COVARIANT \((\alpha, \beta)\), TIME-FREQUENCY, AND \((a, b)\) REPRESENTATIONS

Franz Hlawatsch and Teresa Twaroch

INTHT, Technische Universität Wien, Gusshausstrasse 25/389, A-1040 Vienna, Austria
email: fhlawats@email.tuwien.ac.at

ABSTRACT—We present a general theory of bilinear/quadratic signal representations that are covariant to time-frequency (TF) displacement operators \(D_{\alpha, \beta}\). Covariant representations are developed in the \((\alpha, \beta)\) domain and in the TF domain. In the conjugate case, covariant representations can also be formulated in an eigenfunction parameter domain \((\langle a, b \rangle)\). The relations connecting the \((\alpha, \beta)\) TF, and \((a, b)\) domains are epitomized by the displacement function and the localization function. For conjugate operators, \((\alpha, \beta)\) covariance and \((a, b)\) covariance are equivalent.

1 INTRODUCTION AND OUTLINE

Important classes of bilinear/quadratic time-frequency representations (BTFRs) \([1]-[9]\) can be defined by covariance properties with respect to families of unitary operators causing time-frequency (TF) displacements. This fact has motivated the development of a covariance theory of TF analysis \([10]-[13]\), which is based on the concept of "TF displacement operators." In addition to providing a unified framework for important existing BTFR classes, the covariance theory also allows the systematic construction of new BTFR classes. This paper presents an extended view of the covariance principle. After reviewing TF displacement operators (DOs) \(D_{\alpha, \beta}\), Section 2 characterizes the class of all bilinear "\((\alpha, \beta)\) representations" covariant to a given DO. Section 3 defines the displacement function of a DO and shows how it can be used to convert covariant \((\alpha, \beta)\) representations into covariant BTFRs. For the important "conjugate" case, Section 4 develops covariant "\((a, b)\) representations" (where \(a\) and \(b\) are eigenfunction parameters), shows how these can be used for deriving covariant BTFRs, and argues that these BTFRs are identical to the covariant BTFRs from Section 3.

2 COVARIANT \((\alpha, \beta)\) REPRESENTATIONS

TF Displacement Operators. We consider a family of linear operators \(D_{\alpha, \beta} = D_{\alpha, \beta}^\dagger\) defined on a linear space \(\mathcal{X} \subseteq L_2(\mathbb{R})\), where \(L_2(\mathbb{R})\) is the space of finite-energy signals. The operator family \(\{D_{\alpha, \beta}\}\) is indexed by the 2-D "displacement parameter" \(\theta = (\alpha, \beta) \in \mathcal{D} \subseteq \mathbb{R}^2\). We assume that there exists an operation \(\circ\) such that \((D, \circ)\) forms a (possibly non-commutative) group \([14]-[16]\) with identity element \(e \in \mathcal{D}\). We shall call \(\{D_{\alpha, \beta}\}\) a TF displacement operator (DO) if \(D_{\alpha}\) is unitary \([17]\) for all \(\theta \in \mathcal{D}\), and if it satisfies the composition property \([10]-[13]\)

\[
D_{\beta}D_{\alpha} = e^{j\psi(\theta_1, \theta_2)}D_{e^{j\phi_1} \theta_2}D_{\alpha} \tag{1}
\]

where \(\psi(\theta_1, \theta_2) = \psi(\theta_2, \theta) = 0 \bmod 2\pi\). Thus, a displacement by \(\theta_1\) followed by a displacement by \(\theta_2\) is equivalent, up to a constant phase factor, to a displacement by \(\theta_2\). Therefore, \(\theta = (\alpha, \beta)\) is \((\alpha, \beta)\) covariant. This means that \(D_{\alpha, \beta}^\dagger\) can be undone, up to a constant phase factor, by a displacement by the group-inverse parameter \(\theta^{-1}\).

