The Power Classes—Quadratic Time–Frequency Representations with Scale Covariance and Dispersive Time-Shift Covariance

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***Abstract*— We consider scale-covariant quadratic time–frequency representations (QTFR's) specifically suited for the analysis of signals passing through dispersive systems. These QTFR's satisfy a scale covariance property that is equal to the scale covariance property satisfied by the continuous wavelet transform and a covariance property with respect to generalized time shifts. We derive an existence/representation theorem that shows the exceptional role of time shifts corresponding to group delay functions that are proportional to powers of frequency. This motivates the definition of the *power classes* (PC's) of QTFR's. The PC's contain the affine QTFR class as a special case, and thus, they extend the affine class. We show that the PC's can be defined axiomatically by the two covariance properties they satisfy, or they can be obtained from the affine class through a warping transformation. We discuss signal transformations related to the PC's, the description of the PC's by kernel functions, desirable properties and kernel constraints, and specific PC members. Furthermore, we consider three important PC subclasses, one of which contains the Bertrand P_k distributions. Finally, we comment on the discrete-time implementation of PC QTFR's, and we present simulation results that demonstrate the potential advantage of PC QTFR's.**

I. Introduction

Quadratic time–frequency representations (QTFR's) are powerful tools for analyzing time-varying signals [1]–[4]. Various QTFR classes are best suited for analyzing signals with certain types of properties. For example, the *shift-covariant class* (Cohen's class with signal-independent kernels) [1]–[6] consists of QTFR's that are covariant to constant (nondispersive) time–frequency shifts of the signal and feature a time–frequency analysis resolution that is independent of the analysis time and frequency ("constant-bandwidth analysis"). As an alternative, the *affine class* [7]–[13] consists of QTFR's that are covariant to time–frequency scalings and constant time shifts. The affine class is a framework for a "constant-Q" or proportional-bandwidth time–frequency analysis, similar conceptually to the analysis performed by the wavelet transform, whose time resolution (frequency resolution) is proportional (inversely proportional) to the analysis frequency. An alternative framework for constant-Q time–frequency analysis is the *hyperbolic class* [14]–[17], which consists of QTFR's that are covariant to time–frequency scalings and hyperbolic time shifts.

In this paper, we extend the affine QTFR class by replacing conventional time shifts with generalized (and potentially dispersive) time shifts. These time shifts are ideally matched to signals and systems with corresponding group delay functions [18]. Let $T_X(t,f)$ denote a QTFR of a signal $x(t)$ with Fourier transform $X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$. In many applications, it is desirable that the QTFR's used be covariant to time-frequency scalings, i.e.,

$$T_{\mathcal{C}_aX}(t,f)=T_X\left(at,\frac{f}{a}\right) \quad \text{with } (\mathcal{C}_aX)(f)=\frac{1}{\sqrt{|a|}}X\left(\frac{f}{a}\right) \quad \forall a\in\mathbb{R}\backslash 0. \qquad (1)$$

Within this class of applications, we consider the specific situation where a signal is passed through a linear, time–invariant system with given group delay characteristic. In particular, this situation arises in physical or technical applications when a wave propagates through a dispersive medium. Here, we desire that the QTFR be covariant to generalized time shifts corresponding to the group delay function of the system or medium, i.e.,

$$T_{\mathcal{D}_cX}(t,f)=T_X(t-c\tau(f),f) \quad \text{with}$$
$$(\mathcal{D}_cX)(f)=e^{-j2\pi c\xi(f/f_r)}X(f), \quad \forall c\in\mathbb{R} \qquad (2)$$

where $f_r>0$ is an arbitrary but fixed reference (normalization) frequency, $\xi(b)$ is a given phase function, and the group delay function of the system, up to a factor c, is

$$\tau(f)=\frac{d}{df}\xi\left(\frac{f}{f_r}\right)=\frac{1}{f_r}\xi'\left(\frac{f}{f_r}\right) \quad \text{with } \xi'(b)=\frac{d}{db}\xi(b).$$

The generalized time-shift operator $\mathcal{D}_c$ can be viewed as an allpass filter with group delay $c\tau(f)$. The real-valued parameter c is a factor multiplying the basic group delay law $\tau(f)$; it expresses the "amount of dispersion," which depends on the

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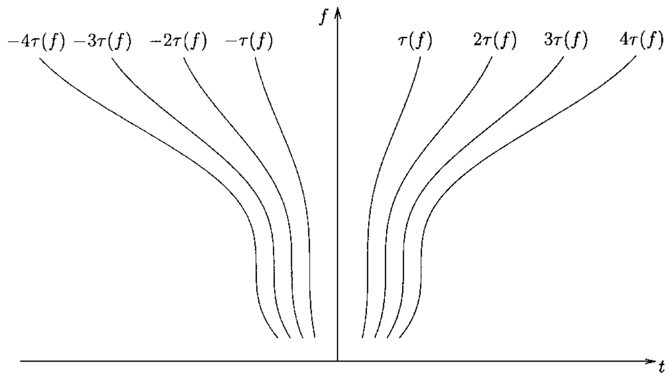

Fig. 1. Generalized time-shift operator $\mathcal{D}_c$ corresponds to a group delay curve $t = c\tau(f)$ in the time–frequency plane. (In this figure, c assumes various positive and negative values.)

length of the path along which the wave propagated through the dispersive medium (see Fig. 1). For example, when an object immersed in water is illuminated by a plane wave [3], [19]–[22], $\tau(f)$ depends on the properties of the medium, whereas c depends on a number of factors, e.g., the object's circumference, its distance from the transmitter/receiver, and the number of times the wave oscillates around the object before it is reflected back to the transmitter/receiver.

This paper considers QTFR's that satisfy the scale covariance property (1) and the generalized time-shift covariance property (2). The scale covariance property of these QTFR's is equal to the scale covariance property satisfied by the continuous wavelet transform. It will be shown that *power-function* group delays play a particularly important role for this type of QTFR's since they are the only functions that lead to maximally wide QTFR classes, i.e., QTFR classes that are parameterized by an arbitrary two-dimensional (2-D) kernel function (see Section II). This will motivate the definition of the *power classes* (PC's) of QTFR's, for which the group delay $\tau(f)$ is described by a power function[1] [16], [17], [23]–[25]. Subsequently, the PC's will be studied in some detail.

We note that two QTFR classes previously introduced are special cases of this generalized framework of scale covariant and generalized time-shift covariant QTFR's:

- The *affine QTFR class* [7]–[13] is obtained with the linear phase function $\xi(b) = b$ and the constant group delay function $\tau(f) \equiv 1/f_r$. In this case, the generalized time-shift covariance property in (2) simplifies to the constant (nondispersive) time-shift covariance property. The Bertrand P_k distributions, which are a subclass of the affine class, also satisfy (2) with a power-function group delay [8], [9].
- The *hyperbolic QTFR class* [14]–[17] is obtained with the logarithmic phase function $\xi(b) = \ln b$ and the hyperbolic group delay function $\tau(f) = 1/f$. In this case, the generalized time-shift covariance property in (2) simplifies to the hyperbolic time-shift covariance property [14].

[1] At this point, we note that alternative methods for a time–frequency analysis of signals localized along a power-law (or, more generally, polynomial) group delay are the time–frequency representations proposed in [26]–[28]. These representations are fundamentally different from PC representations in that they are nonquadratic (i.e., not QTFR's) or defined on a multidimensional frequency space and do not necessarily satisfy the covariance properties (1) and (2).

The paper is organized as follows. In Section II, we consider the general framework of scale covariant and generalized time-shift covariant QTFR's. An existence/representation theorem provides a condition for the existence as well as general expressions of such QTFR's, and it motivates the special emphasis placed on the PC's (satisfying scale and power-dispersive time-shift covariances) in the remainder of this paper. In Section III, we develop the basic theory of the PC's. We show that each PC can be derived from the affine class by a unitary "power warping" mapping similar conceptually to the warping method used to derive the hyperbolic class from Cohen's class [14], [16], [17], [29]–[31]. We discuss the relation of the PC's to a signal expansion into "power impulses," which generalizes the conventional Fourier transform. Finally, we provide explicit expressions ("normal forms") of the PC QTFR's in terms of 2-D kernel functions. Section IV provides a list of desirable properties we might want PC QTFR's to satisfy, along with the corresponding constraints on the 2-D kernels. Section V considers specific PC members. In Section VI, we discuss "localized-kernel" subclasses of the PC's and the intersection of the PC's with the affine class as well as the hyperbolic class. The discrete-time implementation of PC QTFR's is briefly described in Section VII. Finally, simulation results are presented in Section VIII.

II. Scale Covariance and Generalized Time-Shift Covariance

In this section, we consider QTFR's $T_X(t,f)$ satisfying the scale covariance property (1) as well as the generalized time-shift covariance property (2). We first note that any QTFR $T_X(t,f)$ of a signal $X(f)$ can be written in terms of a four-dimensional (4-D) kernel $K_T(t,f;f_1,f_2)$ as [9], [32]

$$T_X(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_T(t,f;f_1,f_2)X(f_1)X^*(f_2)\,df_1\,df_2. \tag{3}$$

The following theorem [23], whose proof is outlined in Appendix A, gives a condition for the existence of and general expressions for QTFR's satisfying the covariance properties in (1) and (2). These expressions correspond to a special format of the 4-D kernel $K_T(t,f;f_1,f_2)$ that is parameterized in terms of a 2-D function $\Gamma_T(b_1,b_2)$.

Theorem 1: A QTFR satisfying the scale covariance (1) and the generalized time-shift covariance (2) for a given phase function $\xi(b)$ or, equivalently, for a given group delay function proportional to $\tau(f) = \frac{1}{f_r}\xi'(\frac{f}{f_r})$ exists if and only if there exists a function $\Gamma_T(b_1,b_2)$ such that

$$\Gamma_T(b_1,b_2)\frac{\xi(\alpha b_1)-\xi(\alpha b_2)}{\alpha\xi'(\alpha)} \equiv \Gamma_T(b_1,b_2)\frac{\xi(b_1)-\xi(b_2)}{\xi'(1)} \tag{4}$$

for all b_1, b_2, α. If (4) is satisfied, then this QTFR satisfying

(1) and (2) is given by[2]

$$T_X(t,f) = \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_T\left(\frac{f_1}{f},\frac{f_2}{f}\right) \times e^{j2\pi\frac{t}{\tau(f)}[\xi(\frac{f_1}{f_r})-\xi(\frac{f_2}{f_r})]}X(f_1)X^*(f_2)\,df_1\,df_2 \quad (5)$$

$$= \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_T\left(\frac{f_1}{f},\frac{f_2}{f}\right) \times e^{j2\pi\frac{tf}{f_r\tau(f_r)}[\xi(\frac{f_1}{f})-\xi(\frac{f_2}{f})]}X(f_1)X^*(f_2)\,df_1\,df_2. \quad (6)$$

The general class of QTFR's satisfying scale covariance and generalized time-shift covariance is thus equivalently formulated by (5) or (6) with the "kernel" $\Gamma_T(b_1,b_2)$ constrained by (4). The equivalence of (5) and (6) follows from (4). Three cases can now be distinguished: $\xi(b)$ may be such that (4) holds for all $\Gamma_T(b_1,b_2)$, or (4) holds for only special choices of $\Gamma_T(b_1,b_2)$, or (4) does not hold for any $\Gamma_T(b_1,b_2)$.

Case 1) A maximally wide QTFR class for a given phase function $\xi(b)$ is obtained when (4) holds for *all* kernels $\Gamma_T(b_1,b_2)$. This means that the phase function $\xi(b)$ must satisfy

$$\frac{\xi(\alpha b_1)-\xi(\alpha b_2)}{\alpha\xi'(\alpha)} \equiv \frac{\xi(b_1)-\xi(b_2)}{\xi'(1)} \quad \forall b_1,b_2,\alpha. \quad (7)$$

It is shown in Appendix B that (7) is satisfied if and only if $\xi(b)$ is proportional to either a power function or a logarithm [33]; in both cases, the basic group delay $\tau(f)=\frac{1}{f_r}\xi'(\frac{f}{f_r})$ will be a power function. The logarithmic form of $\xi(b)$ leads to the *hyperbolic class* considered in [14]–[17]. The power function $\xi(b)$, corresponding to the power-law group delay function $\tau(f)=\frac{\kappa}{f_r}|\frac{f}{f_r}|^{\kappa-1}$ with $\kappa \neq 0$, leads to the *power classes* studied in Sections III through VIII.

Case 2) If the phase function does *not* satisfy (7) but it satisfies (4) for some specific type of kernel $\Gamma_T(b_1,b_2)$, this results in a covariant QTFR class that is less wide than in Case 1 since in Case 1, the kernel $\Gamma_T(b_1,b_2)$ was unconstrained. An important example is the phase function $\xi(b) = b\ln b$ corresponding to the group delay function $\tau(f)=\frac{1}{f_r}(1+\ln\frac{f}{f_r})$. This phase function only satisfies (4) for the special kernel

$$\Gamma_{P_1}(b_1,b_2) = \int_{-\infty}^{\infty}\delta(b_1-\lambda_1(u))\delta(b_2-\lambda_1(-u))\mu(u)\,du \quad (8)$$

where $\lambda_1(u) = \exp(1+\frac{ue^{-u}}{e^{-u}-1})$, and $\mu(u)$ is a real and even function [8], [9]. This kernel is not arbitrary but has a specific form; it satisfies (4) with $\xi(b) = b\ln b$ since it is constrained to have a specific "delta function structure," which is nonzero only at certain combinations of b_1 and b_2 [namely, $b_1=\lambda_1(u)$, $b_2=\lambda_1(-u)$ where $\lambda_1(u)$ is matched to the given phase function $\xi(b)=b\ln b$]. As a consequence, the choice of the 2-D kernel $\Gamma_T(b_1,b_2)$ reduces to the choice of a 1-D weighting function $\mu(u)$. Substituting (8) into (5) or (6) and simplifying yields the class of *Bertrand P_1 distributions* [8], [9]

$$P_{1_X}(t,f) = f\int_{-\infty}^{\infty}X(f\lambda_1(u))X^*(f\lambda_1(-u)) \times e^{j2\pi tf[\lambda_1(u)-\lambda_1(-u)]}\mu(u)\,du, \quad f>0$$

which satisfies the scale covariance and the generalized time-shift covariance for $\xi(b)=b\ln b$. We note that the Bertrand P_1 distributions are also members of the affine class, and hence, they also satisfy the conventional (nondispersive) time-shift covariance property.

