TIME-FREQUENCY FORMULATION AND DESIGN OF OPTIMAL DETECTORS

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Abstract—We present time-frequency (TF) formulations of optimal detectors for various nonstationary detection scenarios involving underspread processes. These TF formulations yield a better understanding of optimal detectors and simple TF design procedures.

1 INTRODUCTION

We consider the detection of a signal $s(t)$ corrupted by nonstationary noise $n(t)$. This can be formulated as a hypothesis test $H_0$: $r(t) = n(t)$ versus $H_1$: $r(t) = s(t) + n(t)$, where $r(t)$ is the observed signal. The noise is assumed to be Gaussian, zero-mean, and circular complex with correlation function $R_n(t, t') = E[n(t) n^*(t')]$ or, equivalently, correlation operator $R_n$. We note that $n(t) \in S_0$, where the noise space $S_0 = \mathbb{R}(R_n)$ is the range of $R_n$.

It is well known that the optimal detectors use the observed signal $r(t)$ to form a (sufficient) test statistic $A(r)$ which is compared to a threshold to obtain the actual decision $[1] - [3]$. We shall distinguish the following two cases.

Case 1: Deterministic signal. The signal $s(t)$ is modeled as deterministic and known. We assume that $s(t) \in S_0$ since otherwise perfect detection would be possible [2]. Hence, there is also $r(t) \in S_0$, so that the noise space $S_0$ is simultaneously the observation space. The optimal detector in this case is the likelihood ratio detector [1] - [3] whose test statistic is the linear functional

$$A_0(r) = \text{Re}(R_n^{-1} r, s).$$

(1)

The performance of this detector is completely characterized by its detection $d_2^2 = \mathbb{E}(R_n^{-1} s, s).$ (2)

Case 2: Random signal. The signal $s(t)$ is a non-stationary, zero-mean, circular complex random process with correlation operator $R_s$. Furthermore, $s(t)$ and $n(t)$ are uncorrelated. To exclude perfect detection, we assume $S_0 \subseteq S_1$ where $S_1 = \mathbb{R}(R_s)$ is the signal space.

There exist two important optimal detectors. The likelihood ratio detector assumes $s(t)$ to be Gaussian; the corresponding test statistic is the quadratic form

$$A_l(r) = (H_l r, r)$$

(3)

with the operator $H_l$ given by

$$H_l = R_n^{-1} - (R_s + R_n)^{-1} = R_n^{-1} R_s (R_s + R_n)^{-1},$$

(4)

or $H_l = R_n^{-1} H_{opt}$ where $H_{opt} = R_s (R_s + R_n)^{-1}$ is the nonstationary Wiener filter [1, 5]. The second optimal test statistic, defined as the quadratic test statistic maximizing the detection, is given by the quadratic form

$$A_d(r) = \langle H_d r, r \rangle$$

(5)

This detector does not assume the signal to be Gaussian. The (maximum) detection achieved is given by

$$d_2^2 = \mathbb{tr}\{r^* \}$$

(6)

where $r$ is known as SNR operator [3].

Stationary case. In the limiting (asymptotic) case of stationary processes, the test statistics (1), (3), and (5) and the deflections (2) and (6) can be expressed in the frequency domain as

$$A_0(r) = \int \frac{\text{Re}\{R(f) S^*(f)\}}{P_s(f)} \, df,$$

$$d_2^2 = 2 \int \mathbb{E}(|S(f)|^2) \, df,$$

(7)

$$A_l(r) = \int \frac{P_s(f)}{P_s(f) + P_n(f)} |R(f)|^2 \, df,$$

(8)

$$A_d(r) = \int \frac{P_s(f)}{P_s(f) + P_n(f)} |R(f)|^2 \, df,$$

(9)

where $R(f)$ and $S(f)$ are the Fourier transforms of $r(t)$ and $s(t)$, respectively, and $P_s(f)$ and $P_n(f)$ are the power spectral densities of $s(t)$ and $n(t)$, respectively. These frequency-domain expressions involve simple scalar products and inverses (instead of operator products and inverses) and are hence easily interpreted. They also allow a simple frequency-domain design of optimal detectors.

Outline of paper. Most previous papers on the subject (e.g., [6, 11]) have described exact time-frequency (TF) implementations of test statistics. In contrast, we propose approximate TF formulations which extend the simple frequency-domain expressions (7)-(9) obtained in the stationary case to the practically important class of "underspread" nonstationary processes [8, 9, 5]. Section 2 summarizes the TF tools required. Sections 3 and 4 discuss the TF formulation and TF design of the optimal detectors for a deterministic signal and for a random signal, respectively. Optimal signal design is reformulated as a TF signal synthesis problem. Section 5 extends the TF detector design to the case of a random signal with reduced a priori knowledge. The close-to-optimal performance of the TF designed detectors is verified using computer simulations.