Example. The TF shift operator \(S_{\tau, \nu} = F T_{\tau, \nu}\), where \((T_{\tau, \nu}(x))(t) = x(t - \tau)\) and \((F T_{\tau, \nu}(x))(t) = x(t)e^{j2\pi t\nu}\), is a DO with \(\theta = (\tau, \nu)\), \(\mathcal{D} = \mathbb{R}^2\), \((\tau_1, \nu_1) \circ (\tau_2, \nu_2) = (\tau_1 + \tau_2, \nu_1 + \nu_2)\), \(\theta_0 = (0, 0)\), \(\theta^{-1} = (-\tau, -\nu)\), and \(\psi(\theta, \theta_2) = 2\pi \nu_1 \tau_2\). Thus, (1) reads \(S_{\tau_1, \nu_2}S_{\tau_1, \nu_1} = e^{-j2\pi \nu_1 \tau_2}S_{\tau_1, \nu_1 + \nu_2}^\dagger\).

Covariant Bilinear \((\alpha, \beta)\) Representations. Although we are primarily interested in covariant BTFRs, we first consider bilinear \((\alpha, \beta)\) representations (or \(\theta\) representations).

Definition 1 \([13]\). A bilinear \(\theta\) representation \(B_{\alpha, \beta}(\theta)\) will be called covariant to a DO \(D_\theta\) if

\[
B_{\alpha, \beta}(\theta) = B_{\alpha, \beta}^\circ \theta \circ \theta^{-1} \tag{2}
\]

where \(\theta = \theta \circ \theta^{-1}\) is equivalent, up to a constant phase factor, to a displacement by \(\theta\). This fact has motivated the development of a covariance theory of TF analysis \([10]-[13]\), which is based on the concept of "TF displacement operators." In addition to providing a unified framework for important existing BTFR classes, the covariance theory also allows the systematic construction of new BTFR classes.

Example. For the TF shift operator \(S_{\tau, \nu}\), the covariance \((2)\) is \(B_{\alpha, \beta}(\tau, \nu) = B_{\alpha, \beta}(\tau - \nu, \nu - \nu)\), and, with \((3)\), the class of all covariant bilinear \((\alpha, \beta)\) representations is \(B_{\alpha, \beta}(\tau, \nu) = \int_{\tau_1} \int_{\tau_2} x(t_1) y(t_2) h^\circ(t_1 - \tau, t_2 - \nu) \exp(-j2\pi \nu_1 \tau_2) dt_1 dt_2\), where \(h^\circ = H H^\dagger\), and \(H = \theta\) is the linear operator whose kernel is \(h(t_1, t_2)\) in \((3)\).

3 COVARIANT TF REPRESENTATIONS

We shall now show how to convert covariant bilinear \(\theta\) representations into covariant BTFRs.

Displacement Function. If \(x(t)\) is localized, in the TF plane, about some (fixed) reference TF point \(z_0 = (t_0, \nu_0)\),

\[
\int_{(t_0, \nu_0)}^z (D_\theta x)(t) = \int_{(t_0, \nu_0)}^z \left(D_\theta x(t_1, t_2) \right) h(t_1, t_2) dt_1 dt_2,
\]

where \(D_\theta x(t_1, t_2)\) is the kernel of \(D_\theta\), i.e., \((D_\theta x)(t) = \int_{(t_0, \nu_0)}^z (D_\theta x(t_1, t_2) \right) h(t_1 - \tau, t_2 - \nu) \exp(j2\pi \nu_1 \tau_2).\)
then \((D_0z)(t)\) will be localized about some other TF point \(z = (t, f)\) that depends on \(\theta\), i.e., \(z = d(\theta)\) where \(d(\cdot)\) is the displacement function (DF) of the DO \(D_0\) [10]-[13]. In order to construct the DF, we define the family of signals \(x_\theta(t) \triangleq D_\theta(t,\bar{t})\) where \(D_\theta(t,\bar{t})\) is the kernel of \(D_\theta\) and \(\bar{t}\) is some fixed time. Similarly, we define the family \(\{y_\theta(t)\}_{\theta \in \mathbb{R}}\) in the frequency domain as \(Y_\theta(f) \triangleq D_\theta(f, \bar{f})\) where \(D_\theta(f, \bar{f}) = \int_0^\infty D_\theta(t, t') e^{-j2\pi(t' - \bar{f})} dt'\) is the DO's frequency-domain kernel\(^3\) and \(\bar{f}\) is some fixed frequency. Eq. (1) then implies \((D_\theta z)(t) = e^{j\theta_0(t)}x_{\theta_0}(t)\) and \((D_\theta y_\theta)(t) = e^{j\theta_0(t)}y_{\theta_0}(t)\), i.e., application of \(D_\theta\) to a member of the family \(\{x_\theta(t)\}_{\theta \in \mathbb{R}}\) or \(\{y_\theta(t)\}_{\theta \in \mathbb{R}}\) yields another member of the respective family \((up to a constant phase factor). In the following, we shall use the group delay \(x_\theta(t)\) and \(y_\theta(t)\) of \(z(t), v_\theta(t) = \frac{d}{dt} \arg\{y_\theta(t)\}\), and the instantaneous frequency of \(v_\theta(t)\) defined by the system of equations

