IEEE 1996 Int. Sympos. Time-Frequency Time-Scale Analysis, Paris, France, June 1996, pp. 525-528

DISCRETE-TIME WILSON EXPANSIONS*

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Abstract—Recently it has been shown that continuous-time orthonormal Wilson bases with good time-frequency localization can be constructed. We introduce and discuss *discrete-time Wilson function sets and frames*, and we show that Wilson sets and frames (potentially oversampled) can be derived from Weyl-Heisenberg sets and frames. We also show that discrete-time Wilson expansions correspond to a new class of cosine-modulated filter banks.

1. INTRODUCTION

The Gabor expansion [1]-[9] is an important linear signal representation. Unfortunately, there do not exist orthonormal Gabor function sets (Weyl-Heisenberg sets) with good localization in both time and frequency [6, 7]. The recently proposed *Wilson expansion* [10]-[12] is a simple variation on the Gabor expansion that overcomes this drawback.

So far, only continuous-time Wilson bases have been considered [10]-[12]. This paper introduces discrete-time Wilson sets and frames with critical sampling and oversampling. Extending the derivation of orthonormal continuous-time Wilson bases from tight Weyl-Heisenberg (WH) frames [10], we show that discrete-time Wilson sets/frames oversampled by a factor K (K odd) can be derived from WH sets/frames oversampled by $2K$. Specifically, tight Wilson frames can be derived from tight WH frames.

We also show that discrete-time Wilson expansions correspond to a new type of cosine-modulated filter banks [13].

2. WEYL-HEISENBERG SETS AND FRAMES

The *discrete-time Gabor expansion* [1]-[9] of a signal[1] $x[n] \in l^2(\mathbb{Z})$ is given by

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} \langle x, \tilde{g}_{l,m} \rangle \, g_{l,m}[n] \,,$$

where the *Weyl-Heisenberg (WH) sets* $\{g_{l,m}[n]\}$ and $\{\tilde{g}_{l,m}[n]\}$ are defined as $g_{l,m}[n] = g[n-lM]\, e^{j2\pi \frac{m}{N} n}$ and $\tilde{g}_{l,m}[n] = \tilde{g}[n-lM]\, e^{j2\pi \frac{m}{N} n}$, with $g[n]$ a suitable "synthesis window" and $\tilde{g}[n]$ a suitable "analysis window," and the parameters $M, N \in \mathbb{N}$ are the grid constants [2, 14].

The WH set $\{g_{l,m}[n]\}$ is a *WH frame* for $l^2(\mathbb{Z})$ if for all $x[n] \in l^2(\mathbb{Z})$

$$A\|x\|^2 \;\le\; \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} |\langle x, g_{l,m} \rangle|^2 \;\le\; B\|x\|^2$$

with the *frame bounds* $A > 0$ and $B < \infty$ [6]. The frame property is desirable since it guarantees completeness of the WH set $\{g_{l,m}[n]\}$ and potentially good numerical properties of the Gabor expansion (characterized by the frame bounds A and B). Furthermore, the minimum-norm analysis window [2, 3, 4, 14, 9] can be calculated as $\tilde{g}^0[n] = (\mathbf{S}^{-1} g)[n]$, where $\mathbf{S}^{-1}$ is the inverse of the *frame operator* defined as $(\mathbf{S}x)[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{N-1} \langle x, g_{l,m} \rangle \, g_{l,m}[n]$ (systematic methods for constructing $\tilde{g}^0[n]$ can be found in [2, 3, 8], [14]-[16]). Best numerical properties are obtained for *tight* WH frames, for which $A = B$.

In the *oversampled* case $N > M$, one observes better numerical stability of the Gabor expansion and better time-frequency concentration of the windows $g[n]$ and $\tilde{g}^0[n]$ than in the case of critical sampling where $N = M$; however, this comes at the cost of redundant (and non-unique) expansion coefficients. In the *undersampled* case $N < M$, the Gabor expansion will not exist for all $x[n] \in l^2(\mathbb{Z})$.

The *discrete-time Zak transform* (ZT) [17]-[20]

$$\mathcal{Z}_x(n,\theta) = \sum_{l=-\infty}^{\infty} x[n+lM]\, e^{-j2\pi\theta l} \tag{1}$$

is of fundamental importance in Gabor (WH frame) theory since it allows an efficient computation of the analysis window $\tilde{g}^0[n]$ and the frame bounds A, B [17, 6, 3, 8, 15].