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$^1$The correlation operator $R_n$ is the linear operator whose kernel is the correlation function $R_n(t, t')$.

$^3$The inner product is defined as $\langle x, y \rangle = \int x(t) y^*(t) \, dt$. Integrals are from $-\infty$ to $\infty$ unless stated otherwise. All inverse operators (e.g., $R_n^{-1}$) are to be understood as (pseudo-)inverses on the observation space $S_0$.

$^4$The deflection of a test statistic $A(r)$ is defined as $d^2 = \mathbb{E}(A(r)^2) - \mathbb{E}(A(r))^2$ where $\mathbb{E}$ and $\mathbb{V}$ are the conditional expectation and variance, respectively, under hypothesis $H_i$ [1, 4].
2 TIME-FREQUENCY FUNDAMENTALS

Weyl symbol and spreading function. Our TF formulations will be based on the Weyl symbol (WS). The WS of a linear operator (linear, time-varying system) \( H \) is a "time-frequency response" defined as [10]-[12]

\[
L_H(t, f) = \int H \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) e^{-j2\pi \tau f} d\tau
\]

where \( H(t, t') \) is the kernel (impulse response) of \( H \). This kernel can be re-obtained from \( L_H(t, f) \) according to

\[
H(t, t') = \int L_H \left( t + \frac{t'}, \frac{t + t'}{2} \right) e^{j2\pi f(t-t')} df.
\]

Using the WS, bilinear forms can be expressed as

\[
\langle H, y \rangle = \langle L_H, W \rangle = \int L_H(t, f) W^*_y(t, f) dt df,
\]

where \( W_y(t, f) = \int y(t + \frac{\tau}{2}) \left( \frac{\tau}{2} \right) e^{-j2\pi \tau f} d\tau \) is the cross Wigner distribution of \( y(t) \) and \( x(t) \) [13].

The spreading function of an operator \( H \),

\[
S_H(\tau, \nu) = \int L_H \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) e^{-j2\pi \nu \tau} d\tau,
\]

is the 2-D Fourier transform of the WS and can be used to describe the TF displacements caused by \( H \) [14].

Wigner-Ville spectrum and expected ambiguity function. If \( H = R_z \) is the correlation operator of a non-stationary random process \( x(t) \), then its WS is known as the Wigner-Ville spectrum (WVS) of \( x(t) \) [15],

\[
W_x(t, f) = L_{R_z}(t, f) = \int R_z(t + \frac{\tau}{2}, t - \frac{\tau}{2}) e^{-j2\pi \nu \tau} d\tau.
\]

The WVS describes the average TF energy distribution of \( x(t) \) [15]. Furthermore, the spreading function of \( R_z \) equals the expected ambiguity function (EAF) of \( x(t) \) [8],

\[
\tilde{A}_x(\tau, \nu) = S_{R_z}(\tau, \nu) = \int R_z \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) e^{-j2\pi \nu \tau} d\tau,
\]

which can be interpreted as a TF correlation function [8].

Underspread operators. An linear operator \( H \) is underspread if its spreading function \( S_H(\tau, \nu) \) is effectively supported within a small region \( \Delta H \) about the origin of the \((\tau, \nu)\)-plane [8, 9]. An underspread operator causes only small TF displacements. Two operators are jointly underspread if their spreading functions are effectively supported within the same small region about the origin. For jointly underspread operators \( H_1, H_2 \), it can be shown that \[9\]

\[
L_{H_1 H_2}(t, f) \approx L_{H_1}(t, f) L_{H_2}(t, f).
\]

Simulation results suggest that an underspread operator \( H \) is jointly underspread with its pseudo-inverse \( H^{-1} \), so that \( L_{H H^{-1}}(t, f) \approx L_{H^{-1} H}(t, f) L_{H}(t, f) \) in the TF region \( G_{\gamma} \) which corresponds to the range \( S_{\gamma} \) of \( H \) in the sense of [16]. Combining with \( H^{-1} H = 1 \), where 1 is the identity operator on \( S_H \), we obtain

\[
L_{H^{-1} H}(t, f) \approx L_{H}(t, f) \quad (t, f) \in G_{\gamma}.
\]

For underspread noise where (15) is a good approximation, there is \( A_{\gamma}(r) \approx A_{\gamma}(r) \) so that the TF designed detector will perform nearly as well as the optimal detector. The deflection of the TF designed detector is easily shown to be the average TF SNR,

\[
\tilde{d}_0 = \frac{1}{G_{\gamma}} \int_{G_{\gamma}} \frac{W_r(t, f)}{W_n(t, f)} dt df = \frac{1}{G_{\gamma}} \int_{G_{\gamma}} SNR(t, f) dt df.
\]