\[
v_\theta(t) = f, \quad v_\theta(t) = f,
\]

depends on \(\theta\), i.e., \(z = d(\theta)\). This defines the DF provided that the above system of equations has a unique solution \(z = (t, f)\) for any \(\theta \in \mathbb{R}\).

According to its construction above, the DF \(z = d(\theta)\) yields the TF point \(z = (t, f)\) that is obtained when displacing the initial reference TF point \(z_0 = (t_0, f_0)\) by the displacement parameter \(\theta\) (see Fig. 1). Note that \(d(\theta) = z_0\). Let \(Z \subset \mathbb{R}^2\) denote the range of the DF, i.e., the set of all TF points \(z = d(\theta)\) with \(\theta \in \mathbb{R}\). We assume that the DF is a one-to-one, differentiable, and area-preserving mapping from \(Z\) onto \(\mathbb{R}^2\).

**Extended Displacement Function.** The DF also describes the displacement of arbitrary TF points. Indeed, the TF point, denoted \(z_2 = (t_2, f_2)\), that is obtained by displacing a given TF point \(z_1 = (t_1, f_1) \in Z\) by \(\theta\) is easily seen to be given by \(z_2 = d((t_1, f_1) \mu \theta)\). The function

\[
e(z, \theta) \triangleq d((t_1 - \bar{t}) \mu \theta)
\]

will be called the extended displacement function of the DO \(D_\theta\). It can be shown that \(e(z, \theta) = d(\theta), e(z, \theta_0) = z, e(e(z, \theta), \theta_0) = e(z, \theta_0), d^{-1}(e(z, \theta)) = d^{-1}(e(z, \theta_0)) = z, e(z, \theta) \neq e(z', \theta')\) or equivalently \(e(e(z, \theta), \theta') \neq z\). We assume (or conjecture) that for any \(\theta \in \mathbb{R}\), the "TF coordinate transform" \(z \rightarrow z' = e(z, \theta)\) is a one-to-one, differentiable, and area-preserving mapping from \(Z\) onto \(\mathbb{R}^2\).

**Example.** The invariant signals of \(S_{\nu}\) are obtained as \(x_{\tau,\nu}(t) = d(t - \tau)e^{j2\pi \nu t}\), with group delay \(\tau_{\nu}(f) = \tau\), and \(y_{\nu}(f) = f\), with instantaneous frequency \(\nu_{\nu}(t) = \nu\) (we have \(t = 0 = \bar{f}\)). It follows that the DF of \(S_{\nu}\) is simply \(d(\tau) = \tau\), with range \(Z = \mathbb{R}^2\) (the entire TF plane). The extended DF of \(S_{\nu}\) is \(e(t, f; \tau) = (t + \tau, f + \nu)\).

**Covariant BTFRs.** Using the extended DF, we are now able to define a covariance property in the TF domain.

**Definition.** [10]-[13]. A BTFR \(B_{\tau,\nu}(z) = B_{\nu,\tau}(z, f, t)\) will be called covariant to a DO \(D_\theta\) if

\[
B_{\tau,\nu}(z) = B_{\nu,\tau}(e(z, \theta^{-1})). \tag{4}
\]

This is identical to the \(\theta\) representation covariance (2) up to a mapping \(z = d(\theta)\); hence the class of all covariant BTFRs is obtained by remapping the class of covariant bilinear \(\theta\) representations \(B_{\nu,\tau}(\theta)\) (3) as

\[
B_{\tau,\nu}(z) = B_{\nu,\tau}(\theta)_{\mu d^{-1}(z)} \tag{5}
\]

This yields the following corollary to Theorem 1:

**Corollary.** [10]-[13]. A BTFR is covariant to a DO \(D_\theta\) if and only if it can be written as

\[
B_{\nu,\tau}(z) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} x(t_1) y^*(t_2) \left| D_{\theta^{-1} (t_1, t_2)}^\circ \right| H(t_1, t_2) dt_1 dt_2, \tag{6}
\]

where the \(h(t_1, t_2)\) is some 2-D function independent of \(x(t_1), y(t_2)\).

