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9.4 TIME-VARYING POWER SPECTRA OF NONSTATIONARY RANDOM PROCESSES[0]

9.4.1 Nonstationary Random Processes

The second-order statistics of a (generally nonstationary) random process[1] $x(t)$ are characterized by the correlation function $r_x(t,t') = \mathrm{E}\{x(t)\,x^*(t')\}$ (with $\mathrm{E}\{\cdot\}$ denoting expectation). In the special case of a (wide-sense) *stationary* random process, the correlation function is of the form $r_x(t,t') = \tilde{r}_x(t-t')$ and the Fourier transform of $\tilde{r}_x(\tau)$,

$$P_x(f) = \int_{-\infty}^{\infty} \tilde{r}_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \;\geq\; 0\,, \tag{9.4.1}$$

is known as the *power spectral density* (PSD) [11]. The PSD describes the distribution of the process' mean power over frequency f and is extremely useful in statistical signal processing. The time-frequency dual of stationary processes is given by *white* processes with correlation functions of the form $r_x(t,t') = q_x(t)\,\delta(t-t')$. Here, the *mean instantaneous intensity* $q_x(t) \geq 0$ is the time-frequency dual of the PSD.

In many applications, the random signals under analysis are nonstationary and thus do not possess a PSD. Various extensions of the PSD to the nonstationary case have been proposed, such as the *generalized Wigner-Ville spectrum* [1] [3] [4] [5] [6] [8] and the *generalized evolutionary spectrum* [8] [9]. In this article, we will briefly discuss these "time-varying power spectra" and show that they yield satisfactory descriptions for the important class of *underspread* nonstationary processes.

9.4.2 The Generalized Wigner-Ville Spectrum

The *generalized Wigner-Ville spectrum* (GWVS) of a nonstationary process $x(t)$ is defined as [1] [3] [4] [5] [6] [8]

$$\overline{W}_x^{(\alpha)}(t,f) \;\triangleq\; \int_{-\infty}^{\infty} r_x^{(\alpha)}(t,\tau)\, e^{-j2\pi f\tau} d\tau$$

with

$$r_x^{(\alpha)}(t,\tau) \;\triangleq\; r_x\!\left(t + \left(\frac{1}{2}-\alpha\right)\tau,\, t - \left(\frac{1}{2}+\alpha\right)\tau\right), \tag{9.4.2}$$

where α is a real-valued parameter. The GWVS equals the generalized Weyl symbol (see Article 4.7) of the correlation operator $\mathbf{R}_x$ (the linear operator whose kernel

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[1]In what follows, all random processes are assumed to be real or circular complex as well as zero-mean.

is the correlation function $r_x(t,t') = \mathrm{E}\{x(t)\,x^*(t')\}$) and, under mild assumptions, it equals the expectation of the generalized Wigner distribution [5] of $x(t)$. For $\alpha = 0$, the GWVS becomes the ordinary *Wigner-Ville spectrum*, and for $\alpha = 1/2$ it reduces to the *Rihaczek spectrum* [1] [3] [4] [5] [6] [7] [8]:

$$\overline{W}_x^{(0)}(t,f) = \int_{-\infty}^{\infty} r_x\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau\,,$$
$$\overline{W}_x^{(1/2)}(t,f) = \int_{-\infty}^{\infty} r_x(t,t-\tau)\, e^{-j2\pi f\tau}\, d\tau\,.$$

The GWVS $\overline{W}_x^{(\alpha)}(t,f)$ is a complete characterization of the second-order statistics of $x(t)$ since the correlation function $r_x(t,t')$ can be recovered from it. Integration of the GWVS gives the marginal properties

$$\int_{-\infty}^{\infty} \overline{W}_x^{(\alpha)}(t,f)\, dt = \mathrm{E}\{|X(f)|^2\}\,, \qquad \int_{-\infty}^{\infty} \overline{W}_x^{(\alpha)}(t,f)\, df = \mathrm{E}\{|x(t)|^2\}\,,$$

provided that the expectations on the right-hand sides exist. In this sense, the GWVS can be considered as a time-frequency (TF) distribution of the mean energy of $x(t)$. However, in general the GWVS is not real-valued; for $\alpha = 0$, $\overline{W}_x^{(0)}(t,f)$ is real-valued though possibly not everywhere nonnegative. For further interesting properties of the GWVS, see [1] [3] [4] [5] [6] [7] [8].

