $(\tilde{\mathcal{B}},\tilde{*})$ of $(\mathcal{B},*)$.

The eigenfunctions $u^{B}_{a}(t)$ and $u^{A}_{b}(t)$ yield a concept for the definition of BTFRs that is different, in general, from the covariance approach discussed so far: a bilinear (a,b) *representation* (a function of the eigenfunction parameters a and b) is constructed and subsequently mapped into a BTFR. We shall, for the moment, assume that the (a,b) representation is given (see [1], [20]-[23], [25, 26] and further below for constructions of (a,b) representations), and study the subsequent mapping $(a,b)\rightarrow(t,f)$. Let $\tau^{B}_{a}(f)=-\frac{1}{2\pi}\frac{d}{df}\arg\{U^{B}_{a}(f)\}$ and $\nu^{A}_{b}(t)=\frac{1}{2\pi}\frac{d}{dt}\arg\{u^{A}_{b}(t)\}$ denote the group delay of $u^{B}_{a}(t)$ and the instantaneous frequency of $u^{A}_{b}(t)$, respectively. For any $c=(a,b)\in\tilde{\mathcal{B}}\times\tilde{\mathcal{A}}$, the corresponding functions $\tau^{B}_{a}(f)$ and $\nu^{A}_{b}(t)$ are assumed[5] to intersect in a unique TF point $z=(t,f)$. This intersection TF point $z=(t,f)$, defined by $\tau^{B}_{a}(f)=t$ and $\nu^{A}_{b}(t)=f$, depends on $c=(a,b)$, i.e., $z=l(c)$ where $l(\cdot)$ is the *localization function* (LF) of the separable DO $\mathbf{D}_\theta=\mathbf{B}_\beta\mathbf{A}_\alpha$ [11, 18, 19] (a similar construction has previously been used in [27, 28] in the context of TF warpings). According to its construction above, the LF assigns to each eigenfunction parameter pair $c=(a,b)$ the TF point $z=(t,f)$ where the effective TF loci of the eigenfunctions $u^{B}_{a}(\cdot)$ and $u^{A}_{b}(\cdot)$ intersect. We assume that the LF is one-to-one, i.e., the inverse LF $l^{-1}(\cdot)$ exists.

Example. The TF shift operator $\mathbf{S}_{\tau,\nu}=\mathbf{F}_\nu\mathbf{T}_\tau$ is a separable DO. The eigenvalues/functions of $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ are $\lambda^{T}_{\tau,f}=e^{-j2\pi\tau f}$, $u^{T}_{f}(t)=e^{j2\pi ft}$ and $\lambda^{F}_{\nu,t}=e^{j2\pi\nu t}$, $U^{F}_{t}(f)=e^{-j2\pi tf}$, respectively. With the instantaneous frequency of $u^{T}_{f}(t)$, $\nu^{T}_{f}(t)=f$, and the group delay of $U^{F}_{t}(f)$, $\tau^{F}_{t}(f)=t$, the LF of $\mathbf{S}_{\tau,\nu}=\mathbf{F}_\nu\mathbf{T}_\tau$ is obtained as $l(t,f)=(t,f)$.

Using the LF, we can now construct a BTFR by remapping a given bilinear (a,b) representation $B'_{x,y}(c)$ according to

$$B_{x,y}(z)\;=\;B'_{x,y}(c)\big|_{c=l^{-1}(z)}\,. \qquad (7)$$

These BTFRs will be different, in general, from the BTFRs constructed using the covariance approach in Section 3; in particular, they will not be covariant to the DO $\mathbf{D}_\theta$ in general. An important exception will be considered next.

The Conjugate Case. The *conjugate case* is defined by a specific relation between the PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$:

Definition 3 [11, 18, 19]. Two PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are said to be *conjugate* if $(\tilde{\mathcal{A}},\tilde{\bullet})=(\mathcal{B},*)$, $(\tilde{\mathcal{B}},\tilde{*})=(\mathcal{A},\bullet)$, and

$$\big(\mathbf{B}_\beta\, u^{A}_{b}\big)(t)=u^{A}_{b*\beta}(t)\,,\qquad \big(\mathbf{A}_\alpha\, u^{B}_{a}\big)(t)=u^{B}_{a\bullet\alpha}(t)\,. \qquad (8)$$