The class of all BTFRs covariant to a DO is seen to be parameterized by a 2-D kernel \(h(t_1, t_2)\). The covariant BTFRs (6) can also be expressed as the bilinear form

\[
B_{\nu,\tau}(z) = (x, H D_{\theta^{-1}(t_1, t_2)} y) H^\circ D_{\theta^{-1}(t_1, t_2)}^\circ, \tag{6}
\]

where \(H\) is the linear operator whose kernel is \(h(t_1, t_2)\) in (6).

**Example.** For \(S_{\nu}\), (4) is the TF shift covariance

\[
B_{\nu,\tau}(z) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} x(t_1) y^*(t_2) h^*(t_1 - t, t_2 - t) e^{-j2\pi f(t_1 - t_2)} dt_1 dt_2, \tag{6}
\]

which is Cohen's class.

**4 THE CONJUGATE CASE**

In the "conjugate" or "dual" case [11], [18]-[22], there exists an alternative approach to the definition of covariant BTFRs.

**The Separable Case.** Under certain conditions regarding the group operation \(\circ [11]\), a DO \(D_\theta\) is separable

\[
D_\theta = B_{\nu,\tau} A_\theta, \tag{7}
\]

where the partial DOs (PDOS) \(A_\alpha B_{\beta} = A_\alpha B_{\beta}\) and \(B_{\beta} A_\alpha = B_{\beta} A_\alpha\) are families of unitary, linear operators indexed by the 1-D displacement parameters \(\alpha \in A\) and \(\beta \in B\), respectively (note that \(D = A \times B\)). The PDOS satisfy the composition properties \(A_\alpha B_{\beta} A_{\alpha'} B_{\beta'} = A_\alpha A_{\alpha'} B_{\beta} B_{\beta'} = B_{\beta} B_{\beta'} A_\alpha A_{\alpha'}\), where \((A, B)\).
and \((B, \ast)\) are commutative groups with identity elements \(a_0\) and \(\beta_0\), respectively (note that \(\theta_0 = (a_0, \beta_0)\)).

**Localization Function.** The (generalized) eigenvalues\(^4\) \(\lambda_{a,b}^\text{TF}\) and eigenfunctions \(\psi_{a,b}^\text{TF}(t)\) of \(A_{a,b}\) are defined by the eigenequation \((A_{a,b}\psi_{a,b}^\text{TF}) = \lambda_{a,b}^\text{TF} \psi_{a,b}^\text{TF}\); they are indexed by a “dual” parameter \(b\) which belongs to the commutative dual group \((A, \ast)\) of \((A, \bullet)\) [23, 15, 16, 24]. A similar definition applies to the eigenvalues \(\lambda_{a,b}^\text{TF}\) and eigenfunctions \(\psi_{a,b}^\text{TF}(t)\) of \(B_{a,b}\), where \(a\) belongs to the dual group \((B, \ast)\) of \((B, \bullet)\).

The eigenfunctions \(\psi_{a,b}^\text{TF}(t)\) and \(\psi_{a,b}^\text{TF}(t)\) yield a concept for the definition of BTFRs that is different, in general, from the covariance approach discussed so far: a bilinear \((a, b)\) representation (a function of the eigenfunction parameters \(a\) and \(b\)) is constructed using the covariance approach in Section 3; in particular, they will not be covariant to the DO \(\theta\) under the PDO equals the group of the respective PDO, and if \(a\) and \(b\) belong to the same group \((A, \bullet)\) and \(B_{a,b}\) respectively (note that \(a\) and \(b\) belong to the same group \((A, \bullet)\) and \(B_{a,b}\) respectively. For any \((a, b)\) representation in (3), \((a, b)\) representations in (3). Nevertheless, it might be possible that mapping an \((a, b)\) representation into the TF domain (via the LF) is not equivalent to the TF domain (via the DF). This issue will be investigated next.

