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9.5 TIME-FREQUENCY CHARACTERIZATION OF RANDOM TIME-VARYING CHANNELS[0]

9.5.1 Time-Varying Channels

In many practical communication systems, the channel is modeled as linear but *time-varying* and *random*. Examples are the mobile radio, ionospheric, tropospheric, and underwater acoustic channels [2] [6] [10] [11] [12] [13] [14]. In this article, we will discuss time-frequency (TF) descriptions of both the channel (see also Article 4.7 for more details) and its second-order statistics.

The input-output relation of a linear, time-varying, random channel $\mathbf{H}$ is

$$r(t) = (\mathbf{H}s)(t) = \int_{-\infty}^{\infty} h(t,t')\, s(t')\, dt' = \int_{-\infty}^{\infty} \tilde{h}(t,\tau)\, s(t-\tau)\, d\tau\,, \tag{9.5.1}$$

where $s(t)$ is the transmit signal, $r(t)$ is the received signal, $h(t,t')$ is the (random) kernel of $\mathbf{H}$, and $\tilde{h}(t,\tau) = h(t,t-\tau)$ is the (random) impulse response of $\mathbf{H}$. Two major physical phenomena underlying practical channels are *multipath propagation* and *Doppler spreading*. Multipath propagation (i.e., several different propagation paths from the transmitter to the receiver via various scattering objects) causes the received signal to consist of several delayed versions of the transmit signal. Doppler spreading is due to the movement of transmitter and/or receiver and/or scatterers; for a narrowband transmit signal $s(t)$, it causes the multipath signals to be frequency-shifted. The received signal $r(t)$ thus consists of several TF shifted (i.e., delayed and modulated) versions of the transmit signal $s(t)$ [2] [6] [10] [11],

$$r(t) = \sum_{k=1}^{N} a_k\, s(t-\tau_k)\, e^{j2\pi\nu_k t}\,.$$

Here, N is the number of scatterers and τ_k, ν_k, and a_k are respectively the (random) delay, Doppler frequency, and reflectivity of the kth scatterer. The above relation can be extended to a continuum of scatterers (corresponding to a continuum of delays τ and Doppler frequency shifts ν) as

$$r(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)\, s_{\nu,\tau}^{(\alpha)}(t)\, d\tau\, d\nu\,, \tag{9.5.2}$$

with $s_{\nu,\tau}^{(\alpha)}(t) = s(t-\tau)\, e^{j2\pi\nu t}\, e^{j2\pi\nu\tau(\alpha-1/2)}$, where α is a real-valued parameter[1] that is arbitrary but assumed fixed and $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)$ denotes the *generalized (delay-*

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[1]The parameter α in $s_{\nu,\tau}^{(\alpha)}(t)$ corresponds to the infinitely many ways of defining a joint TF shift by combining time shifts and frequency shifts. In particular, the case $\alpha = 1/2$ corresponds to first shifting in time and then shifting in frequency, whereas $\alpha = -1/2$ corresponds to first shifting in frequency and then in time.

Doppler) spreading function (GSF) of the channel [2] (see also Article 4.7). The GSF is defined as

$$S_{\mathbf{H}}^{(\alpha)}(\nu,\tau) \triangleq \int_{-\infty}^{\infty} h\Big(t+\Big(\frac{1}{2}-\alpha\Big)\tau, t-\Big(\frac{1}{2}+\alpha\Big)\tau\Big)\, e^{-j2\pi\nu t} dt\,. \tag{9.5.3}$$

It can be shown that the input-output relation (9.5.2) is mathematically equivalent to (9.5.1).

In what follows, we will also use the *generalized Weyl symbol* (GWS)

$$L_{\mathbf{H}}^{(\alpha)}(t,f) \triangleq \int_{-\infty}^{\infty} h\Big(t+\Big(\frac{1}{2}-\alpha\Big)\tau, t-\Big(\frac{1}{2}+\alpha\Big)\tau\Big)\, e^{-j2\pi f\tau} d\tau \tag{9.5.4}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)\, e^{j2\pi(\nu t-\tau f)}\, d\tau\, d\nu\,. \tag{9.5.5}$$

The GWS can be interpreted (with certain precautions, see Article 4.7) as a "TF transfer function" of $\mathbf{H}$.