We next discuss the GWVS of three fundamental types of processes.

- The GWVS of a stationary process with correlation function $r_x(t,t') = \tilde{r}_x(t-t')$ reduces to the PSD $P_x(f)$ for all t, i.e., $\overline{W}_x^{(\alpha)}(t,f) \equiv P_x(f)$.
- The GWVS of a (generally nonstationary) white process with correlation function $r_x(t,t') = q_x(t)\delta(t-t')$ reduces to the mean instantaneous intensity $q_x(t)$ for all f, i.e., $\overline{W}_x^{(\alpha)}(t,f) \equiv q_x(t)$.
- The GWVS of a stationary white process with correlation function $r_x(t,t') = \eta\,\delta(t-t')$ is given by $\overline{W}_x^{(\alpha)}(t,f) \equiv \eta$ (i.e., constant mean energy distribution over the entire TF plane).

These results show that the GWVS is consistent with the PSD of stationary processes and the mean instantaneous intensity of white processes.

The GWVS will be further considered in Section 9.4.6. Before that, we consider an alternative definition of a "time-varying power spectrum" in the next section.

9.4.3 The Generalized Evolutionary Spectrum

The PSD of a stationary random process $x(t)$ can alternatively be defined using an innovations system representation. Here, $x(t)$ is viewed as the output of a linear, time-invariant system $\mathbf{H}$ with impulse response $h(\tau)$ (the *innovations system*) that is driven by stationary white noise $n(t)$ with PSD $P_n(f) \equiv 1$, i.e., $x(t) = (\mathbf{H}n)(t) = \int_{-\infty}^{\infty} h(\tau)\, n(t-\tau)\, d\tau$. The PSD of $x(t)$ can then be written as

$$P_x(f) = |H(f)|^2, \tag{9.4.3}$$

where $H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau$ is the transfer function (frequency response) of $\mathbf{H}$.

A similar innovations system representation is also possible in the nonstationary case. The innovations system $\mathbf{H}$ of a nonstationary random process $x(t)$ is a linear, *time-varying* system defined by $\mathbf{H}\mathbf{H}^+ = \mathbf{R}_x$ (here, the superscript $^+$ denotes the adjoint [10]). Note that $\mathbf{H}$ is not uniquely defined; indeed, all innovations systems can be written as $\mathbf{H} = \mathbf{H}_p\mathbf{U}$ where $\mathbf{H}_p$ is the *positive (semi-) definite* [10] innovations system (which is unique) and $\mathbf{U}$ is a linear operator satisfying $\mathbf{U}\mathbf{U}^+ = \mathbf{I}$ [9].

In analogy to the PSD expression in (9.4.3), the *generalized evolutionary spectrum* (GES) of a nonstationary process $x(t)$ is now defined as [8] [9]

$$G_x^{(\alpha)}(t,f) \triangleq |L_{\mathbf{H}}^{(\alpha)}(t,f)|^2. \tag{9.4.4}$$

Here, $L_{\mathbf{H}}^{(\alpha)}(t,f)$ is the generalized Weyl symbol (see Article 4.7) of an innovations system $\mathbf{H}$ of $x(t)$, i.e.,

$$L_{\mathbf{H}}^{(\alpha)}(t,f) \triangleq \int_{-\infty}^{\infty} h\Big(t + \Big(\frac{1}{2} - \alpha\Big)\tau, t - \Big(\frac{1}{2} + \alpha\Big)\tau\Big)\, e^{-j2\pi f\tau}\, d\tau \tag{9.4.5}$$

where $h(t,t')$ is the kernel of $\mathbf{H}$. Note that the nonuniqueness of $\mathbf{H}$ implies a corresponding nonuniqueness of the GES. For $\alpha = 1/2$, $\alpha = -1/2$, and $\alpha = 0$, the GES reduces to the ordinary *evolutionary spectrum*[2] [12] [13], the *transitory evolutionary spectrum* [2] [9], and the *Weyl spectrum* [9], respectively.