3. ORTHONORMAL WILSON EXPANSIONS

We shall first discuss the construction of critically sampled, orthonormal, discrete-time Wilson bases.

3.1. Orthonormal Wilson Bases

The construction of continuous-time Wilson bases proposed in [10] can be transferred (with slight modifications) to the discrete-time case. Let $\{g_{l,m}[n]\}$ ($-\infty < l < \infty$, $m = 0, 1, .., 2M-1$) be a WH set with oversampling factor 2 (i.e., $N = 2M$). We define a discrete-time *Wilson set* as

$$\psi_{l,m}[n] = \begin{cases} g_{2l,0}[n] & m = 0 \\ \frac{1}{\sqrt{2}}\left(g_{l,m}[n] + g_{l,2M-m}[n]\right) & m+l \text{ even},\; m = 1, ..., M-1 \\ \frac{-j}{\sqrt{2}}\left(g_{l,m}[n] - g_{l,2M-m}[n]\right) & m+l \text{ odd},\; m = 1, ..., M-1 \\ g_{2l+r,M}[n] & m = M, \end{cases}$$

where $r = 0$ for M even and $r = 1$ for M odd. We note that for $m = 1, \ldots, M-1$

$$\psi_{l,m}[n] = \begin{cases} \sqrt{2}\, g_{l,0}[n] \cos\!\left(2\pi \frac{m}{2M} n\right) & m+l \text{ even},\; m = 1, \ldots, M-1 \\ \sqrt{2}\, g_{l,0}[n] \sin\!\left(2\pi \frac{m}{2M} n\right) & m+l \text{ odd},\; m = 1, \ldots, M-1. \end{cases}$$

Following an approach suggested in [11] for the continuous-time case, the next theorem provides necessary and sufficient conditions on $g[n]$ such that $\{\psi_{l,m}[n]\}$ is an orthonormal basis for $l^2(\mathbb{Z})$, i.e., the orthonormal *Wilson expansion*

*FUNDING BY FWF GRANTS P10531-ÖPH AND S-7001-MAT AND BY NSF GRANT DMS-9306430.

[1] Here $l^2(\mathbb{Z})$ is the space of square-summable sequences $x[n]$, i.e., $\|x\|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$.

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M} \langle x, \psi_{l,m} \rangle \, \psi_{l,m}[n]$$

exists for all $x[n] \in l^2(\mathbb{Z})$.

Theorem 1. The Wilson set $\{\psi_{l,m}[n]\}$ is an orthonormal basis for $l^2(\mathbb{Z})$ if and only if

$$C_k^{(g)}[n] = \frac{1}{M}\,\delta[k] \qquad \text{and} \qquad D_k^{(g)}[n] = 0$$

for all $k \in \mathbb{Z}$, where

$$C_k^{(g)}[n] = \sum_{l=-\infty}^{\infty} g[n-lM]\, g^*[n-(l+2k)M]$$

$$D_k^{(g)}[n] = \sum_{l=-\infty}^{\infty} (-1)^l g[n-lM]\, g^*[-n-(l-2k-1)M].$$

We now reformulate the two conditions of Theorem 1 in the frequency domain and in the ZT domain.

Corollary 1. The Wilson set $\{\psi_{l,m}[n]\}$ is an orthonormal basis for $l^2(\mathbb{Z})$ if and only if

$$\hat{C}_l^{(g)}(\theta) = 2M\delta[l] \qquad \text{and} \qquad \hat{D}_l^{(g)}(\theta) = 0$$

for $l = 0, 1, ..., M-1$, where[2]

$$\hat{C}_l^{(g)}(\theta) = \sum_{m=0}^{2M-1} G\Big(\theta - \frac{m}{2M}\Big)\, G^*\Big(\theta - \frac{m+2l}{2M}\Big)$$

$$\hat{D}_l^{(g)}(\theta) = \sum_{m=0}^{2M-1} (-1)^m\, G\Big(\theta + \frac{m}{2M}\Big)\, G^*\Big(\theta - \frac{m+2l-1}{2M}\Big).$$