The TF designed test statistic can equivalently be expressed as the linear functional \( A_{\gamma}(r) = Re \{ \tilde{H}_r, s \} \), where \( \tilde{H}_r \) is defined via its WS as \( L_{\tilde{H}_r}(t, f) \triangleq 1/|W_n(t, f), (t, f) \in G_{\gamma} \). The kernel of \( \tilde{H}_r \) can be obtained using (10). This can be viewed as an (approximate) TF implementation of the inversion of \( R_z \) which is computationally attractive as the WS allows an efficient implementation using the FFT. The
TF designed detector has the further advantage that the required a priori knowledge (the noise WVS $W_n(t, f)$) is specified in the intuitively more accessible TF domain.

Optimal signal design. Returning to the optimal test statistic (1), we define the optimal signal $s_\text{opt}(t)$ as the (normalized) signal $s(t)$ maximizing the deflection,

$$s_\text{opt}(t) = \arg \max_{|s| = 1} d_2^2 = \arg \max_{|s| = 1} \langle R_n^{-1} s, s \rangle.$$

Assuming that $R_n$ has finite rank, $s_\text{opt}(t)$ can be shown to be the eigenvector of $R_n$ corresponding to the smallest eigenvalue of $R_n$. With (11), we have equivalently

$$s_\text{opt}(t) = \arg \max_{|s| = 1} \langle L_{R_n^{-1}} s, s \rangle. \quad (18)$$

This can be interpreted as a TF signal synthesis problem [17, 18]. TF signal synthesis is the calculation of the (normalized) signal $s(t)$ whose Wigner distribution, $W_s(t, f)$, is closest to a given "TF model function" $M(t, f)$, i.e.,

$$s_\text{opt}(t) = \arg \min_{|s| = 1} \| M - W_s \| \text{ or, equivalently,}$$

$$s_\text{opt}(t) = \arg \max_{|s| = 1} \langle M, W_s \rangle.$$

This is recognized as our signal design problem (18) with TF model $M(t, f) = L_{R_n^{-1}}^{-1}(t, f)$. For an underspread noise process, there is $L_{R_n^{-1}}^{-1}(t, f) \approx 1/W_n(t, f)$ which shows that the optimum signal will occupy those TF regions where the noise WVS assumes small values.

In a similar manner, we can maximize the deflection (17) of the TF designed detector (16),

$$s_\text{opt}(t) = \arg \max_{|s| = 1} \int \text{SNR}(t, f) \, dt \, df$$

where $\text{SNR}(t, f) = W_s(t, f)/W_n(t, f)$. This corresponds to TF signal synthesis with $M(t, f) = 1/W_n(t, f)$.

4 CASE 2: RANDOM SIGNAL

TF formulation. Assuming that the signal process $s(t)$ and the noise process $n(t)$ are jointly underspread, and reasoning as in the previous section, the WSS of $H_s$ in (4) and $H_n$ in (6) can be approximated as $L_{H_s}(t, f) \approx L_{R_n^{-1}}^{-1}(t, f) = \frac{W_n(t, f)}{W_n(t, f) + W_n(t, f)}$ and $L_{H_n}(t, f) \approx \frac{W_n(t, f)}{W_n(t, f)}$ (for $f \in G_s$), here $G_s \subseteq G_n$ denotes the effective support of $W_n(t, f)$. This results in the approximate TF formulations of the test statistics $\Lambda_i(r)$ in (3) and $\Lambda_d(r)$ in (5),

$$\Lambda_i(r) \approx \int G_s \frac{W_n(t, f)}{W_s(t, f) + W_n(t, f)} W_s(t, f) \, dt \, df, \quad (19)$$

$$\Lambda_d(r) \approx \int G_s \frac{W_s(t, f)}{W_n(t, f)} W_s(t, f) \, dt \, df. \quad (20)$$

Furthermore, the (maximum) deflection achieved by the deflection optimal detector $\Lambda_d(r)$ can be reformulated as

$$d^2 = \int_{G_s} L_t^{2}(t, f) \, dt \, df = \int_{G_s} \text{SNR}^2(t, f) \, dt \, df,$$

with the TF dependent signal to noise ratio defined for $(t, f) \in G_s$ as $\text{SNR}(t, f) = W_s(t, f)/W_n(t, f)$. These expressions generalize the frequency-domain expressions (8) and (9) valid in the stationary case; they reduce to (8) and (9) for stationary signal and noise processes.