Case 3) Finally, the phase function may be such that condition (4) is not satisfied for any kernel $\Gamma_T(b_1,b_2)$. In this case, there does not exist any QTFR that satisfies *both* the scale covariance (1) and the generalized time-shift covariance (2) with the given phase function $\xi(b)$.

Using a warping transformation, it is always possible to construct QTFR's satisfying the generalized time-shift covariance property (2) for arbitrary one-to-one phase functions $\xi(b)$ [16], [17], [24], [34], [35]. However, these QTFR's do not necessarily satisfy the scale covariance property (1).

III. The Power Classes

The discussion in the previous section has shown the exceptional role played by power-law phase and group delay functions. While power-law time shifts and previously proposed hyperbolic time shifts [14]–[17] are not the only types of time shift that are compatible to scalings (as is shown by the Bertrand P_1 distributions), other types of phase or group delay functions lead to narrower classes in which the choice of the 2-D kernel $\Gamma_T(b_1,b_2)$ is restricted *a priori* [e.g., as in (8)].

In the remainder of this paper, we shall therefore consider the specific case where the phase and group delay functions in (4)–(6) are power functions, i.e.,

$$\xi(b) = \xi_\kappa(b) \triangleq \text{sgn}(b)|b|^\kappa, \quad b\in\mathbb{R}$$
$$\tau(f) = \tau_\kappa(f) \triangleq \frac{\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}, \quad f\in\mathbb{R} \quad (9)$$

with $\kappa\neq 0$, where sgn(b) is -1 for $b<0$ and 1 for $b>0$. The functions $\xi_\kappa(b)$ and $\tau_\kappa(f)$ are indexed by a real-valued, nonzero power parameter κ. The phase function $\xi_\kappa(b)$ is a power function extended to $b<0$ such that $\xi_\kappa(b)$ is an odd, strictly monotonic function constituting an invertible mapping from $\mathbb{R}$ to $\mathbb{R}$. The inverse phase function $\xi_\kappa^{-1}(b)$, which is defined by $\xi_\kappa^{-1}(\xi_\kappa(b)) = b$, is $\xi_\kappa^{-1}(b) = \text{sgn}(b)|b|^{1/\kappa} = \xi_{1/\kappa}(b)$. The group delay function $\tau_\kappa(f)$ is an even power function, with power parameter $\kappa-1$. Fig. 2 depicts $\tau_\kappa(f)$

[2] For simplicity, we consider here only "auto-representations" $T_X(t,f)$. However, the extension to "cross-representations" $T_{X,Y}(t,f)$ of two signals $x(t)$ and $y(t)$ can be done in a straightforward manner by replacing $X(f_1)X^*(f_2)$ with $X(f_1)Y^*(f_2)$ in (3), (5), (6), and subsequent QTFR expressions.

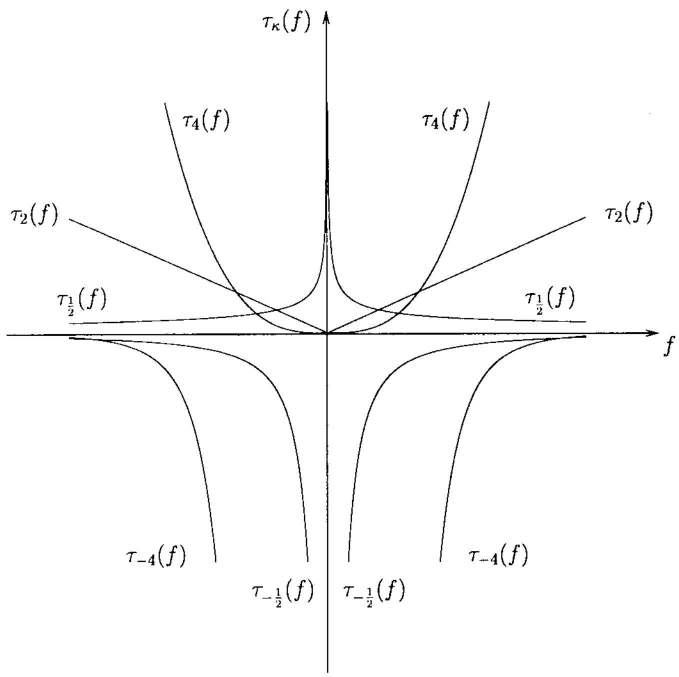

Fig. 2. Power group delay function $\tau_\kappa(f)$ for various choices of the power parameter κ.

for various choices of the power parameter κ. Note that κ determines the curvature of the dispersion characteristic expressed by $\tau_\kappa(f)$.

Dispersive group delays that are a power law or similar to a power law occur in various application areas such as the dispersive propagation of a shock wave in a steel beam $(\kappa = 1/2)$ [36]–[38], trans-ionospheric chirps measured by satellites $(\kappa = -1)$ [39], acoustical waves reflected from a spherical shell immersed in water [19], [20], some cetacean mammal whistles [40], and signals/waves that result as solutions of the diffusion equation $(\kappa = 1/2)$ [41] (e.g., waves propagating along uniform distributed RC transmission lines [42]). Furthermore, power laws can be used to roughly approximate other, more complex, group delays[3].

For $\xi(b) = \xi_\kappa(b)$ and $\tau(f) = \tau_\kappa(f)$ in (9), the generalized time-shift covariance (2) becomes the "power time-shift covariance"

$$T_{\mathcal{D}_c^{(\kappa)}X}(t,f) = T_X(t - c\tau_\kappa(f), f) = T_X\left(t - c\frac{\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}, f\right) \quad (10)$$

with the *power time-shift operator* $\mathcal{D}_c^{(\kappa)}$ defined as

$$\left(\mathcal{D}_c^{(\kappa)}X\right)(f) = e^{-j2\pi c\xi_\kappa\left(\frac{f}{f_r}\right)}X(f) = e^{-j2\pi c\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^\kappa}X(f). \quad (11)$$

It is easily shown that $\xi_\kappa(b)$ satisfies (7) for any $\kappa \neq 0$. Hence, by Theorem 1, there exists a class of QTFR's satisfying the scale covariance (1) and the power time-shift covariance (10) for any given power parameter $\kappa \neq 0$. This QTFR class will be called the *power class associated with the power parameter* κ and abbreviated as PC_κ [16], [17], [23]–[25]. According to (6), QTFR's of PC_κ can be written as

$$T_X^{(\kappa)}(t,f) = \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_T^{(\kappa)}\left(\frac{f_1}{f},\frac{f_2}{f}\right) \times e^{j2\pi\frac{tf}{\kappa}\left[\xi_\kappa\left(\frac{f_1}{f}\right)-\xi_\kappa\left(\frac{f_2}{f}\right)\right]}X(f_1)X^*(f_2)\,df_1\,df_2 \quad (12)$$

where the 2-D kernel $\Gamma_T^{(\kappa)}(b_1,b_2)$ uniquely characterizes the PC_κ QTFR $T^{(\kappa)}$. This kernel is not restricted *a priori*; specific choices of $\Gamma_T^{(\kappa)}(b_1,b_2)$ define specific QTFR's $T^{(\kappa)} \in \mathrm{PC}_\kappa$.

An important special case of the PC's is the *affine class* [7]–[13], which is obtained when $\kappa = 1$, corresponding to the linear phase function $\xi_1(b) = b$ and constant (nondispersive) group delay function $\tau_1(f) \equiv 1/f_r$.

A. Power Warping

The next theorem [23] establishes an important relation between any power class PC_κ and the affine class PC_1. This relation is useful since it permits us to derive the theory of the PC's from the well-known theory of the affine class. The theorem can be verified in a straightforward manner.

Theorem 2: If $T_X^{(A)}(t,f) = T_X^{(1)}(t,f)$ is a QTFR of the affine class PC_1, then the QTFR $T_X^{(\kappa)}(t,f)$ obtained by warping $T_X^{(1)}(t,f)$ according to

$$T_X^{(\kappa)}(t,f) = T_{\mathcal{W}_\kappa X}^{(1)}\left(\frac{t}{f_r\tau_\kappa(f)}, f_r\xi_\kappa\left(\frac{f}{f_r}\right)\right) = T_{\mathcal{W}_\kappa X}^{(1)}\left(\frac{t}{\kappa\left|\frac{f}{f_r}\right|^{\kappa-1}}, f_r\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^\kappa\right) \quad (13)$$

is a member of PC_κ. Here, the frequency axis warping operator $\mathcal{W}_\kappa$ is given by

$$(\mathcal{W}_\kappa X)(f) = \frac{1}{\sqrt{f_r\left|\tau_\kappa\left(f_r\xi_\kappa^{-1}\left(\frac{f}{f_r}\right)\right)\right|}}X\left(f_r\xi_\kappa^{-1}\left(\frac{f}{f_r}\right)\right) = \frac{1}{\sqrt{|\kappa|}\left|\frac{f}{f_r}\right|^{\frac{\kappa-1}{2\kappa}}}X\left(f_r\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^{\frac{1}{\kappa}}\right). \quad (14)$$

Conversely, if $T_X^{(\kappa)}(t,f) \in \mathrm{PC}_\kappa$, then the QTFR

$$T_X^{(1)}(t,f) = T_{\mathcal{W}_\kappa^{-1}X}^{(\kappa)}\left(\frac{t}{f_r\tau_{1/\kappa}(f)}, f_r\xi_{1/\kappa}\left(\frac{f}{f_r}\right)\right) = T_{\mathcal{W}_\kappa^{-1}X}^{(\kappa)}\left(t\kappa\left|\frac{f}{f_r}\right|^{\frac{\kappa-1}{\kappa}}, f_r\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^{\frac{1}{\kappa}}\right) \quad (15)$$

obtained by inversely warping $T_X^{(\kappa)}(t,f)$ is a member of PC_1 (i.e., the affine class). Here, the inverse frequency axis warping operator $\mathcal{W}_\kappa^{-1}$ defined by $(\mathcal{W}_\kappa^{-1}\mathcal{W}_\kappa X)(f) = X(f)$ is given by

$$\left(\mathcal{W}_\kappa^{-1}X\right)(f) = (\mathcal{W}_{1/\kappa}X)(f) = \sqrt{|\kappa|}\left|\frac{f}{f_r}\right|^{\frac{\kappa-1}{2}}X\left(f_r\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^\kappa\right). \quad (16)$$

[3]Whereas polynomial approximations will usually be more accurate than power-law approximations, they are not compatible with the scale covariance property (1). QTFR's and nonquadratic time–frequency representations suited to more general group delays are discussed in [16], [17], [24], [26]–[28], [34], and [35].

Thus, there exists a one-to-one mapping between the affine class PC_1 and all other PC's (see [30] and [31] for a discussion of the underlying principle of "unitary equivalence"). To any affine class QTFR, there exists a corresponding QTFR of PC_κ, and vice versa. This mapping connecting corresponding QTFR's, which is called the *power warping* in the following, consists of i) a unitary, linear signal transform $\mathcal{W}_\kappa$ [see (14)] that is a frequency axis warping according to a power law and ii) an area-preserving,[4] nonlinear time–frequency coordinate transform $(t,f) \rightarrow (\frac{t}{f_r \tau_\kappa(f)}, f_r \xi_\kappa(\frac{f}{f_r}))$ of the QTFR. We note that the transform $\mathcal{W}_\kappa$ has been previously introduced in [43] and [44] in relation to wavelet transforms and that generalizations of the power warping concept are discussed in [16], [17], [24], [30], [31], [34], and [35].

The inverse power warping in (15) equals the power warping in (13) with the power κ replaced by $1/\kappa$. Thus, an algorithm implementing the power warping (13) and (14) can also be used to implement the inverse power warping (15) and (16) simply by substituting $1/\kappa$ for κ. Furthermore, a power warping and an inverse power warping can be combined to obtain a unitary one-to-one mapping from PC_κ to PC_m $(\kappa, m \neq 0)$. This gives the following result: If $T_X^{(\kappa)}(t,f)$ is a member of PC_κ, then

$$T_X^{(m)}(t,f) = T^{(\kappa)}_{\mathcal{W}_{m/\kappa}X}\left(\frac{t}{f_r \tau_{m/\kappa}(f)}, f_r \xi_{m/\kappa}\left(\frac{f}{f_r}\right)\right) = T^{(\kappa)}_{\mathcal{W}_{m/\kappa}X}\left(t\frac{\kappa}{m}\left|\frac{f}{f_r}\right|^{\frac{\kappa-m}{\kappa}}, f_r \operatorname{sgn}(f)\left|\frac{f}{f_r}\right|^{\frac{m}{\kappa}}\right)$$

is a member of PC_m. Thus, there is a one-to-one relation between any two different power classes.