Thus, two PDOs are conjugate if (i) the dual group of one PDO equals the group of the respective other PDO, and if (ii) application of one PDO to an eigenfunction of the respective other PDO merely produces a shift of the eigenfunction parameter (note that (i) is necessary for (ii)). The conjugateness property has a number of remarkable consequences [11], [18]-[20]. In particular, the 2-D group $(\mathcal{D},\circ)$ is the (commutative) *direct sum* [29] of the two 1-D groups $(\mathcal{A},\bullet)$ and $(\mathcal{B},*)$, i.e., $\theta_1\circ\theta_2=(\alpha_1\bullet\alpha_2,\beta_1*\beta_2)=\theta_2\circ\theta_1$ and $\theta^{-1}=(\alpha^{-1},\beta^{-1})$ where α^{-1} and β^{-1} denote the inverse elements in the partial groups $(\mathcal{A},\bullet)$ and $(\mathcal{B},*)$, respectively.

Example. The PDOs $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ are conjugate: $\big(\mathbf{F}_\nu\, u^{T}_{f}\big)(t)=u^{T}_{f+\nu}(t)$ and $\big(\mathbf{T}_\tau\, u^{F}_{t}\big)(t')=u^{F}_{t+\tau}(t')$. The 2-D group underlying the DO $\mathbf{S}_{\tau,\nu}=\mathbf{F}_\nu\mathbf{T}_\tau$ is the commutative group $(\mathcal{D},\circ)=(\mathbb{R}^2,+)$, i.e., $(\tau_1,\nu_1)\circ(\tau_2,\nu_2)=(\tau_1+\tau_2,\,\nu_1+\nu_2)$.

Covariant (a,b) Representations. The assumed conjugateness of the PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ (specifically, the fact that a and α belong to the same group $(\mathcal{A},\bullet)$ and b and β belong to the same group $(\mathcal{B},*)$) allows the formulation of a simple covariance property in the (a,b) domain:

Definition 4. Let $\mathbf{D}_\theta=\mathbf{B}_\beta\mathbf{A}_\alpha$ be a separable DO with *conjugate* PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. A bilinear (a,b) representation $B_{x,y}(c)=B_{x,y}(a,b)$ will be called *covariant to* $\mathbf{D}_\theta$ if

$$B_{\mathbf{D}_\theta x,\mathbf{D}_\theta y}(c)\;=\;B_{x,y}\big(c\circ\theta^{-1}\big)\;=\;B_{x,y}\big(a\bullet\alpha^{-1},b*\beta^{-1}\big).$$

Apart from the additional conjugateness assumption, this definition is fully equivalent to Definition 1 (θ in (2) is formally replaced by c here). Hence, Theorem 1 yields

Corollary 2. Let $\mathbf{D}_\theta=\mathbf{B}_\beta\mathbf{A}_\alpha$ be a separable DO with *conjugate* PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. Then, a bilinear (a,b) representation is covariant to $\mathbf{D}_\theta$ if and only if it can be written as

$$B_{x,y}(c)=\int_{t_1}\int_{t_2}x(t_1)\,y^*(t_2)\,\big(\mathbf{D}^{\otimes}_{c}h\big)^*(t_1,t_2)\,dt_1dt_2\,, \qquad (9)$$

where $h(t_1,t_2)$ is some 2-D function independent of $x(t)$, $y(t)$.

These covariant (a,b) representations are seen to have the same form as the covariant (α,β) representations in (3). Nevertheless, it might be possible that mapping an (a,b) representation into the TF domain (via the LF) is not equivalent to mapping an (α,β) representation into the TF domain (via the DF). This issue will be investigated next.

Covariant BTFRs Based on Covariant (a,b) Representations. Pursuing our study of the conjugate case, we now map the covariant bilinear (a,b) representation in (9) into the TF domain using the LF (see (7)). This yields

$$B_{x,y}(z)=\int_{t_1}\int_{t_2}x(t_1)\,y^*(t_2)\,\big(\mathbf{D}^{\otimes}_{l^{-1}(z)}h\big)^*(t_1,t_2)\,dt_1dt_2\,. \qquad (10)$$

These BTFRs have the same form as the covariant BTFRs in (6), however with the DF $d(\cdot)$ in (6) replaced by the LF $l(\cdot)$. They will satisfy a TF covariance property analogous to (4), again with the DF replaced by the LF. However, we conjecture that, in the conjugate case considered, the DF and LF are equivalent [11, 13]. The geometry underlying our reasoning is depicted in *Fig. 2*.