An equivalent set of conditions is (cf. (1))

$$|\mathcal{Z}_g(n,\theta)|^2 + \left|\mathcal{Z}_g\Big(n, \theta - \frac{1}{2}\Big)\right|^2 = \frac{2}{M}$$

and

$$\mathcal{Z}_g\Big(n, \theta - \frac{1}{2}\Big)\, \mathcal{Z}_g^*(-n,\theta) \;-\; \mathcal{Z}_g(n,\theta)\, \mathcal{Z}_g^*\Big(-n, \theta - \frac{1}{2}\Big) \;=\; 0.$$

The first condition of Theorem 1 (equivalently, of Corollary 1) simplifies when $g[n]$ has finite length or finite bandwidth. If $g[n]$ is compactly supported within $0 \le n \le 2M-1$, then $C_k^{(g)}[n] = \frac{1}{M}\delta[k]$ reduces to

$$|g[n]|^2 + |g[n+M]|^2 = \frac{1}{M} \qquad \text{for } 0 \le n \le M-1.$$

Similarly, if $G(\theta)$ is supported within $0 \le \theta < \frac{1}{M}$, then $\hat{C}_l^{(g)}(\theta) = 2M\delta[l]$ (see Corollary 1) reduces to

$$|G(\theta)|^2 + \left|G\Big(\theta + \frac{1}{2M}\Big)\right|^2 = 2M \qquad \text{for } 0 \le \theta < \frac{1}{2M}.$$

3.2. Relation with Tight WH Frames

An important relation between orthonormal Wilson bases and tight WH frames exists in the case of a conjugate even window $g[n]$. (This relation has previously been noted in [10] for the continuous-time case.)

[2] Here $G(\theta) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j2\pi\theta n}$ is the discrete-time Fourier transform of $g[n]$.

Corollary 2. For $g^*[-n] = g[n]$, the Wilson set $\{\psi_{l,m}[n]\}$ is an orthonormal basis for $l^2(\mathbb{Z})$ if and only if $\{g_{l,m}[n]\}$ constitutes a tight WH frame with oversampling factor 2 and $\|g\| = 1$.

Thus, the construction of orthonormal Wilson bases reduces to finding a tight WH frame oversampled by 2 with conjugate even window $g[n]$. We note that for $g^*[-n] = g[n]$ the second condition of Theorem 1 (equivalently, of Corollary 1) is always satisfied.

We next consider the construction of tight WH frames with oversampling factor 2. If $\{g_{l,m}[n]\}$ is a (nontight) WH frame for $l^2(\mathbb{Z})$, then it can be shown [6] that the WH set $\{g_{T\,l,m}[n]\}$ generated by $g_T[n] = (\mathbf{S}^{-1/2} g)[n]$ is a tight WH frame for $l^2(\mathbb{Z})$ with frame bounds $A = B = 1$. Here, $\mathbf{S}^{-1/2}$ is the positive definite square root of the inverse frame operator $\mathbf{S}^{-1}$. For time-limited or band-limited windows $g[n]$, this construction reduces to simple divisions in the time or frequency domain, respectively. If $g[n]$ is compactly supported within an interval of length $2M$, then [15]

$$g_T[n] = \frac{g[n]}{\sqrt{2MC_0^{(g)}[n]}}.$$

Similarly, if $G(\theta)$ is band-limited within an interval of length $\frac{1}{M}$, then a tight window $g_T[n]$ can be constructed in the frequency domain according to [15]

$$G_T(\theta) = \frac{G(\theta)}{\sqrt{\frac{1}{M}\hat{C}_0^{(g)}(\theta)}}.$$

If none of these support restrictions are satisfied, a tight window $g_T[n]$ can be constructed in the ZT domain as [15]

$$\mathcal{Z}_{g_T}(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\sqrt{\lambda_g(n,\theta)}},$$

where $\lambda_g(n,\theta) = M\left[|\mathcal{Z}_g(n,\theta)|^2 + |\mathcal{Z}_g(n, \theta - \frac{1}{2})|^2\right]$. In all three cases, $g^*[-n] = g[n]$ entails $g_T^*[-n] = g_T[n]$.

4. WILSON SETS AND FRAMES

So far, we have discussed the derivation of critically sampled, orthonormal Wilson bases from WH sets and frames with oversampling factor 2. We now extend our results to the construction of general Wilson sets and frames (critically sampled or oversampled).