B. Power Signal Expansion

The time-frequency geometry underlying the PC's is related to the family of *power impulses* defined as [23]

$$I_c^{(\kappa)}(f) \triangleq \sqrt{|\tau_\kappa(f)|}\, e^{-j2\pi c \xi_\kappa\left(\frac{f}{f_r}\right)} = \sqrt{\frac{|\kappa|}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}}\, e^{-j2\pi c \operatorname{sgn}(f)\left|\frac{f}{f_r}\right|^{\kappa}} \tag{17}$$

with spectral energy density $|I_c^{(\kappa)}(f)|^2 = |\tau_\kappa(f)|$ and group delay $c\tau_\kappa(f)$. For $\kappa = 1$ (corresponding to the affine class), we obtain the Fourier transform of conventional Dirac impulses $I_c^{(1)}(f) = (1/\sqrt{f_r})e^{-j2\pi cf/f_r}$. Frequency scaling the power impulse $I_c^{(\kappa)}(f)$ results in another power impulse with a scaled parameter c, i.e., $(\mathcal{C}_a I_c^{(\kappa)})(f) = |a|^{-\kappa/2} I^{(\kappa)}_{c/\xi_\kappa(a)}(f)$, whereas power time-shifting $I_c^{(\kappa)}(f)$ results in another power impulse with a shifted parameter, i.e., $(\mathcal{D}_{c_0}^{(\kappa)} I_c^{(\kappa)})(f) = I^{(\kappa)}_{c+c_0}(f)$. The power impulses satisfy the completeness property $\int_{-\infty}^{\infty} I_c^{(\kappa)}(f_1) I_c^{(\kappa)*}(f_2)\, dc = \delta(f_1 - f_2)$ and the orthogonality property $\int_{-\infty}^{\infty} I_{c_1}^{(\kappa)}(f) I_{c_2}^{(\kappa)*}(f)\, df = \delta(c_1 - c_2)$ [23]. From the completeness property, it follows that any square-integrable signal $X(f)$ can be expanded into power impulses as

$$X(f) = \int_{-\infty}^{\infty} \rho_X^{(\kappa)}(c) I_c^{(\kappa)}(f)\, dc = \sqrt{\frac{|\kappa|}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}} \int_{-\infty}^{\infty} \rho_X^{(\kappa)}(c) e^{-j2\pi c \operatorname{sgn}(f)\left|\frac{f}{f_r}\right|^{\kappa}}\, dc \tag{18}$$

where the *power coefficient function* $\rho_X^{(\kappa)}(c)$ is the inner product of $X(f)$ with the power impulse $I_c^{(\kappa)}(f)$:

$$\rho_X^{(\kappa)}(c) = \int_{-\infty}^{\infty} X(f) I_c^{(\kappa)*}(f)\, df = \int_{-\infty}^{\infty} X(f)\sqrt{\frac{|\kappa|}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}}\, e^{j2\pi c \operatorname{sgn}(f)\left|\frac{f}{f_r}\right|^{\kappa}}\, df. \tag{19}$$

It follows that $\rho_X^{(\kappa)}(c) = \frac{1}{\sqrt{f_r}} \int_{-\infty}^{\infty} (\mathcal{W}_\kappa X)(f) e^{j2\pi c \frac{f}{f_r}}\, df$, i.e., the power coefficient function is proportional to the inverse Fourier transform of the frequency warped signal $(\mathcal{W}_\kappa X)(f)$ in (14).

The "power signal expansion" in (18) and (19) constitutes a unitary, linear signal transform $X(f) \leftrightarrow \rho_X^{(\kappa)}(c)$, which reduces to the Fourier transform when $\kappa = 1$ [23], [45]. The relevance of the power expansion and the power coefficient function $\rho_X^{(\kappa)}(c)$ to the PC's will become evident in our subsequent discussion. Basic properties of the power signal expansion are the following:

- Unitarity, which implies the equality of inner products $\int_{-\infty}^{\infty} \rho_{X_1}^{(\kappa)}(c) \rho_{X_2}^{(\kappa)*}(c)\, dc = \int_{-\infty}^{\infty} X_1(f) X_2^*(f)\, df$ (cf. Parseval's theorem).
- Frequency scaling the signal scales the power coefficient function, i.e., $\rho^{(\kappa)}_{\mathcal{C}_a X}(c) = |a|^{\kappa/2} \rho_X^{(\kappa)}(\xi_\kappa(a)c)$.
- Power time shifting the signal produces a simple shift of the power coefficient function, i.e., $\rho^{(\kappa)}_{\mathcal{D}^{(\kappa)}_{c_0} X}(c) = \rho_X^{(\kappa)}(c - c_0)$.
- The power coefficient function of a power impulse $I_{c_0}^{(\kappa)}(f)$ is a Dirac function centered at $c = c_0$, i.e., $\rho^{(\kappa)}_{I^{(\kappa)}_{c_0}}(c) = \delta(c - c_0)$.

C. The Normal Forms

The power warping described in Theorem 2 allows us to reformulate the theory of the affine class [7]–[13] for other PC's. In particular, it has proven convenient to express affine class QTFR's $T_X^{(A)}(t,f) = T_X^{(1)}(t,f)$ in the four "normal forms" [7], [46]

$$T_X^{(1)}(t,f) = |f| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T(f(t'-t), f\tau) u_X(t',\tau)\, dt'\, d\tau \tag{20}$$

$$= \frac{1}{|f|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T\left(\frac{f'}{f}, \frac{\nu}{f}\right) U_X(f',\nu) e^{j2\pi t\nu}\, df'\, d\nu \tag{21}$$

[4] *Area-preserving* means that the coordinate transform's Jacobian is ± 1.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T\left(f(t'-t), \frac{f'}{f}\right) W_X(t', f')\, dt'\, df' \quad (22)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T\left(f\tau, \frac{\nu}{f}\right) A_X(\tau, \nu) e^{j2\pi t\nu}\, d\tau\, d\nu \quad (23)$$

with the "signal products" $u_X(t,\tau) = x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})$ and $U_X(f,\nu) = X(f+\frac{\nu}{2})X^*(f-\frac{\nu}{2})$, the Wigner distribution [47], [48] $W_X(t,f) = \int_{-\infty}^{\infty} u_X(t,\tau)e^{-j2\pi f\tau}\, d\tau$, and the narrowband ambiguity function $A_X(\tau,\nu) = \int_{-\infty}^{\infty} u_X(t,\tau)e^{-j2\pi\nu t}\, dt$. The *kernel functions* $\phi_T(c,\zeta)$, $\Phi_T(b,\beta)$, $\psi_T(c,b)$, and $\Psi_T(\zeta,\beta)$ are interrelated by Fourier transforms as

$$\psi_T(c,b) = \int_{-\infty}^{\infty} \phi_T(c,\zeta)e^{j2\pi b\zeta}\, d\zeta = \int_{-\infty}^{\infty} \Phi_T(b,\beta)e^{-j2\pi c\beta}\, d\beta \quad (24)$$

$$\Psi_T(\zeta,\beta) = \int_{-\infty}^{\infty} \phi_T(c,\zeta)e^{j2\pi\beta c}\, dc = \int_{-\infty}^{\infty} \Phi_T(b,\beta)e^{-j2\pi\zeta b}\, db. \quad (25)$$

Any one of these kernels completely characterizes the affine class QTFR $T_X^{(1)}(t,f)$.

Inserting the above normal forms of the affine class into the warping relation (13) yields the following *normal forms of* PC_κ:

$$T_X^{(\kappa)}(t,f) = \left|\xi_\kappa\left(\frac{f}{f_r}\right)\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T\left(\xi_\kappa\left(\frac{f}{f_r}\right)\left[c - \frac{t}{\tau_\kappa(f)}\right], \xi_\kappa\left(\frac{f}{f_r}\right)\zeta\right) v_X^{(\kappa)}(c,\zeta)\, dc\, d\zeta \quad (26)$$

$$= \frac{1}{\left|\xi_\kappa\left(\frac{f}{f_r}\right)\right|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T\left(\frac{b}{\xi_\kappa\left(\frac{f}{f_r}\right)}, \frac{\beta}{\xi_\kappa\left(\frac{f}{f_r}\right)}\right) \times V_X^{(\kappa)}(b,\beta) e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\, db\, d\beta \quad (27)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T\left(\xi_\kappa\left(\frac{f}{f_r}\right)\left[\frac{t'}{\tau_\kappa(f')} - \frac{t}{\tau_\kappa(f)}\right], \xi_\kappa\left(\frac{f'}{f}\right)\right) \times W_X^{(\kappa)}(t',f')\, dt'\, df' \quad (28)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T\left(\xi_\kappa\left(\frac{f}{f_r}\right)\zeta, \frac{\beta}{\xi_\kappa\left(\frac{f}{f_r}\right)}\right) \times A_X^{(\kappa)}(\zeta,\beta) e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\, d\zeta\, d\beta. \quad (29)$$

Here

$$v_X^{(\kappa)}(c,\zeta) = \frac{1}{f_r} u_{\mathcal{W}_\kappa X}\left(\frac{c}{f_r}, \frac{\zeta}{f_r}\right) = \rho_X^{(\kappa)}\left(c + \frac{\zeta}{2}\right)\rho_X^{(\kappa)*}\left(c - \frac{\zeta}{2}\right)$$

$$V_X^{(\kappa)}(b,\beta) = f_r U_{\mathcal{W}_\kappa X}(f_r b, f_r\beta) = \frac{f_r}{|\kappa|\left|b^2 - \frac{\beta^2}{4}\right|^{\frac{\kappa-1}{2\kappa}}} X\left(f_r\xi_\kappa^{-1}\left(b + \frac{\beta}{2}\right)\right) \times X^*\left(f_r\xi_\kappa^{-1}\left(b - \frac{\beta}{2}\right)\right)$$

where $\mathcal{W}_\kappa$ is the power warping operator in (14), and $\rho_X^{(\kappa)}(c)$ is the power coefficient function in (19). Furthermore $W_X^{(\kappa)}(t,f)$, which is called the "power Wigner distribution" in what follows, is the power-warped version of the Wigner distribution [16], [17], [23] (see also Section V), i.e.,

$$W_X^{(\kappa)}(t,f) = W_{\mathcal{W}_\kappa X}\left(\frac{t}{f_r\tau_\kappa(f)}, f_r\xi_\kappa\left(\frac{f}{f_r}\right)\right) \quad (30)$$

and $A_X^{(\kappa)}(\zeta,\beta)$ is essentially the ambiguity function of the warped signal, i.e., $A_X^{(\kappa)}(\zeta,\beta) = A_{\mathcal{W}_\kappa X}(\frac{\zeta}{f_r}, f_r\beta)$. While the above normal forms are fully equivalent to the general expression (12), they are sometimes more convenient to use. The kernels $\Phi_T(b,\beta)$ in (27) and $\Gamma_T^{(\kappa)}(b_1,b_2)$ in (12) are related as

$$\Gamma_T^{(\kappa)}(b_1,b_2) = \left|\xi'_\kappa(\sqrt{|b_1b_2|})\right|\ \Phi_T\left(\frac{\xi_\kappa(b_1)+\xi_\kappa(b_2)}{2}, \xi_\kappa(b_1) - \xi_\kappa(b_2)\right) \quad (31)$$

for all κ. The 2-D kernel functions $\phi_T(c,\zeta)$, $\Phi_T(b,\beta)$, $\psi_T(c,b)$, and $\Psi_T(\zeta,\beta)$ that uniquely characterize each PC member are the *same* kernels as the ones used in the normal forms (20)–(23) to uniquely characterize the corresponding affine class member. They are, thus, interrelated by Fourier transforms as in (24) and (25). The functional form of the PC kernels in (26)–(29) is independent of the power parameter κ; only the arguments of the kernels in (26)–(29) depend on κ. Consequently, corresponding QTFR's of two PC classes with different κ values are characterized by the same kernels. On the other hand, the kernel $\Gamma_T^{(\kappa)}(b_1,b_2)$ in (12) depends on κ [see (31)].

IV. QTFR PROPERTIES AND KERNEL CONSTRAINTS

Some desirable properties that we might want QTFR's of PC_κ to satisfy are listed in Table I, along with associated constraints on the kernels $\phi_T(c,\zeta)$ or $\Phi_T(b,\beta)$. A QTFR of PC_κ satisfies a specific QTFR property in the first column of Table I if its kernel satisfies the associated kernel constraint in the second column. For example, the power Wigner distribution $W_X^{(\kappa)}(t,f)$ satisfies properties $\mathcal{P}_1$–$\mathcal{P}_{12}$ in Table I since its kernel $\Phi_{W^{(\kappa)}}(b,\beta) = \delta(b-1)$ satisfies all of the corresponding kernel constraints. (This holds for any $\kappa \neq 0$.) Several of the QTFR properties are closely related to the power-law time–frequency geometry described by the group delay function $\tau_\kappa(f)$, the power impulse $I_c^{(\kappa)}(f)$, and the power coefficient function $\rho_X^{(\kappa)}(c)$:

- The *power marginal property* $\mathcal{P}_6$ states that integration of the QTFR $T_X^{(\kappa)}(t,f)$ over a curve $t = c\tau_\kappa(f)$ yields the squared magnitude of $\rho_X^{(\kappa)}(c)$.
- The *power localization property* $\mathcal{P}_8$ states that the QTFR of a power impulse $I_c^{(\kappa)}(f)$ is perfectly concentrated along the group delay curve $t = c\tau_\kappa(f)$.