In the mobile communications literature, the parameter α is usually chosen as 1/2. In this case, (9.5.3) and (9.5.4) become

$$S_{\mathbf{H}}^{(1/2)}(\nu,\tau) = \int_{-\infty}^{\infty} \tilde{h}(t,\tau)\, e^{-j2\pi\nu t}\, dt\,, \qquad L_{\mathbf{H}}^{(1/2)}(t,f) = \int_{-\infty}^{\infty} \tilde{h}(t,\tau)\, e^{-j2\pi f\tau}\, d\tau\,.$$

9.5.2 WSSUS Channels

Since the channel $\mathbf{H}$ is random, its GSF $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)$, GWS $L_{\mathbf{H}}^{(\alpha)}(t,f)$, and impulse response $\tilde{h}(t,\tau)$ are 2-D random functions (random processes). Hereafter, these random processes will be assumed zero-mean. The second-order statistics of $\mathbf{H}$ are characterized by the 4-D correlation functions $\mathrm{E}\{S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)\, S_{\mathbf{H}}^{(\alpha)*}(\nu',\tau')\}$, $\mathrm{E}\{L_{\mathbf{H}}^{(\alpha)}(t,f)\, L_{\mathbf{H}}^{(\alpha)*}(t',f')\}$, and $\mathrm{E}\{\tilde{h}(t,\tau)\, \tilde{h}^*(t',\tau')\}$, which are all mathematically equivalent.

Definition and description of WSSUS channels. An important simplification results from the assumption of *wide-sense stationary uncorrelated scattering (WSSUS)* [2] [6] [11] [13] [14]. For WSSUS channels, by definition, the reflectivities of scatterers corresponding to paths with different delay or Doppler are uncorrelated. In terms of the GSF $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)$, this means

$$\mathrm{E}\{S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)\, S_{\mathbf{H}}^{(\alpha)*}(\nu',\tau')\} = C_{\mathbf{H}}(\nu,\tau)\, \delta(\nu-\nu')\, \delta(\tau-\tau')\,, \tag{9.5.6}$$

i.e., $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)$ is a *wide-sense white* random process. The mean intensity function of this white random process, $C_{\mathbf{H}}(\nu,\tau) \geq 0$, is known as the *scattering function* [1] [2] [6] [10] [11] [12] [13] [14].

Together with the Fourier transform relation (9.5.5), the WSSUS relation (9.5.6) implies that

$$\mathrm{E}\{L_{\mathbf{H}}^{(\alpha)}(t,f)\, L_{\mathbf{H}}^{(\alpha)*}(t',f')\} = R_{\mathbf{H}}(t-t', f-f')\,, \tag{9.5.7}$$

with the *TF correlation function* [2] [6] [10] [11] [12] [13] [14]

$$R_{\mathbf{H}}(\Delta t, \Delta f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{\mathbf{H}}(\nu, \tau)\, e^{j2\pi(\nu\Delta t - \tau\Delta f)}\, d\tau\, d\nu\,. \tag{9.5.8}$$

The TF correlation function satisfies $\left|R_{\mathbf{H}}(\Delta t, \Delta f)\right| \leq R_{\mathbf{H}}(0,0)$ and $R^*_{\mathbf{H}}(-\Delta t, -\Delta f) = R_{\mathbf{H}}(\Delta t, \Delta f)$. Eq. (9.5.7) shows that the GWS $L^{(\alpha)}_{\mathbf{H}}(t,f)$ of a WSSUS channel is a 2-D *wide-sense stationary* process. According to (9.5.8), the scattering function $C_{\mathbf{H}}(\nu,\tau)$ is the Fourier transform of the correlation function $R_{\mathbf{H}}(\Delta t, \Delta f)$ of $L^{(\alpha)}_{\mathbf{H}}(t,f)$. Thus, $C_{\mathbf{H}}(\nu,\tau)$ can be interpreted as the *power spectral density* of $L^{(\alpha)}_{\mathbf{H}}(t,f)$. The *path loss* [10] is defined as

$$\rho^2_{\mathbf{H}} \triangleq R_{\mathbf{H}}(0,0) = \mathrm{E}\big\{\big|L^{(\alpha)}_{\mathbf{H}}(t,f)\big|^2\big\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{\mathbf{H}}(\nu,\tau)\, d\tau\, d\nu\,.$$