In contrast to the GWVS, the GES is a nonnegative real-valued function. However, it is not a complete second-order description of $x(t)$ since in general the correlation function $r_x(t,t')$ cannot be recovered from it. For $\alpha = \pm 1/2$ and normal innovations system (i.e., $\mathbf{H}$ satisfies $\mathbf{H}\mathbf{H}^+ = \mathbf{H}^+\mathbf{H}$ [10]; note, in particular, that $\mathbf{H}_p$ is always normal), the GES satisfies the marginal properties, i.e.,

$$\int_{-\infty}^{\infty} G_x^{(\pm 1/2)}(t,f)\, dt \;=\; \mathrm{E}\{|X(f)|^2\}, \qquad \int_{-\infty}^{\infty} G_x^{(\pm 1/2)}(t,f)\, df \;=\; \mathrm{E}\{|x(t)|^2\}.$$

Other properties of the GES are discussed in [8] [9].

Next, we consider the GES of our three fundamental types of processes, assuming that the positive (semi-) definite innovations system $\mathbf{H}_p$ is used in the GES definition (9.4.4).

[2]We note that Priestley's original definition of the evolutionary spectrum was based on a conceptually different approach using "oscillatory processes" [12] [13].

- For a stationary process with PSD $P_x(f)$, $\mathbf{H}_p$ is time-invariant with frequency response $H_p(f) = \sqrt{P_x(f)}$. Here, the GES reduces to the PSD $P_x(f)$ for all t, i.e., $G_x^{(\alpha)}(t,f) \equiv P_x(f)$.
- For a (generally nonstationary) white process with mean instantaneous intensity $q_x(t)$, $\mathbf{H}_p$ is "frequency-invariant" with kernel $h_p(t,t') = \sqrt{q_x(t)}\,\delta(t-t')$. The GES here reduces to $q_x(t)$ for all f, i.e., $G_x^{(\alpha)}(t,f) \equiv q_x(t)$.
- For a stationary and white process with correlation function $r_x(t,t') = \eta\,\delta(t-t')$, we have $\mathbf{H}_p = \sqrt{\eta}\,\mathbf{I}$ with $\mathbf{I}$ the identity operator. Thus, the GES is given by $G_x^{(\alpha)}(t,f) \equiv \eta$.

In Section 9.4.5, we shall consider conditions allowing the interpretation of the GWVS and GES as a "time-varying power spectrum." The formulation of these conditions will be based on a further TF representation of nonstationary processes, to be discussed next.

9.4.4 The Generalized Expected Ambiguity Function

The *generalized expected ambiguity function* (GEAF) is defined as [6] [8] [9]

$$\bar{A}_x^{(\alpha)}(\nu,\tau) \triangleq \int_{-\infty}^{\infty} r_x^{(\alpha)}(t,\tau)\, e^{-j2\pi\nu t}\, dt\,,$$

with $r_x^{(\alpha)}(t,\tau)$ as in (9.4.2). The interpretation of the GEAF is quite different from that of a "time-varying power spectrum:" For a given frequency lag ν and a given time lag τ, the GEAF $\bar{A}_x^{(\alpha)}(\nu,\tau)$ quantifies the statistical correlations of all process components separated in frequency by ν and in time by τ [6]. Hence, the extension of $\bar{A}_x^{(\alpha)}(\nu,\tau)$ about the origin of the (ν,τ) plane indicates the amount of "TF correlations" of $x(t)$. In particular, if $\bar{A}_x^{(\alpha)}(\nu,\tau)$ extends far in the ν direction, this indicates that $x(t)$ has a large spectral correlation width (i.e., $x(t)$ is highly nonstationary), and if $\bar{A}_x^{(\alpha)}(\nu,\tau)$ extends far in the τ direction, this indicates that $x(t)$ has a large temporal correlation width.