Our reference TF point z_0 corresponds via the LF to some eigenfunction parameter point c_0, i.e., $z_0=l(c_0)$. Since in the conjugate case the group underlying c equals the group $(\mathcal{D},\circ)$ underlying the displacement parameter θ, we can always parameterize the eigenfunctions $u^{B}_{a}(t)$ and $u^{A}_{b}(t)$ such that $c_0=(a_0,b_0)=(\alpha_0,\beta_0)=\theta_0$ where θ_0 is the identity element of $(\mathcal{D},\circ)$. We then have $z_0=l(\theta_0)$, i.e., z_0 is the intersection of the group delay $\tau^{B}_{\alpha_0}(f)$ of $u^{B}_{\alpha_0}(t)$ and the

[5] Sometimes this assumption holds for the group delay of $u^{A}_{b}(t)$ and the instantaneous frequency of $u^{B}_{a}(t)$; here, an analogous theory can be formulated. Other variations are also possible [13].

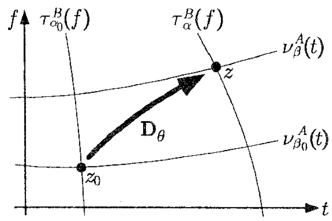

Fig. 2. Equivalence of LF and DF in the conjugate case. (This figure should be compared to Fig. 1.)

instantaneous frequency $\nu^A_{\beta_0}(t)$ of $u^A_{\beta_0}(t)$. Let us now displace the eigenfunctions $u^B_{\alpha_0}(t)$ and $u^A_{\beta_0}(t)$ by θ. Using the eigenequations and (8), we obtain the displaced eigenfunctions as $\left(\mathbf{D}_\theta\, u^B_{\alpha_0}\right)(t) = \left(\mathbf{B}_\beta \mathbf{A}_\alpha\, u^B_{\alpha_0}\right)(t) = \left(\mathbf{B}_\beta\, u^B_{\alpha_0 \bullet \alpha}\right)(t) = \left(\mathbf{B}_\beta\, u^B_\alpha\right)(t) = \lambda^B_{\beta,\alpha}\, u^B_\alpha(t)$ and similarly $\left(\mathbf{D}_\theta\, u^A_{\beta_0}\right)(t) = \lambda^A_{\alpha,\beta_0}\, u^A_\beta(t)$. The group delay of $\left(\mathbf{D}_\theta\, u^B_{\alpha_0}\right)(t) = \lambda^B_{\beta,\alpha}\, u^B_\alpha(t)$ is $\tau^B_\alpha(f)$, and the instantaneous frequency of $\left(\mathbf{D}_\theta\, u^A_{\beta_0}\right)(t) = \lambda^A_{\alpha,\beta_0}\, u^A_\beta(t)$ is $\nu^A_\beta(t)$. Hence, the TF point where the effective TF loci of the displaced eigenfunctions intersect is the intersection of $\tau^B_\alpha(f)$ and $\nu^A_\beta(t)$, which is $z = l(\alpha,\beta) = l(\theta)$. On the other hand, the TF point z has been obtained by displacing the reference TF point z_0 by θ (see Fig. 2), and thus $z = d(\theta)$. Hence, in the conjugate case considered, the LF and the DF are identical, $l(\cdot) \equiv d(\cdot)$. This derivation is not mathematically strict but a plausibility argument that relies on our geometric interpretation of the DF and LF.

From the equivalence of the LF and DF, it finally follows that the BTFRs in (10) are identical to the covariant BTFRs in (6). Thus, in the conjugate case, the construction of BTFRs via covariant (a,b) representations is equivalent to the construction of BTFRs via covariant (α,β) representations.

Example. $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ being conjugate, we can construct covariant bilinear (a,b) representations for $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$. Since here $a = t$ and $b = f$, these (a,b) representations are actually BTFRs. With (9), they are obtained as the (α,β) representations ((τ,ν) representations) from the example in Section 2, with τ and ν replaced by t and f, respectively. This gives Cohen's class, $B_{x,y}(t,f) = \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\; h^*(t_1 - t, t_2 - t)\; e^{-j2\pi f(t_1 - t_2)}\, dt_1 dt_2$. Since $l(t,f) = (t,f)$, the LF mapping (7) from the (a,b) domain into the TF domain has no effect, and the final BTFR class obtained is still Cohen's class, i.e., precisely the BTFR class derived in the example in Section 3 via the DF mapping (5) from the (α,β) domain ((τ,ν) domain). Thus, the (α,β) domain and the (a,b) domain lead to the same class of covariant BTFRs.

We finally note that the application of our covariance theory to other DOs besides $\mathbf{S}_{\tau,\nu}$ is described in [10]-[13], [19]. Furthermore, a "unitary equivalence" approach to covariant BTFRs is discussed in [30].

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