Finally, in terms of the impulse response $\tilde{h}(t,\tau)$, the WSSUS property is expressed as

$$\mathrm{E}\big\{\tilde{h}(t,\tau)\, \tilde{h}^*(t',\tau')\big\} = r_{\mathbf{H}}(t-t',\tau)\, \delta(\tau-\tau')\,,$$

where $r_{\mathbf{H}}(\Delta t,\tau)$ is related by Fourier transforms to $C_{\mathbf{H}}(\nu,\tau)$ and $R_{\mathbf{H}}(\Delta t, \Delta f)$. Thus, $\tilde{h}(t,\tau)$ is wide-sense stationary with respect to time t and uncorrelated for different delays τ. Note that this stationarity with respect to t refers to the second-order *statistics* of the channel and does not imply that the channel's realizations are time-invariant systems (cf. the first of the examples given below).

The 2-D functions $C_{\mathbf{H}}(\nu,\tau)$, $R_{\mathbf{H}}(\Delta t, \Delta f)$, and $r_{\mathbf{H}}(\Delta t,\tau)$ are *mathematically equivalent* descriptions of the second-order statistics of a WSSUS channel. They are related by Fourier transforms and do not dependent on α.

The composition (series connection) $\mathbf{H}_2\mathbf{H}_1$ of two statistically independent WSSUS channels $\mathbf{H}_1$ and $\mathbf{H}_2$ can be shown to be again a WSSUS channel. Its scattering function and TF correlation function are given by

$$\begin{aligned} C_{\mathbf{H}_2\mathbf{H}_1}(\nu,\tau) &= \big(C_{\mathbf{H}_2} ** C_{\mathbf{H}_1}\big)(\nu,\tau) \\ R_{\mathbf{H}_2\mathbf{H}_1}(\Delta t, \Delta f) &= R_{\mathbf{H}_2}(\Delta t, \Delta f)\, R_{\mathbf{H}_1}(\Delta t, \Delta f)\,, \end{aligned}$$

where $**$ denotes 2-D convolution. Note that $\mathbf{H}_2\mathbf{H}_1$ and $\mathbf{H}_1\mathbf{H}_2$ have the same second-order statistics.

From the 2-D functions $C_{\mathbf{H}}(\nu,\tau)$ and $R_{\mathbf{H}}(\Delta t, \Delta f)$, several 1-D channel descriptions can be derived. In particular, the *delay power profile* and *Doppler power profile* are respectively defined as [10]

$$P_{\mathbf{H}}(\tau) \triangleq \int_{-\infty}^{\infty} C_{\mathbf{H}}(\nu,\tau)\, d\nu\,, \qquad Q_{\mathbf{H}}(\nu) \triangleq \int_{-\infty}^{\infty} C_{\mathbf{H}}(\nu,\tau)\, d\tau\,.$$

Their Fourier transforms,

$$\begin{aligned} p_{\mathbf{H}}(\Delta f) &\triangleq \int_{-\infty}^{\infty} P_{\mathbf{H}}(\tau)\, e^{-j2\pi\tau\Delta f}\, d\tau = R_{\mathbf{H}}(0, \Delta f)\,, \\ q_{\mathbf{H}}(\Delta t) &\triangleq \int_{-\infty}^{\infty} Q_{\mathbf{H}}(\nu)\, e^{j2\pi\nu\Delta t}\, d\nu = R_{\mathbf{H}}(\Delta t, 0)\,, \end{aligned}$$

are known as *time correlation function* and *frequency correlation function*, respectively. Often, for the sake of simplicity, a separable model is assumed for the scattering function and the TF correlation function, i.e., $C_{\mathbf{H}}(\nu,\tau) = \frac{1}{\rho_{\mathbf{H}}^2} Q_{\mathbf{H}}(\nu)\, P_{\mathbf{H}}(\tau)$ and $R_{\mathbf{H}}(\Delta t, \Delta f) = \frac{1}{\rho_{\mathbf{H}}^2} q_{\mathbf{H}}(\Delta t)\, p_{\mathbf{H}}(\Delta f)$.