The GEAF equals the generalized spreading function (see Article 4.7) of the correlation operator $\mathbf{R}_x$. Like the GWVS, the GEAF is a complete second-order statistic. GEAFs with different α values differ merely by a phase factor, i.e.,

$$\bar{A}_x^{(\alpha_2)}(\nu,\tau) = \bar{A}_x^{(\alpha_1)}(\nu,\tau)\, e^{j2\pi(\alpha_1-\alpha_2)\nu\tau}\,.$$

Therefore, the GEAF magnitude is independent of α, $\left|\bar{A}_x^{(\alpha_1)}(\nu,\tau)\right| = \left|\bar{A}_x^{(\alpha_2)}(\nu,\tau)\right|$, and we may thus simply write $|\bar{A}_x(\tau,\nu)|$. GWVS and GEAF are related by a 2-D Fourier transform,

$$\overline{W}_x^{(\alpha)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{A}_x^{(\alpha)}(\nu,\tau)\, e^{-j2\pi(\tau f-\nu t)}\, d\nu\, d\tau\,; \qquad (9.4.6)$$

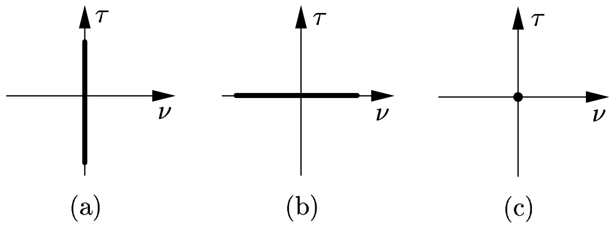


Figure 9.1: Schematic representation of the GEAF magnitude of some (classes of) random processes: (a) stationary process, (b) white process, (c) stationary white process.

this extends the Wiener-Khintchine relation (9.4.1) to the nonstationary case.

Again, it is instructive to consider our three process types (see Fig. 9.1; this figure should be compared to that in Section 4.7.3):

- The GEAF of a stationary process $x(t)$ with correlation function $r_x(t,t') = \tilde{r}_x(t-t')$ is given by $\bar{A}_x^{(\alpha)}(\nu,\tau) = \delta(\nu)\,\tilde{r}_x(\tau)$ (i.e., only temporal correlations that are characterized by $\tilde{r}_x(\tau)$).

- The GEAF of a (generally nonstationary) white process $x(t)$ with correlation function $r_x(t,t') = q_x(t)\,\delta(t-t')$ is given by $\bar{A}_x^{(\alpha)}(\nu,\tau) = Q_x(\nu)\,\delta(\tau)$, where $Q_x(\nu)$ is the Fourier transform of the mean instantaneous intensity $q_x(t)$ (i.e., only spectral correlations that are characterized by $Q_x(\nu)$).

- The GEAF of a stationary and white process $x(t)$ with correlation function $r_x(t,t') = \eta\,\delta(t-t')$ is given by $\bar{A}_x^{(\alpha)}(\nu,\tau) = \eta\,\delta(\nu)\delta(\tau)$ (i.e., neither temporal nor spectral correlations).

9.4.5 Underspread Processes

A nonstationary random process is said to be *underspread* if its GEAF is well concentrated about the origin of the (ν,τ) plane, thus implying a small "TF correlation width." In contrast, a process with large TF correlation width is termed *overspread*. We will see in Section 9.4.6 that the GWVS and GES of an underspread process are approximately equivalent and can be interpreted as "time-varying power spectra."

There are two alternative mathematical definitions of underspread processes [6] [8]. The first one [6] assumes that the GEAF $\bar{A}_x^{(\alpha)}(\nu,\tau)$ is supported in a compact region $\mathcal{G}_x$ about the origin of the (ν,τ) plane, i.e., $\big|\bar{A}_x(\nu,\tau)\big| = 0$ for $(\nu,\tau) \notin \mathcal{G}_x$. Let $\nu_x \triangleq \max_{(\nu,\tau)\in\mathcal{G}_x} |\nu|$ and $\tau_x \triangleq \max_{(\nu,\tau)\in\mathcal{G}_x} |\tau|$ denote the maximum frequency lag and time lag, respectively, for which the process $x(t)$ features TF correlations. The *TF correlation spread* of $x(t)$ is defined as $\sigma_x \triangleq 4\nu_x\tau_x$, which is the area of the rectangle $[-\nu_x,\nu_x]\times[-\tau_x,\tau_x]$ enclosing $\mathcal{G}_x$. The process $x(t)$ is considered underspread if $\sigma_x \ll 1$ [6].