**Definition 3** [11, 18, 19]. Two PDOs \(A_{a,b}\) and \(B_{c,d}\) are said to be conjugate if \((A, \ast) = (B, \ast)\), \((B, \ast) = (A, \bullet)\), and

\[
(B_{a,b}\psi_{a,b}^\text{TF}(t)) = \psi_{b,a}^\text{TF}(t), \quad (A_{a,b}\psi_{a,b}^\text{TF}(t)) = \psi_{a,b}^\text{TF}(t).
\]

Thus, two PDOs are conjugate if \((i)\) the dual group of one PDO equals the group of the respective PDO, and if if

\(^4\)Sometimes this assumption holds for the group delay of \(\psi_{a,b}^\text{TF}(t)\) and the instantaneous frequency of \(\psi_{a,b}^\text{TF}(t)\); here, an analogous theory can be formulated. Other variations are also possible [13].
place the eigenfunctions \( v_{\beta,0}(t) \) and \( u_{\alpha,0}(t) \) by \( \theta \). Using the
eigenvalues \( \lambda_{\alpha,0} \) and \( \lambda_{\beta,0} \), we obtain the displaced
eigenfunctions \( u_{\alpha,0}(\theta) = \lambda_{\alpha,0} u_{\alpha,0}(t) \) and similarly \( v_{\beta,0}(\theta) = \lambda_{\beta,0} v_{\beta,0}(t) \). The group delay of \( D_{\alpha} u_{\alpha,0}(\theta) \) is \( \lambda_{\alpha,0}^2 \), and the instantaneous frequency of \( D_{\alpha} u_{\alpha,0}(\theta) \) is \( \lambda_{\alpha,0}^2 \). Hence, the TF point where the effective
TF locus of the displaced eigenfunctions intersect is the intersection of \( \tau^x(f) \) and \( \tau^y(f) \), which is \( \tau = (\alpha, \beta) = (\theta) \). On the other hand, the TF point \( z \) has been obtained by
converting from\( \theta \) to \( z \). Hence, in the conjugate case, considered, the
DF and the LF are identical, \( \tau^z(f) = \tau^y(f) \). This derivation is
not mathematically strict but a plausibility argument that
relies on our geometric interpretation of the DF and LF.

From the equivalence of the LF and DF, it finally follows that
the BTFRs in (10) are identical to the covariant BTFRs in (6). Thus, in the conjugate case, the construction of BTFRs
via covariant \((\alpha, \beta)\) representations is equivalent to the construction of BTFRs via covariant \((\alpha, \beta)\) representations.

Example. \( T_1 \) and \( P_1 \) being conjugate, we can construct covariant bilinear \((\alpha, \beta)\) representations for \( S_{\alpha,\beta} = F_{T_1, P_1}. \) Since here \( a = t \) and \( b = f \), these \((\alpha, \beta)\) representations are actually
BTFRs. With (9), they are obtained as the \((\alpha, \beta)\) representations \((\tau, \nu)\) representations from the example in Section 2, with \( r \) and \( \nu \) replaced by \( t \) and \( f \), respectively. This gives Cohen's class, \( B_{\alpha,\beta}(t, f) = \int_t^\infty \int_{-\infty}^\infty \tau(x) \nu(t - x) \exp(-j2\pi f(t - x)) dt \ dx \). This gives
the LF mapping \((7)\) from the \((\alpha, \beta)\) domain into the TF domain has no effect, and the final BTFR class obtained is still Cohen's class, i.e.,
precisely the BTFR class derived in the example in Section 3 via the DF mapping \((5)\) from the \((\alpha, \beta)\) domain \((\tau, \nu)\) domain. Thus, the \((\alpha, \beta)\) domain and the \((\alpha, \beta)\) domain lead to the same class of covariant BTFRs.

We finally note that the application of our covariance theory to other domains \( S_{\alpha,\beta} \) is described in [10]-[13], [19]. Furthermore, a "unitary equivalence" approach to covariant BTFRs is discussed in [30].

Acknowledgment
We thank R. G. Baraniuk, A. Berthon, K. Nowak, A. Papandreou-Suppappola, and A. M. Sayeed for illuminating discussions.

References