4.1. Complete Wilson Sets

Let $\{g_{l,m}[n]\}$ with $g_{l,m}[n] = g[n-lM]\, e^{j2\pi\frac{m}{2KM}n}$ and $\{h_{l,m}[n]\}$ with $h_{l,m}[n] = h[n-lM]\, e^{j2\pi\frac{m}{2KM}n}$ be WH sets with oversampling factor $2K$ (K odd). We define a Wilson synthesis set $\{\psi_{l,m}[n]\}$ associated to the window $g[n]$ by

$$\psi_{l,m}[n] = \tag{2}$$

$$\begin{cases} g_{2l,0}[n] & m = 0 \\ \sqrt{2}\, g_{l,0}[n] \cos\big(2\pi\frac{m}{2KM}n\big) & m+l \text{ even},\ m = 1,..,KM-1 \\ \sqrt{2}\, g_{l,0}[n] \sin\big(2\pi\frac{m}{2KM}n\big) & m+l \text{ odd},\ m = 1,..,KM-1, \\ g_{2l+r,KM}[n] & m = KM, \end{cases}$$

and a Wilson analysis set $\{\phi_{l,m}[n]\}$ associated to $h[n]$ as

$$\phi_{l,m}[n] = \tag{3}$$

$$\begin{cases} h_{2l,0}[n] & m = 0 \\ \sqrt{2}\, h_{l,0}[n] \cos\big(2\pi\frac{m}{2KM}n\big) & m+l \text{ even},\ m = 1,..,KM-1 \\ \sqrt{2}\, h_{l,0}[n] \sin\big(2\pi\frac{m}{2KM}n\big) & m+l \text{ odd},\ m = 1,..,KM-1, \\ h_{2l+r,KM}[n] & m = KM. \end{cases}$$

Here, $r = 0$ for M even and $r = 1$ for M odd. The resulting *Wilson expansion* reads

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{KM} \langle x, \phi_{l,m} \rangle \, \psi_{l,m}[n] \,. \tag{4}$$

The next theorem provides necessary and sufficient conditions on $g[n]$ and $h[n]$ such that the Wilson expansion (4) exists for all $x[n] \in l^2(\mathbb{Z})$, i.e., the Wilson synthesis set $\{\psi_{l,m}[n]\}$ is complete in $l^2(\mathbb{Z})$ with $\{\phi_{l,m}[n]\}$ being a "dual" Wilson analysis set.

Theorem 2. The Wilson expansion (4) exists for all $x[n] \in l^2(\mathbb{Z})$ if and only if

$$C_k^{(g,h)}[n] = \frac{1}{KM}\,\delta[k] \qquad \text{and} \qquad D_k^{(g,h)}[n] = 0$$

for all $k \in \mathbb{Z}$, where

$$C_k^{(g,h)}[n] = \sum_{l=-\infty}^{\infty} g[n-lM]\, h^*[n-(l+2kK)M]$$

$$D_k^{(g,h)}[n] = \sum_{l=-\infty}^{\infty} (-1)^l g[n-lM]\, h^*[-n-(l-2kK-K)M].$$

We can again formulate frequency-domain and ZT-domain versions of Theorem 2.

Corollary 3. The Wilson expansion (4) exists for all $x[n] \in l^2(\mathbb{Z})$ if and only if

$$\hat{C}_l^{(g,h)}(\theta) = 2M\delta[l] \quad \text{and} \quad \hat{D}_{l,p}^{(g,h)}(\theta) = 0$$

for $l = 0, 1, ..., M-1$, $p = 0, 1, ..., K-1$, where

$$\hat{C}_l^{(g,h)}(\theta) = \sum_{m=0}^{2KM-1} G\Big(\theta - \frac{m}{2KM}\Big)\, H^*\Big(\theta - \frac{m+2Kl}{2KM}\Big)$$

$$\hat{D}_{l,p}^{(g,h)}(\theta) = \sum_{m=0}^{2M-1} (-1)^m\, G\Big(\theta + \frac{mK+p}{2KM}\Big)\, H^*\Big(\theta - \frac{(m-2l+1)K-p}{2KM}\Big).$$