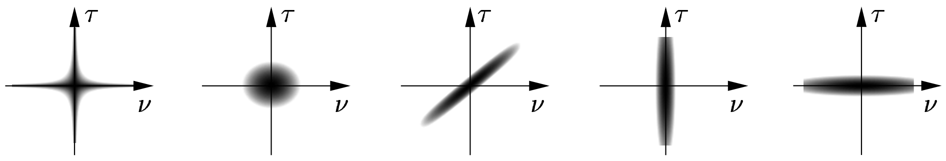


Figure 9.2: Schematic representation of the GEAF magnitude of various types of nonstationary processes: (a) underspread process with small $m_x^{(1,1)}$; (b) underspread process with small $m_x^{(1,0)} m_x^{(0,1)}$; (c) "chirpy" underspread process [8]; (d) quasi-stationary process (small $m_x^{(0,1)}$); (e) quasi-white process (small $m_x^{(1,0)}$).

An alternative description of the GEAF's extension that avoids the assumption of compact GEAF support uses the normalized *weighted GEAF integrals*[3] [8]

$$m_x^{(\phi)} \triangleq \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\nu,\tau)\,|\bar{A}_x(\nu,\tau)|\,d\nu\,d\tau}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\bar{A}_x(\nu,\tau)|\,d\nu\,d\tau}\,.$$

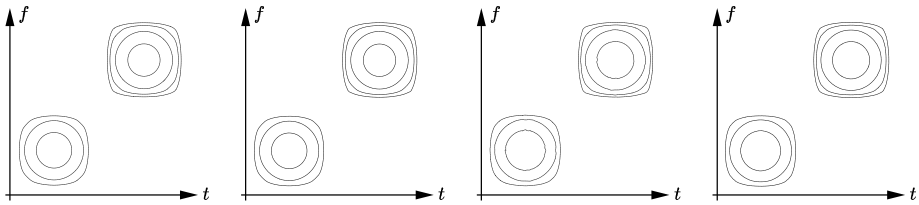
Here, $\phi(\nu,\tau)$ is a nonnegative weighting function satisfying $\phi(\nu,\tau) \geq \phi(0,0) = 0$ and penalizing GEAF contributions located away from the origin. Important special cases are the *GEAF moments* $m_x^{(k,l)} \triangleq m_x^{(\phi_{k,l})}$ obtained with the weighting functions $\phi_{k,l}(\nu,\tau) = |\nu|^l |\tau|^k$ with $k,l \in \mathbb{N}_0$. A random process $x(t)$ can now be considered underspread if suitable weighted GEAF integrals or moments are "small." Processes that are underspread in the compact-support sense considered previously are easily shown to be a special case of this extended, more flexible definition of underspread processes.

Examples of underspread processes are illustrated in Fig. 9.2 (this figure should be compared to that in Section 4.7.4). We caution that the concept of underspread processes is not equivalent to that of quasi-stationary processes: indeed, a quasi-stationary process may be overspread if its temporal correlation width is very large. Finally, note that according to the Fourier transform relation (9.4.6), the GWVS of an underspread process is a *smooth* function.

9.4.6 Time-Varying Spectral Analysis of Underspread Processes

For *underspread* nonstationary processes, the GWVS and GES can be interpreted as "time-varying power spectra" that generalize the PSD of stationary processes and the mean instantaneous intensity of white processes. Indeed, small weighted GEAF integrals $m_x^{(\phi)}$ (or small moments $m_x^{(k,l)}$ or a small TF correlation spread σ_x) ensure the validity of the approximations described in what follows [6] [8] [9].

[3] Further definitions of weighted GEAF integrals and moments can be found in [8].



(a) (b) (c) (d)

Figure 9.3: GWVS and GES of an underspread process $x(t)$: (a) Wigner-Ville spectrum $\overline{W}_x^{(0)}(t,f)$, (b) real part of Rihaczek spectrum $\overline{W}_x^{(1/2)}(t,f)$, (c) Weyl spectrum $G_x^{(0)}(t,f)$, (d) evolutionary spectrum $G_x^{(1/2)}(t,f)$. In this and subsequent simulations, signal duration is 256 samples and normalized frequency ranges from $-1/4$ to $1/4$.