TABLE I

DESIRABLE QTFR PROPERTIES AND CORRESPONDING KERNEL CONSTRAINTS IN THE POWER CLASSES AND LOCALIZED-KERNEL POWER SUBCLASSES. (THE LOCALIZED-KERNEL POWER SUBCLASSES ARE DISCUSSED IN SECTION VI-A.) NOTE THAT $G'_T(0) = \frac{d}{d\beta} G_T(\beta)|_{\beta=0}$, AND $\mathcal{C}_a$, $\mathcal{D}_c^{(\kappa)}$, $\tau_\kappa(f)$, $\rho_X^{(\kappa)}(c)$, AND $I_c^{(\kappa)}(f)$ ARE DEFINED IN (1), (11), (9), (19), AND (17), RESPECTIVELY

QTFR PROPERTY	KERNEL CONSTRAINT	LPC_κ KERNEL CONSTRAINT
$\mathcal{P}_1$ Scale covariance: $T^{(\kappa)}_{\mathcal{C}_a X}(t,f) = T^{(\kappa)}_X\left(at, \frac{f}{a}\right)$	always satisfied	always satisfied
$\mathcal{P}_2$ Power time-shift covariance: $T^{(\kappa)}_{\mathcal{D}_c^{(\kappa)} X}(t,f) = T^{(\kappa)}_X(t - c\tau_\kappa(f), f)$	always satisfied	always satisfied
$\mathcal{P}_3$ Real-valuedness: $T^{(\kappa)}_X(t,f) = T^{(\kappa)*}_X(t,f)$	$\Phi_T(b,\beta) = \Phi_T^*(b,-\beta)$	$F_T(\beta) = F_T(-\beta)$ $G_T(\beta) = G_T^*(-\beta)$
$\mathcal{P}_4$ Energy distribution: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{(\kappa)}_X(t,f)\,dt\,df = \int_{-\infty}^{\infty} \|X(f)\|^2 df$	$\int_{-\infty}^{\infty} \Phi_T(b,0)\frac{db}{\|b\|} = 1$	$G_T(0) = \|F_T(0)\|$
$\mathcal{P}_5$ Frequency marginal: $\int_{-\infty}^{\infty} T^{(\kappa)}_X(t,f)\,dt = \|X(f)\|^2$	$\Phi_T(b,0) = \delta(b-1)$	$G_T(0) = F_T(0) = 1$
$\mathcal{P}_6$ Power marginal: $\int_{-\infty}^{\infty} T^{(\kappa)}_X(c\tau_\kappa(f), f)\,\|\tau_\kappa(f)\|df = \|\rho^{(\kappa)}_X(c)\|^2$	$\int_{-\infty}^{\infty} \Phi_T(b,\alpha b)\frac{db}{\|b\|} = 1, \;\; \forall\alpha$	$\int_{-\infty}^{\infty} \delta(b - F_T(\alpha b))$ $\cdot\, G_T(\alpha b)\frac{db}{\|b\|} = 1$
$\mathcal{P}_7$ Frequency localization: $X(f) = \delta(f - \hat{f}) \;\Rightarrow\; T^{(\kappa)}_X(t,f) = \delta(f - \hat{f})$	$\Phi_T(b,0) = \delta(b-1)$	$G_T(0) = F_T(0) = 1$
$\mathcal{P}_8$ Power localization: $T^{(\kappa)}_{I_c^{(\kappa)}}(t,f) = \|\tau_\kappa(f)\|\,\delta(t - c\tau_\kappa(f))$	$\int_{-\infty}^{\infty} \Phi_T(b,\beta)\,db = 1, \;\; \forall\beta$	$G_T(\beta) = 1$
$\mathcal{P}_9$ Moyal's formula/unitarity: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{(\kappa)}_{X_1}(t,f)\,T^{(\kappa)*}_{X_2}(t,f)\,dt\,df =$ $\left\|\int_{-\infty}^{\infty} X_1(f)\,X_2^*(f)\,df\right\|^2$	$\int_{-\infty}^{\infty} \Phi_T^*(b\beta,\alpha\beta)\,\Phi_T(\beta,\alpha\beta)\,d\beta$ $= \delta(b-1), \quad \forall\alpha$	$\int_{-\infty}^{\infty} \delta(b\beta - F_T(\alpha\beta))$ $\cdot\,\delta(\beta - F_T(\alpha\beta))\,\|G_T(\alpha\beta)\|^2 d\beta$ $= \delta(b-1), \quad \forall\alpha$
$\mathcal{P}_{10}$ Group delay: $\frac{\int_{-\infty}^{\infty} t\,T^{(\kappa)}_X(t,f)\,dt}{\int_{-\infty}^{\infty} T^{(\kappa)}_X(t,f)\,dt} = -\frac{1}{2\pi}\frac{d}{df}\arg X(f)$	$\Phi_T(b,0) = \delta(b-1)$ and $\frac{\partial}{\partial\beta}\Phi_T(b,\beta)\Big\|_{\beta=0} = 0$	$G_T(0) = F_T(0) = 1,$ $G'_T(0) = F'_T(0) = 0$
$\mathcal{P}_{11}$ Finite frequency support: $X(f) = 0$ for $f \notin [f_1, f_2] \;\Rightarrow$ $T^{(\kappa)}_X(t,f) = 0$ for $f \notin [f_1, f_2]$	$\Phi_T(b,\beta) = 0$ for $\|\frac{b-1}{\beta}\| > \frac{1}{2}$	$G_T(\beta) = 0$ for $\|F_T(\beta) - 1\| > \|\frac{\beta}{2}\|$
$\mathcal{P}_{12}$ Finite c-support: $\rho^{(\kappa)}_X(c) = 0$ for $c \notin [c_1, c_2] \;\Rightarrow$ $T^{(\kappa)}_X(t,f) = 0$ for $t \notin [c_1\tau_\kappa(f), c_2\tau_\kappa(f)]$	$\phi_T(c,\zeta) = 0$ for $\|\frac{c}{\zeta}\| > \frac{1}{2}$	$\int_{-\infty}^{\infty} G_T(\beta)\,e^{-j2\pi\zeta F_T(\beta)}$ $\cdot\, e^{-j2\pi c\beta} d\beta = 0$ for $\|\frac{c}{\zeta}\| > \frac{1}{2}$
$\mathcal{P}_{13}$ Time-shift covariance: $T^{(\kappa)}_{\mathcal{S}_\tau X}(t,f) = T^{(\kappa)}_X(t - \tau, f)$ with $(\mathcal{S}_\tau X)(f) = X(f)\,e^{-j2\pi\tau f}$	see Section 6.2	see Section 6.2

- The *finite c-support property* $\mathcal{P}_{12}$ states that the QTFR of a signal whose power coefficient function $\rho_X^{(\kappa)}(c)$ is zero outside an interval $[c_1, c_2]$ is itself zero outside the corresponding time–frequency region defined by $t \in [c_1\tau_\kappa(f), c_2\tau_\kappa(f)]$.

From Table I, it can be seen that the group delay property ($\mathcal{P}_{10}$) corresponds to a simple kernel constraint that can easily be satisfied. The time–frequency dual of the group delay property is the instantaneous frequency property [1]–[3]. However, the kernel constraint that corresponds to the instantaneous frequency property (not included here) is very complex. Therefore, it appears that the instantaneous frequency property does not fit well within the PC_κ framework.

If a QTFR in PC_κ, $T_X^{(\kappa)}(t,f)$, satisfies property $\mathcal{P}_i$ with power parameter κ in Table I, then the corresponding QTFR in PC_m, $T_X^{(m)}(t,f)$, will satisfy the corresponding property $\mathcal{P}_i$ with power parameter m in Table I since the kernel constraints in Table I are independent of the power. This fact is especially useful when constructing QTFR's of PC_κ by applying the power warping mapping [see Section III-A] to well-known QTFR's of the affine class PC_1. All affine class properties map into corresponding properties of PC_κ, and all kernel constraints for PC_κ are identical to corresponding kernel constraints for the affine class. For example, the Wigner distribution, $W_X(t,f)$, which is a member of the affine class, satisfies the temporal marginal property $\int_{-\infty}^{\infty} W_X(t,f)\,df = |x(t)|^2$. This property is the "power marginal property" ($\mathcal{P}_6$) for $\kappa = 1$, i.e., the power marginal property for the PC_κ simplifies to the temporal marginal property for the PC_1 (i.e. the affine class). As a result, the power Wigner distribution $W_X^{(\kappa)}(t,f)$ must satisfy the power marginal property $\mathcal{P}_6$ for all $\kappa \neq 0$ since $W_X^{(\kappa)}(t,f)$ is the PC_κ QTFR corresponding to $W_X(t,f) = W_X^{(1)}(t,f)$.

TABLE II
KERNELS AND PROPERTIES (DEFINED IN TABLE I) OF SOME QTFR'S OF THE POWER CLASSES FOR ALL $\kappa \neq 0$. NOTE THAT $r(c) \leftrightarrow R(\beta)$ AND $h(\zeta) \leftrightarrow H(b)$ ARE FOURIER TRANSFORM PAIRS

PC_κ QTFR	KERNELS	PROPERTIES SATISFIED
Power Wigner distribution, $W^{(\kappa)}$	$\phi_{W^{(\kappa)}}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c)$ $\Phi_{W^{(\kappa)}}(b,\beta) = \delta(b-1)$ $\psi_{W^{(\kappa)}}(c,b) = \delta(c)\,\delta(b-1)$ $\Psi_{W^{(\kappa)}}(\zeta,\beta) = e^{-j2\pi\zeta}$	$\mathcal{P}_1$–$\mathcal{P}_{12}$
Generalized power Wigner distribution, $W^{(\kappa)(\alpha)}$	$\phi_{W^{(\kappa)(\alpha)}}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c-\alpha\zeta)$ $\Phi_{W^{(\kappa)(\alpha)}}(b,\beta) = \delta(b-1+\alpha\beta)$ $\psi_{W^{(\kappa)(\alpha)}}(c,b) = \frac{1}{\lvert\alpha\rvert}\,e^{j2\pi\frac{c(b-1)}{\alpha}}$ $\Psi_{W^{(\kappa)(\alpha)}}(\zeta,\beta) = e^{j2\pi\alpha\zeta\beta}\,e^{-j2\pi\zeta}$	$\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_4$–$\mathcal{P}_9$, $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$ if $\lvert\alpha\rvert < \frac{1}{2}$
Smoothed pseudo power Wigner distribution, $SPW^{(\kappa)}$	$\phi_{SPW^{(\kappa)}}(c,\zeta) = r(c)\,h(-\zeta)$ $\Phi_{SPW^{(\kappa)}}(b,\beta) = H(b)\,R(-\beta)$ $\psi_{SPW^{(\kappa)}}(c,b) = r(c)\,H(b)$ $\Psi_{SPW^{(\kappa)}}(\zeta,\beta) = h(-\zeta)\,R(-\beta)$	$\mathcal{P}_1$, $\mathcal{P}_2$ $\mathcal{P}_3$ if $H(b), r(c) \in R$ $\mathcal{P}_4$ if $R(0)\int_{-\infty}^{\infty} H(b)\frac{db}{\lvert b\rvert} = 1$
Powergram, $S^{(\kappa)}$ with analysis wavelet H	$\phi_{S^{(\kappa)}}(c,\zeta) = v_H^{(\kappa)}(c,-\zeta)$ $\Phi_{S^{(\kappa)}}(b,\beta) = V_H^{(\kappa)}(b,-\beta)$ $\psi_{S^{(\kappa)}}(c,b) = W_H^{(\kappa)}\Big(\tau_\kappa\big(f_r\xi_{1/\kappa}(b)\big)c, f_r\xi_{1/\kappa}(b)\Big)$ $\Psi_{S^{(\kappa)}}(\zeta,\beta) = A_H^{(\kappa)}(-\zeta,-\beta)$	$\mathcal{P}_1$–$\mathcal{P}_3$, $\mathcal{P}_4$ if $\int\limits_{-\infty}^{\infty} \left\lvert\frac{f_r}{f}\right\rvert^\kappa \lvert H(f)\rvert^2 df = 1$
Bertrand distributions, P_κ	$\Gamma_{P_\kappa}^{(\kappa)}(b_1,b_2) =$ $\int_{-\infty}^{\infty} \delta\big(b_1-\lambda_\kappa(u)\big)\,\delta\big(b_2-\lambda_\kappa(-u)\big)\,\mu(u)\,du$ with $\lambda_\kappa(u)$ defined in (34)	$\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{13}$ (further properties: see [9] for constraints in terms of $\mu(u)$)
Power Bertrand distribution, $P_0^{(\kappa)}$	$\phi_{P_0^{(\kappa)}}(c,\zeta) = \int_{-\infty}^{\infty} G_{P_0^{(\kappa)}}(\beta)\,e^{-j2\pi(c\beta+\zeta F_{P_0^{(\kappa)}}(\beta))}\,d\beta$ $\Phi_{P_0^{(\kappa)}}(b,\beta) = G_{P_0^{(\kappa)}}(\beta)\,\delta\big(b-F_{P_0^{(\kappa)}}(\beta)\big)$ $\psi_{P_0^{(\kappa)}}(c,b) = \int_{-\infty}^{\infty} G_{P_0^{(\kappa)}}(\beta)\,\delta\big(b-F_{P_0^{(\kappa)}}(\beta)\big)\,e^{-j2\pi c\beta}\,d\beta$ $\Psi_{P_0^{(\kappa)}}(\zeta,\beta) = G_{P_0^{(\kappa)}}(\beta)\,e^{-j2\pi\zeta F_{P_0^{(\kappa)}}(\beta)}$ with $F_{P_0^{(\kappa)}}(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$, $G_{P_0^{(\kappa)}}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$	$\mathcal{P}_1$–$\mathcal{P}_5$, $\mathcal{P}_7$, $\mathcal{P}_9$–$\mathcal{P}_{11}$. Also $\mathcal{P}_{13}$ but only for $\kappa = 1$
Power Flandrin D distribution, $D^{(\kappa)}$	same as $P_0^{(\kappa)}$, but with $F_{P_0^{(\kappa)}}(\beta)$, $G_{P_0^{(\kappa)}}(\beta)$ replaced by $F_{D^{(\kappa)}}(\beta) = 1+(\frac{\beta}{4})^2$, $G_{D^{(\kappa)}}(\beta) = 1-(\frac{\beta}{4})^2$	$\mathcal{P}_1$–$\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_{10}$
Power passive Unterberger distribution, $PUD^{(\kappa)}$	same as $P_0^{(\kappa)}$, but with $F_{P_0^{(\kappa)}}(\beta)$, $G_{P_0^{(\kappa)}}(\beta)$ replaced by $F_{PUD^{(\kappa)}}(\beta) = \sqrt{1+(\frac{\beta}{2})^2}$, $G_{PUD^{(\kappa)}}(\beta) = \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}$	$\mathcal{P}_1$–$\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$
Power active Unterberger distribution, $AUD^{(\kappa)}$	same as $P_0^{(\kappa)}$, but with $F_{P_0^{(\kappa)}}(\beta)$, $G_{P_0^{(\kappa)}}(\beta)$ replaced by $F_{AUD^{(\kappa)}}(\beta) = \sqrt{1+(\frac{\beta}{2})^2}$, $G_{AUD^{(\kappa)}}(\beta) = 1$	$\mathcal{P}_1$–$\mathcal{P}_5$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$

V. MEMBERS OF THE POWER CLASSES

We next consider specific QTFR's and QTFR families of the PC's [16], [17], [23]. The kernels and properties of these PC QTFR's are summarized in Table II. Most of these QTFR's correspond, via the power warping discussed in Section III-A, to important QTFR's of the affine class.