An equivalent set of conditions is

$$\sum_{k=0}^{2K-1} \mathcal{Z}_g\Big(n, \theta - \frac{k}{2K}\Big)\, \mathcal{Z}_h^*\Big(n, \theta - \frac{k}{2K}\Big) = \frac{2}{M}$$

and

$$\mathcal{Z}_g\Big(n, \theta - \frac{k}{2K} - \frac{1}{2}\Big)\, \mathcal{Z}_h^*\Big(-n, \theta - \frac{k}{2K}\Big) - \mathcal{Z}_g\Big(n, \theta - \frac{k}{2K}\Big)\, \mathcal{Z}_h^*\Big(-n, \theta - \frac{k}{2K} - \frac{1}{2}\Big) = 0$$

for $k = 0, 1, ..., K-1$. These conditions again simplify in the case of finite-length or bandlimited windows $g[n]$, $h[n]$.

4.2. Wilson Frames

The Wilson synthesis set $\{\psi_{l,m}[n]\}$ is complete in $l^2(\mathbb{Z})$ if the conditions of Theorem 2 or Corollary 3 are satisfied; however, it need not possess the (desirable) frame property. We now discuss the derivation of *Wilson frames* from WH frames, assuming a conjugate even synthesis window $g[n]$.

Corollary 4. Let $\{g_{l,m}[n]\}$ be a WH frame for $l^2(\mathbb{Z})$ with $N = 2KM$ (K odd), $g^*[-n] = g[n]$, and frame bounds A and B. Furthermore, let $\tilde{g}^0[n]$ denote the minimum-norm analysis window as considered in Section 2.

(i) The Wilson synthesis set $\{\psi_{l,m}[n]\}$ associated to $g[n]$ by (2) is a frame with frame bounds $A_w = \frac{A}{2}$ and $B_w = \frac{B}{2}$, i.e., for all $x[n] \in l^2(\mathbb{Z})$

$$\frac{A}{2}\|x\|^2 \le \sum_{l=-\infty}^{\infty} \sum_{m=0}^{KM} |\langle x, \psi_{l,m} \rangle|^2 \le \frac{B}{2}\|x\|^2 \,.$$

(ii) For $h[n] = 2\tilde{g}^0[n]$, the Wilson analysis set $\{\phi_{l,m}[n]\}$ constructed from $h[n]$ according to (3) is the *dual frame* [6] associated to $\{\psi_{l,m}[n]\}$.

The following conclusions can be drawn:

- The frame bounds $A_w = \frac{A}{2}$, $B_w = \frac{B}{2}$ of the Wilson synthesis frame $\{\psi_{l,m}[n]\}$ are trivially related to the frame bounds A, B of the underlying WH synthesis frame $\{g_{l,m}[n]\}$. Since $\frac{B_w}{A_w} = \frac{B}{A}$, the Wilson frame inherits the numerical properties of the WH frame.
- In particular, if the WH frame is tight ($A = B$), then the Wilson frame derived from it is tight as well.
- The dual frame associated to a Wilson frame has again Wilson structure.

With Corollary 4, known techniques for the calculation of the minimum-norm dual window can now be applied for constructing Wilson analysis frames for given Wilson synthesis frames. We note that the methods for the calculation of tight WH frames (see Section 3.2) can be generalized to the case of integer oversampling [15].

5. WILSON FILTER BANKS

Signal expansions and frames are intimately related to filter banks [21]-[27]. Hence, it is not surprising that the idea underlying the construction of Wilson bases introduced in [10, 11] and generalized in this paper has been used in filter bank theory for many years (see [13], Chapter 8).

We shall now show that discrete-time Wilson expansions correspond to perfect-reconstruction filter banks of the cosine-modulated type [13]. For the sake of simplicity, we shall discuss only the critical case $K = 1$ although an extension to the oversampled case is possible. Consider a critically sampled filter bank with $2M$ channels and decimation factor $2M$ in each subband, and with the $2M$ analysis filter impulse responses given by

$$h_k[n] = \begin{cases} h^*[-n] & k = 0 \\ \sqrt{2}\, h^*[-n] \cos\big(2\pi \frac{k}{2M} n\big) & k \text{ even},\ k = 1, ..., M-1 \\ \sqrt{2}\, h^*[-n-M] \cos\big(2\pi \frac{k}{2M} n\big) & k \text{ odd},\ k = 1, ..., M-1 \\ h^*[-n-rM]\,(-1)^n & k = M \end{cases}$$

and

$$h'_k[n] = \begin{cases} -\sqrt{2}\, h^*[-n-M] \sin\big(2\pi \frac{k}{2M} n\big) & k \text{ even},\ k = 1, ..., M-1 \\ -\sqrt{2}\, h^*[-n] \sin\big(2\pi \frac{k}{2M} n\big) & k \text{ odd},\ k = 1, ..., M-1. \end{cases}$$