Approximate equivalence. In general, the GWVS and GES of a given process $x(t)$ may yield very different results which, moreover, may strongly depend on the parameter α used. However, for an underspread process $x(t)$, all these results are approximately equal, i.e.,

$$\begin{aligned} \overline{W}_x^{(\alpha_1)}(t,f) &\approx \overline{W}_x^{(\alpha_2)}(t,f)\,, \\ G_x^{(\alpha_1)}(t,f) &\approx G_x^{(\alpha_2)}(t,f)\,, \\ \overline{W}_x^{(\alpha_1)}(t,f) &\approx G_x^{(\alpha_2)}(t,f)\,. \end{aligned}$$

Indeed, it can be shown [8] that the approximation error $\overline{W}_x^{(\alpha_1)}(t,f) - \overline{W}_x^{(\alpha_2)}(t,f)$ is upper bounded as

$$\left|\overline{W}_x^{(\alpha_1)}(t,f) - \overline{W}_x^{(\alpha_2)}(t,f)\right| \;\leq\; 2\pi\,|\alpha_1 - \alpha_2|\, \|\bar{A}_x\|_1\, m_x^{(1,1)}, \qquad (9.4.7)$$

with $\|\bar{A}_x\|_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\bar{A}_x(\nu,\tau)|\,d\nu\,d\tau$. Thus, for an underspread system where $m_x^{(1,1)}$ is small, $\overline{W}_x^{(\alpha_1)}(t,f)$ and $\overline{W}_x^{(\alpha_2)}(t,f)$ will be approximately equal as long as $|\alpha_1 - \alpha_2|$ is not too large. Similar bounds can be developed for the approximation errors $G_x^{(\alpha_1)}(t,f) - G_x^{(\alpha_2)}(t,f)$ and $\overline{W}_x^{(\alpha_1)}(t,f) - G_x^{(\alpha_2)}(t,f)$ [8].

We can conclude from these results that for an underspread process, the choice of a specific spectrum is not critical. An example is shown in Fig. 9.3. For this example, the maximum normalized differences between the spectra shown are all around 0.03 (e.g. $\max_{t,f}\left|\overline{W}_x^{(0)}(t,f) - G_x^{(0)}(t,f)\right| / \max_{t,f}\left|\overline{W}_x^{(0)}(t,f)\right| = 0.029$). A counterexample involving an *overspread* process is shown in Fig. 9.4. Here, the results obtained with the various spectra are seen to be dramatically different, and indeed the maximum normalized differences range from 1 to 8.5 (e.g. $\max_{t,f}\left|\overline{W}_x^{(1/2)}(t,f) - G_x^{(1/2)}(t,f)\right| / \max_{t,f}\left|\overline{W}_x^{(1/2)}(t,f)\right| = 2.13$). It can be seen that all spectra contain oscillating components (so-called *statistical cross-terms*) which are indicative of TF correlations [8]. Such statistical cross-terms are reduced in extensions of the GWVS and GES that contain a TF smoothing [1] [3] [4] [5] [8].

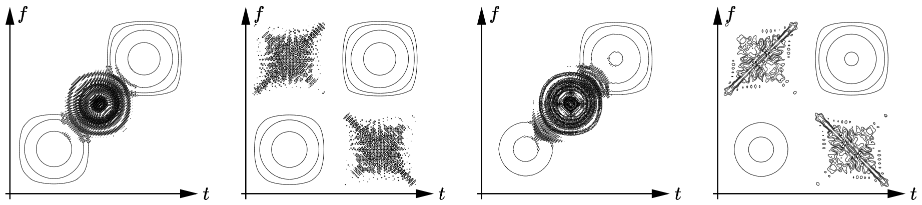


Figure 9.4: GWVS and GES of an overspread process $x(t)$: (a) Wigner-Ville spectrum $\overline{W}_x^{(0)}(t,f)$, (b) real part of Rihaczek spectrum $\overline{W}_x^{(1/2)}(t,f)$, (c) Weyl spectrum $G_x^{(0)}(t,f)$, (d) evolutionary spectrum $G_x^{(1/2)}(t,f)$.

For underspread processes, on the other hand, a TF smoothing does not cause a big difference.