TF design. The TF formulations (19) and (20) suggest a TF design resulting in the quadratic test statistics

$$\Lambda_i(r) \approx \int G_s \frac{W_n(t, f)}{W_s(t, f) + W_n(t, f)} W_s(t, f) \, dt \, df, \quad (21)$$

$$\Lambda_d(r) \approx \int G_s \frac{W_s(t, f)}{W_n(t, f)} W_s(t, f) \, dt \, df. \quad (22)$$

For jointly underspread signal and noise, the TF designed test statistics $\Lambda_i(r)$ and $\Lambda_d(r)$ can be expected to perform nearly as well as the optimal test statistics $\Lambda_x(r)$ and $\Lambda_d(r)$, respectively. The deflection achieved by $\Lambda_d(r)$ is given by

$$d^2 = \int_{G_s} \text{SNR}^2(t, f) \, dt \, df = \frac{\| \text{SNR} \|^2}{\| W_s \|^2}.$$

The above TF designed test statistics can be equivalently implemented as quadratic forms, $\Lambda_i(r) = (H_s r, r)$ and $\Lambda_d(r) = (H_d r, r)$, where $H_s$ and $H_d$ are defined via their WSS as $L_{H_s}(t, f) \approx \frac{W_n(t, f)}{W_n(t, f) + W_n(t, f)}$ and $L_{H_d}(t, f) \approx \frac{W_n(t, f)}{W_n(t, f)}$ for $(t, f) \in G_s$.

The TF designed detectors have the advantage that the a priori knowledge required is specified in the TF domain. Furthermore, the operator inversions of $R_n$ and $R_n + R_n$ in (4) and (5) are replaced by computationally less expensive scalar inverions plus WS transforms.

Simulation results. Fig. 1 compares the performance of the optimal likelihood ratio detector (test statistic $\Lambda_x(r)$) with that of the corresponding TF designed detector (test statistic $\Lambda_d(r)$). The nonstationary processes $s(t)$ and $n(t)$ were generated using the TF synthesis method in [19]. The performance results in parts (a)-(h) were obtained by Monte Carlo simulation. It is seen that the TF designed detector closely approximates the optimal detector.

5 EXTENDED TIME-FREQUENCY DESIGN

We now consider a nonstationary random signal $s(t)$ corrupted by stationary white noise with known intensity (power spectral density) $\eta$. The statistics of the signal process $s(t)$ are assumed to be unknown except for the support $A_s$ of the EAF of $s(t)$ (see (13)). This reduced a priori knowledge suffices to calculate a minimum-variance unbiased estimate of the signal WVS $W_s(t, f)$ given by [20]

$$\widehat{W}_s(t, f) = \langle T_r r, r \rangle - \eta \quad \text{with} \quad T_r r = S_r r T_S r^\dagger.$$

Here, $S_r r$ is the TF shift operator, $(S_r r)(t') = s(t') e^{j2\pi ft'}$, and the operator $T$ is defined via its spreading function (see (12)) such that $T_r r (t, \nu) = I_r(t, \nu)$, where $I_r(t, \nu)$ is the indicator function of the EAF support $A_s$, i.e., $I_r(t, \nu) = 1$ for $(t, \nu) \in A_s$ and 0 elsewhere. This WVS estimator can be shown [20] to be unbiased under hypothesis $H_1$ (i.e., when the signal $s(t)$ is actually present). It can also be shown [20] that the variance of $\widehat{W}_s(t, f)$ will be reasonably small if $s(t)$ is underspread.

Substituting the WVS estimate $\widehat{W}_s(t, f)$ for $W_s(t, f)$ and using $W_n(t, f) = \eta$ in the TF designed test statistics $\Lambda_i(r)$ and $\Lambda_d(r)$ (see (21), (22)) yields the test statistics

$$\Lambda_i(r) \approx \frac{1}{\eta} \int G_s \frac{\widehat{W}_s(t, f)}{\widehat{W}_s(t, f) + \eta} W_s(t, f) \, dt \, df,$$

$$\Lambda_d(r) \approx \frac{1}{\eta^2} \int G_s \frac{\widehat{W}_s(t, f)}{\widehat{W}_s(t, f) + \eta} W_s(t, f) \, dt \, df.$$
For $s(t)$ underspread, these test statistics can be expected to perform reasonably well. Note that $\Lambda_2(r)$ allows an intuitive “estimator-correlator” interpretation: an estimate of the signal WVS is computed which is then correlated with the Wigner distribution of the observation $r(t)$.

6 CONCLUSIONS

We presented a framework for the time-frequency formulation, interpretation, and design of optimal detectors for deterministic and random signals. This framework is based on the Weyl symbol and Wigner-Ville spectrum, and is valid for underspread, nonstationary processes. We also extended the time-frequency detector design to include an estimation of the signal’s Wigner-Ville spectrum.

References