Power Wigner Distribution: The *Wigner distribution* [47], [48] is a prominent member of the affine class (=PC_1). The corresponding QTFR of PC_κ is the *power Wigner distribution* [cf. (30)], whose second normal form in (27) simplifies to

$$W_X^{(\kappa)}(t,f) = \int_{-\infty}^{\infty} V_X^{(\kappa)}\left(\xi_\kappa\left(\frac{f}{f_r}\right),\beta\right) e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\,d\beta$$
$$= \left|\frac{f}{\kappa}\right| \int_{-\infty}^{\infty} X\left(f\xi_\kappa^{-1}\left(1+\frac{\beta}{2}\right)\right) \times X^*\left(f\xi_\kappa^{-1}\left(1-\frac{\beta}{2}\right)\right) e^{j2\pi\frac{tf}{\kappa}\beta}\,\frac{1}{\left|1-\frac{\beta^2}{4}\right|^{\frac{\kappa-1}{2\kappa}}}\,d\beta.$$

Equivalently, the first normal form in (26) simplifies to

$$W_X^{(\kappa)}(t,f) = \int_{-\infty}^{\infty} v_X^{(\kappa)}\left(\frac{t}{\tau_\kappa(f)},\zeta\right) e^{-j2\pi\xi_\kappa(\frac{f}{f_r})\zeta}\,d\zeta$$
$$= \int_{-\infty}^{\infty} \rho_X^{(\kappa)}\left(\frac{t}{\tau_\kappa(f)}+\frac{\zeta}{2}\right) \times \rho_X^{(\kappa)*}\left(\frac{t}{\tau_\kappa(f)}-\frac{\zeta}{2}\right) e^{-j2\pi\xi_\kappa(\frac{f}{f_r})\zeta}\,d\zeta.$$

The power Wigner distribution reduces to the Wigner distribution when $\kappa = 1$. It has particularly simple kernels (see

Table II), and it satisfies all the desirable QTFR properties from Table I, except for the conventional time-shift covariance property, which is satisfied only for $\kappa = 1$. The third normal form in (28) shows that any QTFR of PC_κ can be derived from $W_X^{(\kappa)}(t,f)$ via a 2-D linear transformation.

Generalized Power Wigner Distribution: The PC_κ QTFR corresponding to the *generalized Wigner distribution* [46] $W_X^{(\alpha)}(t,f) = \int_{-\infty}^{\infty} X(f-\nu(\alpha-\frac{1}{2}))X^*(f-\nu(\alpha+\frac{1}{2}))e^{j2\pi t\nu}\,d\nu$ of the affine class is the *generalized power Wigner distribution,* which is given as

$$\begin{aligned} W_X^{(\kappa)(\alpha)}(t,f) &= \int_{-\infty}^{\infty} v_X^{(\kappa)}\left(\frac{t}{\tau_\kappa(f)}+\alpha\zeta,\zeta\right)e^{-j2\pi\xi_\kappa(\frac{f}{f_r})\zeta}\,d\zeta \\ &= \int_{-\infty}^{\infty} V_X^{(\kappa)}\left(\xi_\kappa\left(\frac{f}{f_r}\right)-\alpha\beta,\beta\right)e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\,d\beta \end{aligned}$$

where α is a real-valued parameter. For $\alpha = 0$, the generalized power Wigner distribution simplifies to the power Wigner distribution, i.e., $W_X^{(\kappa)(0)}(t,f) = W_X^{(\kappa)}(t,f)$.

Smoothed Pseudo Power Wigner Distribution: An important, practical QTFR of the affine class is the *affine-smoothed pseudo Wigner distribution* whose smoothing kernel $\psi_T(c,b)$ in (22) is separable [7]. The corresponding PC_κ QTFR, which is called the *smoothed pseudo power Wigner distribution* in the following, is

$$\begin{aligned} \mathrm{SPW}_X^{(\kappa)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r\left(\xi_\kappa\left(\frac{f}{f_r}\right)\left[\frac{t'}{\tau_\kappa(f')}-\frac{t}{\tau_\kappa(f)}\right]\right) \\ \times H\left(\xi_\kappa\left(\frac{f'}{f}\right)\right)W_X^{(\kappa)}(t',f')\,dt'\,df' \end{aligned}$$

where $r(c)$ and $H(b)$ are two 1-D windows that independently control the smoothing characteristics in the time direction and in the direction along the power group delay curves, respectively [25].

Powergram: The *scalogram*, which is another prominent member of the affine class, is defined as the squared magnitude of the time–frequency version of the wavelet transform [7], [49]:

$$S_X(t,f) = \frac{f_r}{|f|}\left|\int_{-\infty}^{\infty} X(f')\tilde{H}^*\left(f_r\frac{f'}{f}\right)e^{j2\pi tf'}\,df'\right|^2. \quad (32)$$

Here, the Fourier transform of the analysis wavelet $\tilde{H}(f)$ is assumed to be concentrated about $f = f_r$. The corresponding QTFR of PC_κ is the *powergram*

$$S_X^{(\kappa)}(t,f) = \frac{f_r}{|f|}\left|\int_{-\infty}^{\infty} X(f')H^*\left(f_r\frac{f'}{f}\right)e^{j2\pi\frac{t}{\tau_\kappa(f)}\xi_\kappa(\frac{f'}{f_r})}\,df'\right|^2$$

with the analysis wavelet[5] $H(f) = (\mathcal{W}_\kappa^{-1}\tilde{H})(f)$ also concentrated about $f = f_r$. The powergram is the squared magnitude of a "power wavelet transform"

$$\begin{aligned} \mathrm{PWT}_X^{(\kappa)}(t,f) &= \int_{-\infty}^{\infty} X(f')\big(\mathcal{D}_{t/\tau_\kappa(f)}^{(\kappa)}\mathcal{C}_{f/f_r}H\big)^*(f')\,df' \\ &= \sqrt{\frac{f_r}{|f|}}\int_{-\infty}^{\infty} X(f')H^*\left(f_r\frac{f'}{f}\right)e^{j2\pi\frac{t}{\tau_\kappa(f)}\xi_\kappa(\frac{f'}{f_r})}\,df' \end{aligned}$$

defined as the inner product of the signal $X(f)$ with the warped wavelet $(\mathcal{D}_c^{(\kappa)}\mathcal{C}_aH)(f)$, where $c = t/\tau_\kappa(f)$ and $a = f/f_r$. Furthermore, the powergram can be written in terms of the power Wigner distributions of the wavelet and of the signal as

$$\begin{aligned} S_X^{(\kappa)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_H^{(\kappa)}\left(\frac{f}{f_r}\left(t'-\frac{\tau_\kappa(f')}{\tau_\kappa(f)}t\right),f_r\frac{f'}{f}\right) \\ \times W_X^{(\kappa)}(t',f')\,dt'\,df' \end{aligned}$$

which amounts to a specific type of smoothing of the signal's power Wigner distribution. For $\kappa = 1$, the powergram reduces to the scalogram in (32).

Bertrand Distributions: The *Bertrand P_k distributions* [8], [9] are affine class (i.e., PC_1) QTFR's defined as

$$\begin{aligned} P_{k_X}(t,f) = f\int_{-\infty}^{\infty} X(f\lambda_k(u))X^*(f\lambda_k(-u)) \\ \times e^{j2\pi tf[\lambda_k(u)-\lambda_k(-u)]}\mu(u)\,du, \\ k \neq 0, \quad f > 0 \quad (33) \end{aligned}$$

where $\mu(u)$ is an arbitrary weighting function and

$$\begin{aligned} \lambda_k(u) &= \left(k\frac{e^{-u}-1}{e^{-ku}-1}\right)^{\frac{1}{k-1}}, \quad k \neq 0,1 \\ \lambda_1(u) &= \exp\left(1+\frac{ue^{-u}}{e^{-u}-1}\right). \end{aligned} \quad (34)$$

Letting $k = \kappa$, the P_κ distributions satisfy the κth power time-shift covariance (10) in addition to the scale covariance and the conventional time-shift covariance underlying the affine class. Hence, they are simultaneously members of the affine class PC_1 and the κth power class[6] PC_κ. Within PC_κ, a remarkable feature of the Bertrand P_κ distributions is that they all satisfy the conventional (constant) time-shift covariance. This will be further discussed in Section VI-B.

Power Bertrand Distribution: The Bertrand unitary P_0 distribution, which is defined as

$$\begin{aligned} P_{0_X}(t,f) &= f\int_{-\infty}^{\infty} X(f\lambda_0(u))X^*(f\lambda_0(-u))e^{j2\pi tfu}\mu_0(u)\,du \\ &= f\int_{-\infty}^{\infty} U_X\big(fF_{P_0}(\beta),f\beta\big)G_{P_0}(\beta)e^{j2\pi tf\beta}\,d\beta, \quad f > 0 \end{aligned}$$

with $\lambda_0(u) = \frac{u/2}{\sinh(u/2)}e^{\frac{u}{2}}$, $\mu_0(u) = \frac{u/2}{\sinh(u/2)}$, $F_{P_0}(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$, and $G_{P_0}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$, is a prominent member

[5] In order to obtain the powergram with a given analysis wavelet $H(f)$ by using (13) to power-warp the scalogram in (32), we must compute the scalogram using the warped wavelet $\tilde{H}(f) = (\mathcal{W}_\kappa H)(f)$. Note, however, that it is still true that the scalogram and the powergram have the same kernel function $\Phi_T(b,\beta) = f_rU_{\tilde{H}}(f_rb,-f_r\beta) = V_H^{(\kappa)}(b,-\beta)$ since they are corresponding members of the affine class PC_1 and the κth power class PC_κ, respectively.

[6] While k is a power parameter just as κ and we may normally write κ instead of k, there are situations calling for two different notations. Specifically, using (13) to warp $P_{k_X}(t,f)$ in (33) produces a new PC_κ QTFR $P_{k_X}^{(\kappa)}(t,f)$. In general, $P_{k_X}(t,f) \neq P_{k_X}^{(\kappa)}(t,f)$ except for $\kappa = 1$.

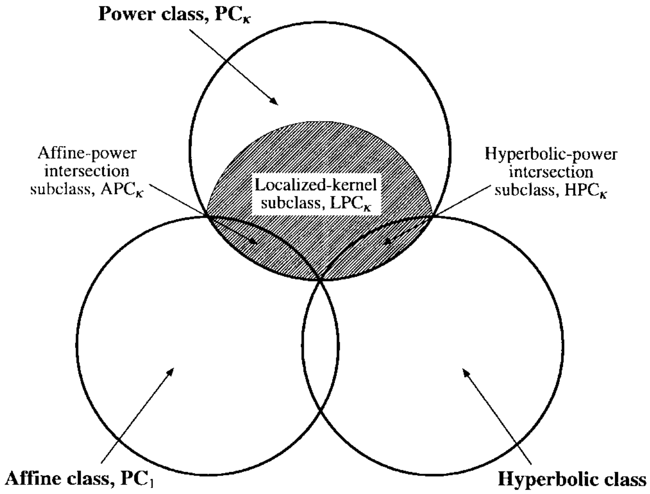

Fig. 3. Pictorial summary of the power classes and their subclasses.

of the affine class [8]–[10], [29], [50], [51]. The corresponding QTFR of PC_κ is the *power Bertrand distribution* [16] whose second normal form in (27) simplifies to

$$P_{0_X}^{(\kappa)}(t,f) = f\int_{-\infty}^{\infty} X\big(f\lambda_0^{(\kappa)}(u)\big)X^*\big(f\lambda_0^{(\kappa)}(-u)\big)e^{j2\pi tfu}\mu_0^{(\kappa)}(u)\,du = \xi_\kappa\left(\frac{f}{f_r}\right)\int_{-\infty}^{\infty} V_X^{(\kappa)}\left(\xi_\kappa\left(\frac{f}{f_r}\right)F_{P_0^{(\kappa)}}(\beta), \xi_\kappa\left(\frac{f}{f_r}\right)\beta\right) \times G_{P_0^{(\kappa)}}(\beta)e^{j2\pi\frac{tf}{\kappa}\beta}\,d\beta, \quad f>0$$

with $\lambda_0^{(\kappa)}(u) = (\frac{\kappa u/2}{\sinh(\kappa u/2)})^{1/\kappa}e^{\frac{u}{2}}$, $\mu_0^{(\kappa)}(u) = (\frac{\kappa u/2}{\sinh(\kappa u/2)})^{1/\kappa}$, $F_{P_0^{(\kappa)}}(\beta) = F_{P_0}(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$, and $G_{P_0^{(\kappa)}}(\beta) = G_{P_0}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$. For $\kappa=1$, $P_{0_X}^{(\kappa)}(t,f)$ reduces to the Bertrand unitary P_0 distribution, i.e., $P_{0_X}^{(1)}(t,f) = P_{0_X}(t,f)$.