Approximate real-valuedness and positivity of the GWVS. The PSD of stationary processes and the mean instantaneous intensity of white processes are real-valued and nonnegative. This is also true for the GES of arbitrary processes. In contrast, the GWVS is real-valued only for $\alpha = 0$ and generally not everywhere nonnegative. In the case of underspread processes, however, it can be shown [8] that the imaginary part of the GWVS is approximately zero and the real part of the GWVS is approximately nonnegative, i.e.,

$$\Im\{\overline{W}_x^{(\alpha)}(t,f)\} \approx 0\,, \qquad \mathbf{R}e\{\overline{W}_x^{(\alpha)}(t,f)\} \gtrsim 0\,.$$

Upper bounds on the associated approximation errors (similar to (9.4.7)) can again be provided [8].

As an example, we reconsider the underspread process from Fig. 9.3. The normalized maximum of the imaginary part of the Rihaczek spectrum (the real part is shown in Fig. 9.3(b)) is $\max_{t,f}\big|\Im\{\overline{W}_x^{(1/2)}(t,f)\}\big| / \max_{t,f}\big|\overline{W}_x^{(1/2)}(t,f)\big| = 0.024$ and the normalized maximum of the negative real part is $\max_{t,f}\big\{-\mathbf{R}e\{\overline{W}_x^{(1/2)}(t,f)\}\big\} / \max_{t,f}\big|\overline{W}_x^{(1/2)}(t,f)\big| = 0.006$.

Approximate input-output relations. If a stationary process $x(t)$ with PSD $P_x(f)$ is passed through a time-invariant linear system with impulse response $k(\tau)$ and transfer function $K(f)$, the output $y(t) = (x * k)(t)$ is also stationary and its PSD equals $P_y(f) = |K(f)|^2\, P_x(f)$. Similarly, the response $y(t) = w(t)\, x(t)$ of a linear frequency-invariant system (see Article 4.7) to a white process $x(t)$ with mean instantaneous intensity $q_x(t)$ is again white with $q_y(t) = |w(t)|^2\, q_x(t)$. A similar input-output relation does not exist for a general nonstationary process $x(t)$ that is passed through a general time-varying linear system $\mathbf{K}$. However, for an underspread process that is passed through an underspread system (i.e., a time-varying linear system introducing only small TF shifts, see Article 4.7), one can show the

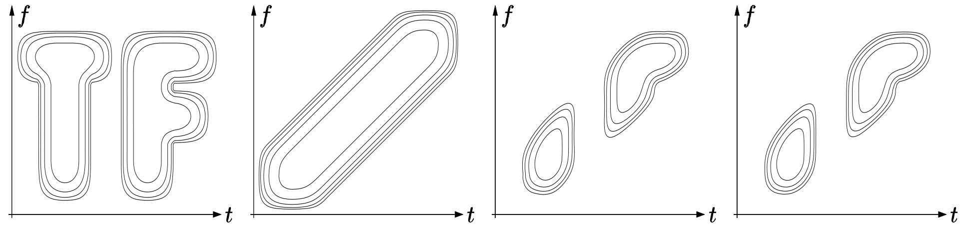


Figure 9.5: Approximate input-output relation for the GWVS: (a) Wigner-Ville spectrum $\overline{W}_x^{(0)}(t,f)$ of input process $x(t)$, (b) Weyl symbol $L_{\mathbf{K}}^{(0)}(t,f)$ of LTV system $\mathbf{K}$, (c) Wigner-Ville spectrum $\overline{W}_y^{(0)}(t,f)$ of filtered process $y(t) = (\mathbf{K}x)(t)$, (d) approximation $|L_{\mathbf{K}}^{(0)}(t,f)|^2\, \overline{W}_x^{(0)}(t,f)$.

following approximate input-output relations of the GWVS and GES:

$$\overline{W}_y^{(\alpha)}(t,f) \approx \big|L_{\mathbf{K}}^{(\alpha)}(t,f)\big|^2\, \overline{W}_x^{(\alpha)}(t,f)\,,$$
$$G_y^{(\alpha)}(t,f) \approx \big|L_{\mathbf{K}}^{(\alpha)}(t,f)\big|^2\, G_x^{(\alpha)}(t,f)\,,$$

with $y(t) = (\mathbf{K}x)(t)$. Note that the generalized Weyl symbol $L_{\mathbf{K}}^{(\alpha)}(t,f)$ of $\mathbf{K}$ (see (9.4.5)) takes the place of the transfer function $K(f)$ or $w(t)$. An example for the Wigner-Ville spectrum (GWVS with $\alpha = 0$) is shown in Fig. 9.5. In this example, the normalized maximum approximation error is $\max_{t,f}\big\{\big|\overline{W}_y^{(0)}(t,f) - \big|L_{\mathbf{K}}^{(0)}(t,f)\big|^2\, \overline{W}_x^{(0)}(t,f)\big|\big\}/\max_{t,f}\big|\overline{W}_y^{(0)}(t,f)\big| = 0.017$.