The corresponding $2M$ synthesis filters are given by

$$f_k[n] = \begin{cases} g[n] & k = 0 \\ \sqrt{2}\, g[n] \cos\big(2\pi \frac{k}{2M} n\big) & k \text{ even},\ k = 1, ..., M-1 \\ \sqrt{2}\, g[n-M] \cos\big(2\pi \frac{k}{2M} n\big) & k \text{ odd},\ k = 1, ..., M-1 \\ g[n-rM]\,(-1)^n & k = M \end{cases}$$

and

$f'_k[n] =$

$$\begin{cases} \sqrt{2}\, g[n-M]\, \sin\!\left(2\pi \frac{k}{2M} n\right) & k \text{ even},\ k = 1, ..., M-1 \\ \sqrt{2}\, g[n]\, \sin\!\left(2\pi \frac{k}{2M} n\right) & k \text{ odd},\ k = 1, ..., M-1, \end{cases}$$

where r is as before. The synthesis filters and Wilson synthesis functions are related as

$$f_k[n] = \begin{cases} \psi_{0,0}[n] & k = 0 \\ \psi_{0,k}[n] & k \text{ even},\ k = 1, ..., M-1 \\ \psi_{1,k}[n] & k \text{ odd},\ k = 1, ..., M-1 \\ \psi_{0,M}[n] & k = M, \end{cases}$$

$$f'_k[n] = \begin{cases} \psi_{1,k}[n] & k \text{ even},\ k = 1, ..., M-1 \\ \psi_{0,k}[n] & k \text{ odd},\ k = 1, ..., M-1. \end{cases}$$

Analogous relations exist between the analysis filters $h_k[n]$, $h'_k[n]$ and the Wilson analysis functions $\phi_{0,k}[n]$, $\phi_{1,k}[n]$.

The Wilson filter bank consists of two partial filter banks, one having $M+1$ channels ($h_k[n]$, $f_k[n]$, $k = 0, ..., M$) and the other one having $M-1$ channels ($h'_k[n]$, $f'_k[n]$, $k = 1, ..., M-1$). The filters $h_0[n]$ and $h_M[n]$, and $f_0[n]$ and $f_M[n]$, have half the bandwidth of the other filters; this is a difference from the cosine-modulated filter banks proposed so far [13]. We note, however, that a similar filter bank has recently been proposed in [28].

A Wilson filter bank corresponding to an orthonormal Wilson basis is paraunitary, i.e., orthogonal, while a Wilson filter bank with different analysis and synthesis prototypes is biorthogonal. It can easily be checked that for $g[n] = h[n]$ a Wilson filter bank is paraunitary, i.e., $h_k^*[-n] = f_k[n]$ and $h_k'^*[-n] = f'_k[n]$. We have seen in Section 4 that tight Wilson frames can be constructed from tight WH frames. Since WH frames correspond to DFT filter banks [22]-[24] and tight frames correspond to paraunitary filter banks [22]-[26], the procedure described in Section 4 can be seen as constructing paraunitary Wilson filter banks from paraunitary DFT filter banks. In a Wilson filter bank the subband signals are guaranteed to be real if the input signal and the analysis prototype $h[n]$ are both real; this is an important difference from DFT filter banks.

6. CONCLUSION

We presented a discrete-time framework for the recently introduced Wilson expansions. Specifically, we demonstrated the derivation of complete Wilson sets and Wilson frames (oversampled or critically sampled) from WH sets and frames, respectively. We also established a correspondence of Wilson expansions to a new class of cosine-modulated filter banks.

We finally note that our construction procedure can be generalized to the derivation of *multi-window* Wilson bases and frames from multi-window WH frames [8].

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