Other PC_κ QTFR's include the power Flandrin D distribution $D_X^{(\kappa)}(t,f)$, the power passive Unterberger distribution $\mathrm{PUD}_X^{(\kappa)}(t,f)$, and the power active Unterberger distribution, $\mathrm{AUD}_X^{(\kappa)}(t,f)$ (see Table II).

VI. Subclasses

In this section, we consider three important subclasses of any power class PC_κ. The first subclass is defined by a special kernel structure, whereas the second and third subclasses are defined by an additional covariance property and form the intersection with the affine and hyperbolic class, respectively (see Fig. 3).

A. Localized-Kernel Subclass

The idea of localized kernels was first introduced by the Bertrands in the context of the affine class [50], [52]; this led to the affine localized-kernel subclass further considered in [7], [10], [12], and [53]. In a similar manner, application of the localized-kernel principle within the hyperbolic class led to the definition of the hyperbolic localized-kernel subclass [15]. Here, we will introduce and discuss the *localized-kernel subclass* of PC_κ, denoted LPC_κ, that is conceptually analogous to the affine and hyperbolic localized-kernel subclasses. The LPC_κ consists of all PC_κ QTFR's whose kernel $\Phi_T(b,\beta)$ in (27) is perfectly localized along a curve $b = F_T(\beta)$ in the (b,β) plane:

$$\Phi_T(b,\beta) = G_T(\beta)\delta(b - F_T(\beta)) \tag{35}$$

where $F_T(\beta) \in \mathbb{R}$ and $G_T(\beta)$ are arbitrary 1-D functions that characterize the specific QTFR $T^{(\kappa)}$. Inserting (35) into (27), it is seen that QTFR's of LPC_κ can be expressed as

$$T_X^{(\kappa)}(t,f) = \left|\xi_\kappa\left(\frac{f}{f_r}\right)\right|\int_{-\infty}^{\infty} V_X^{(\kappa)}\left(\xi_\kappa\left(\frac{f}{f_r}\right)F_T(\beta), \xi_\kappa\left(\frac{f}{f_r}\right)\beta\right) \times G_T(\beta)e^{j2\pi\frac{tf}{\kappa}\beta}\,d\beta = \left|\frac{f}{\kappa}\right|\int_{-\infty}^{\infty} X\left(f\xi_\kappa^{-1}\left(F_T(\beta)+\frac{\beta}{2}\right)\right) \times X^*\left(f\xi_\kappa^{-1}\left(F_T(\beta)-\frac{\beta}{2}\right)\right) \times e^{j2\pi\frac{tf}{\kappa}\beta}\frac{G_T(\beta)}{\left|F_T^2(\beta)-\frac{\beta^2}{4}\right|^{\frac{\kappa-1}{2\kappa}}}\,d\beta. \tag{36}$$

For $\kappa=1$, the LPC_κ reduces to the localized-kernel affine subclass [7], [10], [12], [53]. Note, also, that the LPC_κ is obtained when applying the power warping in (13) to the localized-kernel affine subclass LPC_1. The 1-D kernels $F_T(\beta)$ and $G_T(\beta)$ of corresponding QTFR's in LPC_κ and LPC_1 are identical.

The 2-D kernel $\Phi_T(b,\beta)$ in (35) is parameterized by the two 1-D kernels $F_T(\beta)$ and $G_T(\beta)$. This simplifies the theoretical analysis of QTFR's of LPC_κ (e.g., kernel constraints, see Table I). In addition, the kernel structure (35) is also important as it is necessary for the following QTFR property (cf. [15], [53]): For a signal with phase function $c\xi(b)$, the QTFR $T^{(\kappa)}$ is perfectly concentrated along the corresponding group delay curve $\tau(f) = \frac{c}{f_r}\xi'(\frac{f}{f_r})$, i.e.,

$$X_c(f) = z(f)e^{-j2\pi c\xi(\frac{f}{f_r})} \Rightarrow T_{X_c}(t,f) = z^2(f)\delta\left(t - \frac{c}{f_r}\xi'\left(\frac{f}{f_r}\right)\right) \quad \text{for all } c\in\mathbb{R}. \tag{37}$$

Here, $z(f)\geq 0$ is a given amplitude function, and $\xi(b)\in\mathbb{R}$ is a given phase function. The next theorem, whose proof is similar to the proof of an analogous theorem for the hyperbolic localized-kernel subclass [15], states that (37) can only be satisfied by a PC_κ QTFR if its kernel has the localized-kernel structure in (35).

Theorem 3: Let the functions $z(f)\geq 0$ and $\xi(b)\in\mathbb{R}$ be given, and assume that the function $\Xi_{f,\beta}(b) \triangleq \xi(\frac{f}{f_r}\xi_\kappa^{-1}(b+\frac{\beta}{2})) - \xi(\frac{f}{f_r}\xi_\kappa^{-1}(b-\frac{\beta}{2}))$ is one-to-one and differentiable for any (fixed) f,β. Then, the concentration property (37) is satisfied by a PC_κ QTFR $T^{(\kappa)}$ if and only if the following three conditions are satisfied:

1) There exists a function $F_T(\beta)$ that is independent of f and satisfies

$$\Xi_{f,\beta}(F_T(\beta)) = \frac{1}{\kappa}\frac{f}{f_r}\xi'\left(\frac{f}{f_r}\right)\beta \quad \text{for all } f,\beta\in\mathbb{R}. \tag{38}$$

TABLE III
QTFR's BELONGING TO THE LOCALIZED-KERNEL SUBCLASS OF PC_κ, LPC_κ, AND THEIR CORRESPONDING ONE-DIMENSIONAL KERNELS IN (35) AND (36).* WE HAVE NOT FOUND CLOSED-FORM EXPRESSIONS FOR $F_{P_\kappa}(\beta)$ AND $G_{P_\kappa}(\beta)$

LPC_κ QTFR	$F_T(\beta)$ kernel	$G_T(\beta)$ kernel
Power Wigner distribution, $W^{(\kappa)}$	$F_{W^{(\kappa)}}(\beta)=1$	$G_{W^{(\kappa)}}(\beta)=1$
Generalized power Wigner distribution, $W^{(\kappa)(\alpha)}$	$F_{W^{(\kappa)(\alpha)}}(\beta)=1-\alpha\beta$	$G_{W^{(\kappa)(\alpha)}}(\beta)=1$
Bertrand distributions, P_κ	*	*
Power Bertrand distribution, $P_0^{(\kappa)}$	$F_{P_0^{(\kappa)}}(\beta)=\frac{\beta}{2}\coth(\frac{\beta}{2})$	$G_{P_0^{(\kappa)}}(\beta)=\frac{\beta/2}{\sinh(\beta/2)}$
Power Flandrin D distribution, $D^{(\kappa)}$	$F_{D^{(\kappa)}}(\beta)=1+(\frac{\beta}{4})^2$	$G_{D^{(\kappa)}}(\beta)=1-(\frac{\beta}{4})^2$
Power passive Unterberger distribution, $PUD^{(\kappa)}$	$F_{PUD^{(\kappa)}}(\beta)=\sqrt{1+(\frac{\beta}{2})^2}$	$G_{PUD^{(\kappa)}}(\beta)=\frac{1}{\sqrt{1+(\frac{\beta}{2})}}$
Power active Unterberger distribution, $AUD^{(\kappa)}$	$F_{AUD^{(\kappa)}}(\beta)=\sqrt{1+(\frac{\beta}{2})^2}$	$G_{AUD^{(\kappa)}}(\beta)=1$

2) The ratio $\frac{z^2(f)}{Z_{f,\beta}(F_T(\beta))}$ with $Z_{f,\beta}(b) = |b^2 - \frac{\beta^2}{4}|^{\frac{1-\kappa}{2\kappa}} z(f\xi_\kappa^{-1}(b+\frac{\beta}{2}))z(f\xi_\kappa^{-1}(b-\frac{\beta}{2}))$ is independent of f.
3) The kernel $\Phi_T(b,\beta)$ of the PC_κ QTFR $T^{(\kappa)}$ is

$$\Phi_T(b,\beta) = G_T(\beta)\delta(b - F_T(\beta)) \quad \text{with}$$
$$G_T(\beta) = \frac{z^2(f)}{Z_{f,\beta}(F_T(\beta))}.$$

Hence, in order to see if a PC_κ QTFR satisfies the concentration property (37), we first need to check whether a kernel $F_T(\beta)$ exists that satisfies (38). If $F_T(\beta)$ does not exist, then there does not exist any PC_κ QTFR satisfying (37) for the given phase function $\xi(b)$. If a kernel $F_T(\beta)$ satisfying (38) exists, then we have to form the ratio $\frac{z^2(f)}{Z_{f,\beta}(F_T(\beta))}$ and check Condition 2. We note that Condition 2 is always satisfied if $z(f) = z_0|f|^\alpha$ with $z_0 > 0$ and $\alpha \in \mathbb{R}$, in which case, $G_T(\beta) = |F_T^2(\beta) - \frac{\beta^2}{4}|^{\frac{\kappa-1-2\alpha}{2\kappa}}$. Finally, Condition 3 shows that the PC_κ QTFR $T^{(\kappa)}$ must necessarily be a member of LPC_κ [see (35)]. Two noteworthy examples are the following:

- For $\xi(b) = \xi_\kappa(b)$ in (9), Condition 1 is satisfied for any arbitrary function $F_T(\beta)$. Choosing $z(f) = z_0|f|^\alpha$ in order to satisfy Condition 2, Condition 3 yields[7] $G_T(\beta) = |F_T^2(\beta) - \frac{\beta^2}{4}|^{\frac{\kappa-1-2\alpha}{2\kappa}}$.
- For $\xi(b) = \ln b$, Condition 1 is satisfied for $F_T(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$. Choosing $z(f) = z_0|f|^\alpha$ in order to satisfy Condition 2, Condition 3 yields $G_T(\beta) = |\frac{\beta^2}{4}\coth^2(\frac{\beta}{2}) - \frac{\beta^2}{4}|^{\frac{\kappa-1-2\alpha}{2\kappa}} = |\frac{\beta/2}{\sinh(\beta/2)}|^{\frac{\kappa-1-2\alpha}{\kappa}}$.

Some important QTFR's of LPC_κ and their 1-D kernels are listed in Table III. These kernels equal the respective kernels of the corresponding QTFR's of the localized-kernel affine subclass (e.g., $F_{P_0^{(\kappa)}}(\beta)$ within LPC_κ equals $F_{P_0}(\beta)$ within LPC_1).

The kernel constraints corresponding to various QTFR properties (see the second column of Table I) can be reformulated in terms of $F_T(\beta)$ and $G_T(\beta)$, as summarized in the third column of Table I.

[7] Note that if $\alpha = \frac{\kappa-1}{2}$, then $G_T(\beta) = 1$, which is the LPC_κ kernel constraint for the power localization property $\mathcal{P}_8$ in Table I.

B. Affine-Power Intersection Subclass

The conventional, constant (nondispersive) time-shift covariance property

$$T_{\mathcal{S}_\tau X}(t,f) = T_X(t-\tau,f) \quad \text{with}$$
$$(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f}X(f) \tag{39}$$

is of particular importance in many applications. This covariance property is a special case of (10) with $\kappa = 1$. In this subsection, we will consider the PC_κ subclass that consists of all PC_κ QTFR's satisfying the constant time-shift covariance (39). This PC_κ subclass is easily seen to form the intersection of PC_κ with the affine class PC_1 and will therefore be called the *affine-power intersection subclass* of PC_κ or briefly APC_κ; its members will be denoted as $T_X^{(A\cap\kappa)}(t,f)$. Note that APC_1 is simply the affine class or PC_1.

It has been shown in [9] that with the restriction to analytic signals, the APC_κ is given by the Bertrand P_κ distributions in (33). Suitably extending the derivation in [9], it can be shown that for general (nonanalytic) signals, the APC_κ is characterized as follows.

- For $\kappa > 0$ and $\kappa \neq 1$, any QTFR of APC_κ can be written as

$$T_X^{(A\cap\kappa)}(t,f) = T_X^{(\kappa)++}(t,f) + T_X^{(\kappa)--}(t,f) + T_X^{(\kappa)+-}(t,f) + T_X^{(\kappa)-+}(t,f) \qquad -\infty < f < \infty.$$

Here, the component QTFR's $T_X^{(\kappa)ss'}(t,f)$, where s and s' are $+$ or $-$, are given by the expression at the bottom of the next page, where

$$\lambda_\kappa(u) = \left(\kappa\frac{e^{-u}-1}{e^{-\kappa u}-1}\right)^{\frac{1}{\kappa-1}}$$
$$\tilde{\lambda}_\kappa(u) = \left(\kappa\frac{e^{-u}+1}{e^{-\kappa u}+1}\right)^{\frac{1}{\kappa-1}}, \quad \kappa > 0, \quad \kappa \neq 1.$$

We emphasize that $T_X^{(\kappa)++}(t,f)$ is equal to the Bertrand P_κ distributions $P_{\kappa_X}(t,f)$ in (33). While $T_X^{(\kappa)++}(t,f) = P_{\kappa_X}(t,f)$ assigns positive and negative signal frequencies to positive and negative QTFR frequencies, respectively, the other components $T_X^{(\kappa)--}(t,f)$, $T_X^{(\kappa)+-}(t,f)$, and

$T_X^{(\kappa)-+}(t,f)$ involve a crossover of positive and negative frequencies. Note that the four-component APC$_\kappa$ QTFR's are parameterized in terms of the four 1-D functions $\mu_T^+(u)$, $\mu_T^-(u)$, $\tilde{\mu}_T^+(u)$, and $\tilde{\mu}_T^-(u)$.

- For $\kappa < 0$, any QTFR of APC$_\kappa$ can be written as

$$T_X^{(A\cap\kappa)}(t,f) = T_X^{(\kappa)++}(t,f) + T_X^{(\kappa)--}(t,f) \qquad -\infty < f < \infty$$

with $T_X^{(\kappa)++}(t,f) = P_{\kappa X}(t,f)$ and $T_X^{(\kappa)--}(t,f)$ as defined above. Here, $T_X^{(A\cap\kappa)}(t,f)$ is parameterized in terms of the two 1-D functions $\mu_T^+(u)$ and $\mu_T^-(u)$.