Discussion. The above approximations (more can be found in [6] [8] [9]) corroborate the interpretation of the GWVS and GES of underspread processes as *time-varying power spectra*. A mathematical underpinning of these approximations is provided by explicit upper bounds on the associated approximation errors [8]; these bounds involve the GEAF parameters $m_x^{(\phi)}$, $m_x^{(k,l)}$ or σ_x defined in Section 9.4.5. In the underspread case, these GEAF parameters are small and thus the approximations are guaranteed to be good. On the other hand, we caution that the approximations are *not* valid for overspread processes (cf. Fig. 9.4).

9.4.7 Summary and Conclusions

In this article, we have shown that for the practically important class of underspread processes (i.e., processes with small time-frequency correlations), the generalized Wigner-Ville spectrum and generalized evolutionary spectrum can be interpreted in a meaningful way as time-varying power spectra. Indeed, for underspread processes the generalized Wigner-Ville spectrum and the generalized evolutionary spectrum (approximately) satisfy desirable properties that any reasonable definition of a time-varying power spectrum would be expected to satisfy. We note that applications of the generalized Wigner-Ville spectrum in statistical signal processing are considered in Articles 12.1 and 12.4.

References

[1] M. Amin, "Time-frequency spectrum analysis and estimation for non-stationary random processes," in *Advances in Spectrum Estimation* (B. Boashash, ed.), pp. 208–232, Melbourne: Longman Cheshire, 1992.

[2] C. S. Detka and A. El-Jaroudi, "The transitory evolutionary spectrum," in *Proc. IEEE ICASSP-94*, (Adelaide, Australia), pp. 289–292, April 1994.

[3] P. Flandrin, "Time-dependent spectra for nonstationary stochastic processes," in *Time and Frequency Representation of Signals and Systems* (G. Longo and B. Picinbono, eds.), pp. 69–124, Wien: Springer, 1989.

[4] P. Flandrin and W. Martin, "The Wigner-Ville spectrum of nonstationary random signals," in *The Wigner Distribution — Theory and Applications in Signal Processing* (W. Mecklenbräuker and F. Hlawatsch, eds.), pp. 211–267, Amsterdam (The Netherlands): Elsevier, 1997.

[5] P. Flandrin, *Time-Frequency/Time-Scale Analysis.* San Diego (CA): Academic Press, 1999.

[6] W. Kozek, F. Hlawatsch, H. Kirchauer, and U. Trautwein, "Correlative time-frequency analysis and classification of nonstationary random processes," in *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, (Philadelphia, PA), pp. 417–420, Oct. 1994.

[7] W. Martin and P. Flandrin, "Wigner-Ville spectral analysis of nonstationary processes," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 33, pp. 1461–1470, Dec. 1985.

[8] G. Matz and F. Hlawatsch, "Time-varying spectra for underspread and overspread nonstationary processes," in *Proc. 32nd Asilomar Conf. Signals, Systems, Computers*, (Pacific Grove, CA), pp. 282–286, Nov. 1998.

[9] G. Matz, F. Hlawatsch, and W. Kozek, "Generalized evolutionary spectral analysis and the Weyl spectrum of nonstationary random processes," *IEEE Trans. Signal Processing*, vol. 45, pp. 1520–1534, June 1997.

[10] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science.* New York: Springer, 2nd ed., 1982.

[11] A. Papoulis, *Probability, Random Variables, and Stochastic Processes.* New York: McGraw-Hill, 3rd ed., 1991.

[12] M. B. Priestley, "Evolutionary spectra and non-stationary processes," *J. Roy. Stat. Soc. Ser. B*, vol. 27, no. 2, pp. 204–237, 1965.

[13] M. B. Priestley, *Spectral Analysis and Time Series – Part II.* London: Academic Press, 1981.