It can be shown that the component QTFR's $T_X^{(\kappa)++}(t,f)$ and $T_X^{(\kappa)--}(t,f)$ as well as (assuming invertibility of the even function $\tilde{\lambda}_\kappa(u) + \tilde{\lambda}_\kappa(-u)$ for $u > 0$) $T_X^{(\kappa)+-}(t,f)$ and $T_X^{(\kappa)-+}(t,f)$ are members of the localized-kernel subclass LPC$_\kappa$ introduced in Section VI-A. The QTFR's of APC$_\kappa$, being the sum of all four component QTFR's, are, however, not members of LPC$_\kappa$ in general.

C. Hyperbolic-Power Intersection Subclass

The hyperbolic class [14]–[17] consists of QTFR's that satisfy the scale covariance property in (1) and the hyperbolic time-shift covariance property

$$T_{\mathcal{H}_c X}(t,f) = T_X\big(t - c/f, f\big) \quad \text{with}$$
$$(\mathcal{H}_c X)(f) = e^{-j2\pi c \ln(f/f_r)} X(f), \quad f > 0. \tag{40}$$

This covariance property is a special case of (2) with $\xi(b) = \ln b$ and $\tau(f) = 1/f$. The PC$_\kappa$ subclass that consists of all PC$_\kappa$ QTFR's satisfying the hyperbolic time-shift covariance (40) forms the intersection of the PC$_\kappa$ with the hyperbolic class. It will therefore be called the *hyperbolic-power intersection subclass* of PC$_\kappa$ or, briefly, HPC$_\kappa$ [15]–[17]; its members will be denoted $T_X^{(H\cap\kappa)}(t,f)$. Since the hyperbolic class is only defined for analytic signals and $f > 0$, the HPC$_\kappa$ is likewise defined only for analytic signals and $f > 0$.

By imposing the hyperbolic time-shift covariance (40) on the PC$_\kappa$ QTFR's in (27), the associated condition on the PC$_\kappa$ kernel is obtained as

$$\Phi_T(b,\beta) = G_T(\beta)\delta\left(b - \frac{\beta}{2}\coth\left(\frac{\beta}{2}\right)\right) \tag{41}$$

where the 1-D kernel $G_T(\beta)$ characterizes the HPC$_\kappa$ QTFR $T^{(H\cap\kappa)}$. Note that (41) is a special case of the LPC$_\kappa$ kernel in (35) with $F_T(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$ fixed to the corresponding kernel of the power Bertrand distribution $P_{0_X}^{(\kappa)}(t,f)$ (cf. Table III). Thus, the HPC$_\kappa$ is a subclass of the LPC$_\kappa$ (see Section VI-A) with $G_T(\beta)$ arbitrary and $F_T(\beta) = F_{P_0^{(\kappa)}}(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$. It follows that any HPC$_\kappa$ QTFR can be written as [cf. (36)]

$$T_X^{(H\cap\kappa)}(t,f) = \xi_\kappa\left(\frac{f}{f_r}\right)\int_{-\infty}^{\infty} V_X^{(\kappa)}\left(\xi_\kappa\left(\frac{f}{f_r}\right)\frac{\beta}{2}\coth\left(\frac{\beta}{2}\right)\right.$$
$$\left.\xi_\kappa\left(\frac{f}{f_r}\right)\beta\right) G_T(\beta) e^{j2\pi\frac{tf}{\kappa}\beta}\, d\beta, \quad f > 0.$$

An important QTFR of HPC$_\kappa$ is the power Bertrand distribution $P_{0_X}^{(\kappa)}(t,f)$. It can be shown that any member of HPC$_\kappa$ can be derived from $P_{0_X}^{(\kappa)}(t,f)$ through a scaled temporal convolution:

$$T_X^{(H\cap\kappa)}(t,f) = \frac{f}{|\kappa|}\int_{-\infty}^{\infty} h_T\left(\frac{f}{\kappa}(t-t')\right) P_{0_X}^{(\kappa)}(t',f)\, dt' \qquad f > 0$$

where $h_T(c)$ is a 1-D kernel function whose Fourier transform is $H_T(\beta) = G_T(\beta)/G_{P_0^{(\kappa)}}(\beta)$.

The property kernel constraints for the HPC$_\kappa$ QTFR's equal those for the LPC$_\kappa$ QTFR's in the third column of Table I with $F_T(\beta) = \frac{\beta}{2}\coth(\frac{\beta}{2})$. In particular, the finite frequency support property $\mathcal{P}_{11}$ is always satisfied by all HPC$_\kappa$ QTFR's since $|F_T(\beta) - 1| = |\frac{\beta}{2}\coth(\frac{\beta}{2}) - 1| \le |\beta/2|$.

Fig. 3 provides a pictorial summary of the three PC$_\kappa$ subclasses discussed.

VII. Implementation

Next, we consider the numerical implementation of PC$_\kappa$ QTFR's [25]. Since direct implementation of the QTFR expressions in (26)–(29) using numerical integration is very expensive, we propose to base the discrete implementation of PC$_\kappa$ QTFR's on the warping relations (13) and (14); this allows us to compute PC$_\kappa$ QTFR's using existing efficient algorithms for computing affine class QTFR's [54]. This approach is similar conceptually to that of the Canfield and Jones implementation of hyperbolic class QTFR's [55], [56]. The algorithm we propose consists of discretized versions of the following three steps:

Step 1) a power-law frequency warping of the signal $X(f)$ according to (14);

Step 2) computation of the affine QTFR of the warped signal $T_{\mathcal{W}_\kappa X}^{(1)}(t,f)$;

Step 3) a nonlinear time–frequency coordinate transform according to (13), i.e., $(t,f) \to (\frac{t}{f_r \tau_\kappa(f)}, f_r \xi_\kappa(\frac{f}{f_r}))$.

We note that a generalized discussion of the overall discrete implementation technique can be found in [55].

Step 1) Let[8] $x[n] = x(n\Delta t)$, $n = 0, 1, \ldots, L-1$ be the sampled version of an appropriately bandlimited continuous-time signal $x(t)$ [i.e., $X(f) \approx 0$ for $|f| > 1/(2\Delta t)$]. The corresponding frequency spacing in the length L discrete

[8] We use parentheses $(\cdot)$ to denote continuous time/frequency variables and brackets $[\cdot]$ to denote discrete time/frequency variables.

$$T_X^{(\kappa)ss'}(t,f) = \begin{cases} |f|\int_{-\infty}^{\infty} X(sf\lambda_\kappa(u)) X^*(sf\lambda_\kappa(-u)) e^{j2\pi tfs[\lambda_\kappa(u)-\lambda_\kappa(-u)]}\mu_T^s(u)\, du, & \text{for } s = s' \\ |f|\int_{-\infty}^{\infty} X(sf\tilde{\lambda}_\kappa(u)) X^*(-sf\tilde{\lambda}_\kappa(-u)) e^{j2\pi tfs[\tilde{\lambda}_\kappa(u)+\tilde{\lambda}_\kappa(-u)]}\tilde{\mu}_T^s(u)\, du, & \text{for } s = -s' \end{cases}$$

Fourier transform (FT) is $\Delta f = 1/(L\Delta t)$. In order to reduce approximation errors in the subsequent warping stage, we perform a frequency-domain interpolation by means of time-domain zero padding, i.e., we form the following discrete-time signal of length uL

$$x'[n] = \begin{cases} x(n\Delta t), & n = 0, 1, \ldots, L-1 \\ 0, & n = L, L+1, \ldots, uL-1. \end{cases}$$

The length uL discrete FT of $x'[n]$ is given by $X'[l] \propto X(l/(uL\Delta t))$, $l = 0, 1, \ldots, uL-1$ with reduced frequency sample spacing $\Delta f/u = 1/(uL\Delta t)$.

Next, warped-frequency samples are computed according to (14) using a uniform sampling of the warped-frequency axis. The warped-frequency sample locations are given by $f_m = f_r \xi_{\frac{1}{\kappa}}(\frac{m\Delta v}{f_r})$, $m = 0, 1, \ldots, M-1$, where Δv is the frequency sample spacing of the warped-frequency FT, computed such that $|f_{m+1} - f_m| \le \Delta f, \forall m$ (with Δf as defined above), and M is the resulting number of warped-frequency FT samples required to represent the frequency domain. The FT value at f_m is obtained by linear interpolation of the closest neighbors in the upsampled FT $X'[l]$. We use these FT values to obtain the normalized discrete warped-frequency FT [cf. (14)]

$$Y[m] = \frac{1}{\sqrt{f_r \left|\tau_\kappa\left(f_r \xi_{\frac{1}{\kappa}}\left(\frac{m\Delta v}{f_r}\right)\right)\right|}} X\left(f_r \xi_{\frac{1}{\kappa}}\left(\frac{m\Delta v}{f_r}\right)\right) \qquad m = 0, 1, \ldots, M-1.$$

In order to avoid time-aliasing effects in the subsequent computation of the affine QTFR (e.g. Wigner distribution) in *Step 2)*, we next compress the signal bandwidth to one quarter of the effective sampling rate by inserting M zeros in the center of the frequency period. The resulting warped-frequency FT of length $2M$ will be denoted by $Y'[m]$, $m = 0, 1, \ldots, 2M-1$.

Step 2) A discrete-time, discrete-frequency version $T_{Y'}^{(1)}[q, m]$, $q, m = 0, 1, \ldots, 2M-1$ of the affine QTFR $T_{Y'}^{(1)}(t, f)$ is computed for the warped-frequency signal $Y'[m]$. The underlying time–frequency sampling grid is assumed to be uniform. Efficient algorithms for computing affine QTFR's on uniform grids can be found in [54].

Step 3) We perform a frequency axis warping $T_{Y'}^{(1)}[q, m] \to \hat{T}_{Y'}[q, i]$ and, subsequently, a time axis warping $\hat{T}_{Y'}[q, i] \to \tilde{T}_{Y'}[n, i]$ of the discrete affine QTFR $T_{Y'}^{(1)}[q, m]$ calculated in *Step 2)*. The frequency axis warping $T_{Y'}^{(1)}[q, m] \to \hat{T}_{Y'}[q, i]$ implements a discrete version of the continuous-frequency warping $\hat{T}_{Y'}(t, f) = T_{Y'}^{(1)}(t, f_r \xi_\kappa(\frac{f}{f_r}))$. Hence, for each uniformly spaced frequency sample i, $i = 0, 1, \ldots, L-1$, we need to find the corresponding warped frequency sample m' such that $m'\Delta v = f_r \xi_\kappa(\frac{i\Delta f}{f_r})$. Since the resulting $m' = f_r \xi_\kappa(\frac{i\Delta f}{f_r})/\Delta v$ is not an integer in general, we linearly interpolate $T_{Y'}^{(1)}[q, m]$ to obtain the required nongrid point value.

The subsequent time axis warping $\hat{T}_{Y'}[q, i] \to \tilde{T}_{Y'}[n, i]$ implements a discrete version of the continuous-time warping $\tilde{T}_{Y'}(t, f) = \hat{T}_{Y'}(\frac{t}{f_r \tau_\kappa(f)}, f)$. Thus, we obtain the scaled time sample q' corresponding to each uniformly spaced time sample n, $n = 0, 1, \ldots, L-1$ such that $q'\Delta t = \frac{n\Delta t}{f_r \tau_\kappa(i\Delta f)}$, where i is the discrete frequency index from above. As before, since q' is not an integer in general, we linearly interpolate the discrete QTFR $\hat{T}_{Y'}[q, i]$ to obtain the required nongrid point value. We can now relate this warped QTFR of Y' back to the power QTFR of X as

$$\begin{aligned} T_X^{(\kappa)}[n, i] &= T_X^{(\kappa)}(n\Delta t, i\Delta f) = \tilde{T}_{Y'}(n\Delta t, i\Delta f) \\ &= T_{Y'}^{(1)}\left(\frac{n\Delta t}{f_r \tau_\kappa(i\Delta f)}, f_r \xi_\kappa\left(\frac{i\Delta f}{f_r}\right)\right) \\ &\qquad n, i = 0, 1, \ldots, L-1. \end{aligned}$$

Due to the additional warping steps, the implementation of PC_κ QTFR's is more expensive than that of QTFR's of the affine class (i.e., PC_1 QTFR's). This increased computational complexity is the price to be paid for the better performance of PC_κ QTFR's in analyzing signals with power-dispersive characteristics (see next section).

VIII. Simulation Results

We applied the discrete implementation outlined in the previous section to analyze a two-component signal consisting of two power impulses $I_c^{(\kappa)}(f)$ in (17) with signal power parameter $\kappa_{\text{sig}} = 3$. For computational purposes, the impulses are windowed in the frequency domain. Figs. 4(a) and (b) show the results obtained with the power Wigner distribution and a smoothed pseudo power Wigner distribution with a very short analysis window $r(c)$ (see Section V). Both distributions have power parameter $\kappa_{\text{distr}} = 3$. Note that the power parameter κ_{distr} of the two PC_κ QTFR's is matched to the power parameter κ_{sig} of the power impulses. The power Wigner distribution in Fig. 4(a) has very good time–frequency concentration but large cross terms [25]. These cross terms are effectively suppressed in the smoothed pseudo power Wigner distribution in Fig. 4(b) with hardly any loss of time-frequency concentration. We also show [in Fig. 4(c) and (d)] the results obtained with the Wigner distribution and an affine-smoothed pseudo Wigner distribution, which are both members of the affine class [1], [3], [7] (i.e., both have power parameter $\kappa_{\text{distr}} = 1$). The Wigner distribution in Fig. 4(c) is not matched to the power impulses, displaying complicated cross terms. The affine-smoothed pseudo Wigner distribution in Fig. 4(d) does not remove all the cross terms and has a larger loss of time–frequency concentration than the smoothed pseudo power Wigner distribution in Fig. 4(b). Even though all QTFR's in Fig. 4 are scale covariant, the results of the two PC_3 QTFR's in Fig. 4(a) and (b) are better than those of the corresponding two affine class (PC_1) QTFR's in Fig. 4(c) and (d) because the former two are optimally matched to the $\kappa_{\text{sig}} = 3$ power-law group delays of the power impulse signal components.

In order to further demonstrate the effect of mismatch in the signal and distribution power parameters κ_{sig} and κ_{distr}, Fig. 4(e) and (f) show the results obtained when analyzing the above signal using the power Wigner distribution and a smoothed pseudo power Wigner distribution with power parameter $\kappa_{\text{distr}} = 4$. Note that in Fig. 4(e) and (f), the power parameter of the PC_4 QTFR's, $\kappa_{\text{distr}} = 4$, is different from that of the signal, $\kappa_{\text{sig}} = 3$. The smoothed pseudo power Wigner

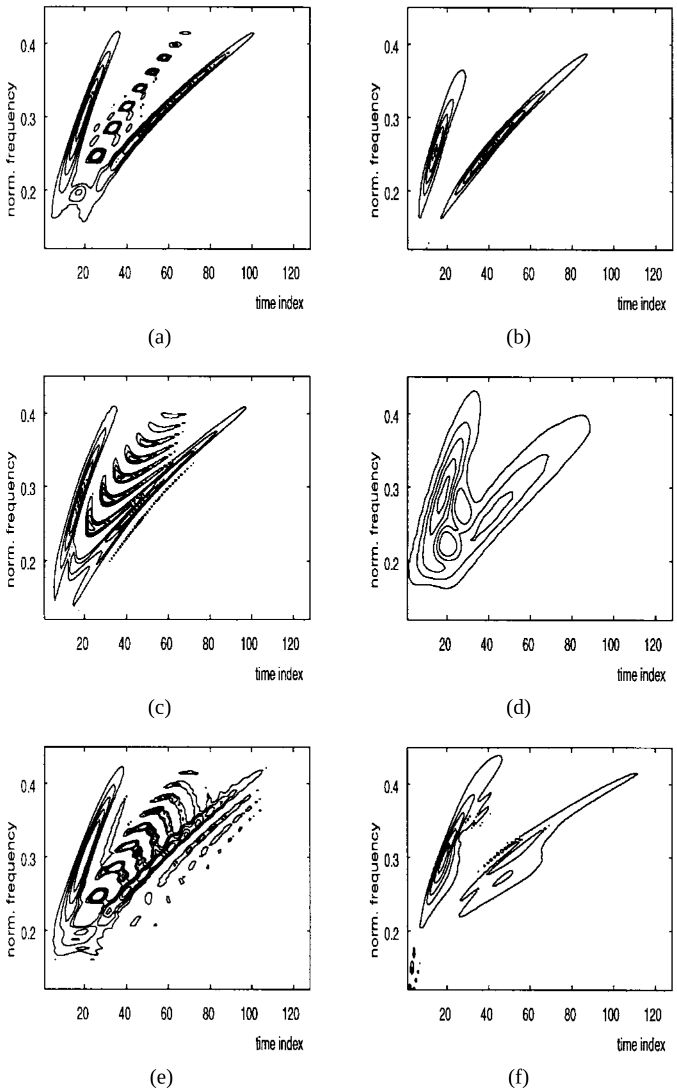

Fig. 4. PC analysis of a two-component analytic signal consisting of two windowed power impulses $I_c^{(\kappa)}(f)$ in (17) with signal power parameter $\kappa_{\rm sig} = 3$. (a) Power Wigner distribution with $\kappa_{\rm distr} = 3$. (b) Smoothed pseudo power Wigner distribution with $\kappa_{\rm distr} = 3$. (c) Wigner distribution $(\kappa_{\rm distr} = 1)$. (d) Affine-smoothed pseudo Wigner distribution $(\kappa_{\rm distr} = 1)$. (e) Power Wigner distribution with $\kappa_{\rm distr} = 4$. (f) Smoothed pseudo power Wigner distribution with $\kappa_{\rm distr} = 4$.

distribution in Fig. 4(f) has better cross term suppression and better time–frequency concentration along the true group delay than the affine-smoothed pseudo Wigner distribution in Fig. 4(d) since the power parameter mismatch in Fig. 4(f) is smaller than in Fig. 4(d) [18], [25].

Next, we demonstrate the use of PC_κ QTFR's in the analysis of real-data signals. Fig. 5 shows two $\mathrm{PC}_{0.35}$ QTFR's and two affine (i.e., PC_1) QTFR's of the measured impulse response[9] of a steel beam with rectangular cross section [36]–[38]. The impulse response was obtained by lightly tapping one end of the steel beam in the direction orthogonal to the flat side of the beam. Bending waves travel along the beam until they are reflected at the free end. They return to the point of impact, are

[9]This impulse response was obtained by J. Woodhouse in a laboratory experiment conducted at Cambridge University. We are grateful to D. Newland and J. Woodhouse for making this data accessible to us.

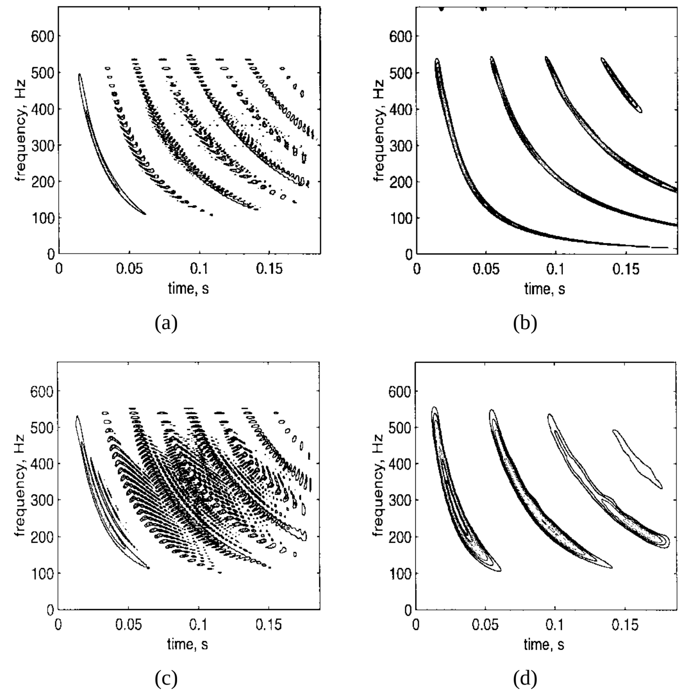

Fig. 5. PC analysis of a bandpass-filtered segment of the measured impulse response of a steel beam. (a) Power Wigner distribution with $\kappa = 0.35$. (b) Smoothed pseudo power Wigner distribution with $\kappa = 0.35$. (c) Wigner distribution $(\kappa = 1)$. (d) Affine-smoothed pseudo Wigner distribution $(\kappa = 1)$. The sampling frequency of the data is 4096 Hz.

reflected again, etc., thereby producing a series of echoes with increasing dispersion. The time–frequency representations in Fig. 5 display a bandpass-filtered segment of the impulse response. As can be seen, the smoothed pseudo power Wigner distribution with $\kappa = 0.35$ in Fig. 5(b) shows better resolution and/or cross term suppression than the other three QTFR's depicted. (Note that the specific value of $\kappa = 0.35$ was chosen empirically to match the time–frequency curvature of the primary reflection.) Short-time power QTFR's were also found to be useful in analyzing cetacean mammal whistles [40].

IX. Conclusion

The *power classes* PC_κ have been introduced as the classes of all quadratic time–frequency representations (QTFR's) satisfying the scale covariance property and a dispersive time-shift covariance property corresponding to a power-law group delay function. This provides a generalized framework for scale covariant time–frequency analysis specifically suited to signals passing through dispersive systems or signals localized along power-law curves in the time-frequency plane. The affine class is a special case of the power classes.

All power classes are "isomorphic" or "unitarily equivalent" [30], [31] in the sense that any QTFR of a given power class is related to a corresponding QTFR of any other power class through a unitary, linear "power warping" mapping. Specifically, any power class can be obtained from the affine class (PC_1) using such a warping. The structural equivalence of all power classes can be attributed to the fact that the composite operator $\mathcal{D}_c^{(\kappa)}\mathcal{C}_a$ on which the respective power classes PC_κ are based is a representation of the same group (the *affine group* [9], [51], [57]–[61]).

Compared with the previously proposed scale covariant QTFR classes (the affine class and the hyperbolic class), the power classes offer increased flexibility of time–frequency analysis. Specifically, as our simulation results have shown, existing *a priori* knowledge about the signal can be put to advantage by appropriately choosing the power parameter κ. On the other hand, there exist some practical limitations of the power classes: They are more expensive to compute than traditional affine class QTFR's, they are matched to a specific time-frequency structure of the signal, and they are not necessarily covariant to conventional (nondispersive) time shifts. Thus, the absolute position of the time origin may be important (see Section VI-B for time-shift covariant PC_κ QTFR's).

We finally note that the aspects of cross term geometry and cross term attenuation through smoothing—aspects that are especially important for practical applications—are discussed in [25]. Furthermore, due to the unitary equivalence of the power classes and the affine class, existing results on regularity and unitarity of affine class QTFR's [13], [62] can directly be applied to PC_κ QTFR's.

Appendix A
Proof of Theorem 1

In order to prove Theorem 1, we first note that the two covariance properties (1) and (2) are strictly equivalent to the combined covariance

$$T_{\mathcal{D}_c \mathcal{C}_a X}(t,f) = T_X\left(a(t - c\tau(f)), \frac{f}{a}\right). \tag{A1}$$

Substituting the general QTFR expression (3) in both sides of (A1), we obtain the kernel relation

$$|a| K_T(t,f;af_1,af_2) e^{j2\pi c[\xi(\frac{af_1}{f_r}) - \xi(\frac{af_2}{f_r})]} = K_T\left(a(t - c\tau(f)), \frac{f}{a}, f_1, f_2\right) \tag{A2}$$

which must be satisfied for all choices of t, f, f_1, f_2, a, c. This is a necessary and sufficient condition for the combined covariance (A1). In particular, setting $a = f/f_r$ and $c = t/\tau(f)$ and carrying out some simple manipulations, (A2) becomes

$$K_T(t,f;f_1,f_2) = \frac{1}{|f|}\Gamma_T\left(\frac{f_1}{f}, \frac{f_2}{f}\right) e^{j2\pi \frac{t}{\tau(f)}[\xi(\frac{f_1}{f_r}) - \xi(\frac{f_2}{f_r})]} \tag{A3}$$

where we have set $\Gamma_T(b_1,b_2) \triangleq f_r K_T(0, f_r; f_r b_1, f_r b_2)$. Note that the kernel expression (A3) yields the QTFR expression in (5). However, (A3) is only a *necessary* condition for (A1) since it was derived from (A2) through a special choice of the parameters t, f, f_1, f_2, a, c. Resubstituting (A3) in (A1), it is seen that (A1) is satisfied only if (4) is met. Thus, if (and only if) there exists a kernel function $\Gamma_T(b_1,b_2)$ satisfying (4), the QTFR constructed according to (A3) or, equivalently, (5) will satisfy the combined covariance (A1). We note that (6) is obtained instead of (5) if we define a combined covariance using the operator composition $\mathcal{C}_a\mathcal{D}_c$ in (A1) instead of $\mathcal{D}_c\mathcal{C}_a$.

Appendix B
Group Delay Functions Satisfying Condition (7)

In the following, it is shown that (7), i.e.,

$$\frac{\xi(\alpha b) - \xi(\alpha b_0)}{\alpha \xi'(\alpha)} \equiv \frac{\xi(b) - \xi(b_0)}{\xi'(1)} \quad \forall b, b_0, \alpha \tag{B1}$$

is only satisfied if the derivative of the phase function $\xi(b)$ is proportional to a power function. The proof given here is a simplified version of a proof due to Flandrin [33].

Taking the derivative with respect to b of both sides of (B1), we obtain $\frac{\alpha\xi'(\alpha b)}{\alpha\xi'(\alpha)} = \frac{\xi'(b)}{\xi'(1)}$, and further, for $\alpha \neq 0$, $\xi'(\alpha b) = \xi'(\alpha)\frac{\xi'(b)}{\xi'(1)}$. Dividing both sides through $\xi'(1)$ yields

$$\overline{\xi'}(\alpha b) = \overline{\xi'}(\alpha)\overline{\xi'}(b) \quad \text{with } \overline{\xi'}(b) \triangleq \frac{\xi'(b)}{\xi'(1)}.$$

This is satisfied if and only if $\overline{\xi'}(b)$ is a power function, i.e., $\overline{\xi'}(b) = b^p$ or $\overline{\xi'}(b) = |b|^p$. We shall adopt the second solution since for $b < 0$ the first solution is not real for all $p \in \mathbb{R}$. It follows that $\xi'(b) = \xi'(1)\overline{\xi'}(b) = K|b|^p$ with $K \neq 0$ arbitrary. For this paper, we substitute $p = \kappa - 1$. For $\kappa \neq 0$, setting $K = \kappa$, and recalling that $\tau(f) = \frac{1}{f_r}\xi'(\frac{f}{f_r})$, we obtain the group delay function of the power classes, $\tau(f) = \frac{\kappa}{f_r}|\frac{f}{f_r}|^{\kappa-1} = \tau_\kappa(f)$. For $\kappa = 0$, setting $K = 1$, we obtain the group delay function $\tau(f) = 1/|f|$; for $f > 0$, this is the group delay function of the hyperbolic class.

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