Examples of WSSUS channels. In the following, we briefly present some special cases and important examples of WSSUS channels.

- *Time-invariant WSSUS channel.* The impulse response of a time-invariant channel has the form $\tilde{h}(t,\tau) = g(\tau)$. The WSSUS property here implies $\mathrm{E}\{g(\tau)\, g^*(\tau')\} = P_{\mathbf{H}}(\tau)\, \delta(\tau - \tau')$. It follows that $C_{\mathbf{H}}(\nu,\tau) = \delta(\nu)\, P_{\mathbf{H}}(\tau)$ and $R_{\mathbf{H}}(\Delta t, \Delta f) = p_{\mathbf{H}}(\Delta f)$.

- *Frequency-invariant WSSUS channel.* Next, we consider a "frequency-invariant" channel with impulse response $\tilde{h}(t,\tau) = w(t)\, \delta(\tau)$, i.e., the input signal is simply multiplied by $w(t)$. With $W(\nu)$ denoting the Fourier transform of $w(t)$, the WSSUS property here implies $\mathrm{E}\{W(\nu)\, W^*(\nu')\} = Q_{\mathbf{H}}(\nu)\, \delta(\nu - \nu')$. It follows that $C_{\mathbf{H}}(\nu,\tau) = Q_{\mathbf{H}}(\nu)\, \delta(\tau)$ and $R_{\mathbf{H}}(\Delta t, \Delta f) = q_{\mathbf{H}}(\Delta t)$.

- *Random TF shift.* The GSF of a channel effecting a random frequency shift by ν_0 and a random time shift by τ_0 is given by $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau) = \delta(\nu - \nu_0)\, \delta(\tau - \tau_0)$. It can be shown that this channel is WSSUS with scattering function $C_{\mathbf{H}}(\nu,\tau) = f_{\nu_0,\tau_0}(\nu,\tau)$, where $f_{\nu_0,\tau_0}(\nu,\tau)$ is the joint probability density function of (ν_0, τ_0) [4]. Furthermore, $R_{\mathbf{H}}(\Delta t, \Delta f) = \Psi_{\nu_0,\tau_0}(\Delta t, -\Delta f)$, with $\Psi_{\nu_0,\tau_0}(\Delta t, \Delta f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\nu_0,\tau_0}(\nu,\tau)\, e^{j2\pi(\nu\Delta t + \tau\Delta f)}\, d\nu\, d\tau$ being the characteristic function of (ν_0, τ_0).

- *Typical mobile radio channel.* A channel model popular in the mobile radio literature [10] uses a separable scattering function $C_{\mathbf{H}}(\nu,\tau) = \frac{1}{\rho_{\mathbf{H}}^2} Q_{\mathbf{H}}(\nu) P_{\mathbf{H}}(\tau)$ with an exponential delay power profile

$$P_{\mathbf{H}}(\tau) = \begin{cases} \frac{\rho_{\mathbf{H}}^2}{\tau_0}\, e^{-\tau/\tau_0}, & \tau \geq 0\,, \\ 0\,, & \tau < 0\,, \end{cases} \tag{9.5.9}$$

and a *Jakes* Doppler power profile

$$Q_{\mathbf{H}}(\nu) = \begin{cases} \frac{\rho_{\mathbf{H}}^2}{\pi\sqrt{\nu_{\max}^2 - \nu^2}}, & |\nu| \leq \nu_{\max}\,, \\ 0\,, & |\nu| > \nu_{\max}\,. \end{cases} \tag{9.5.10}$$

The associated TF correlation function is $R_{\mathbf{H}}(\Delta t, \Delta f) = \frac{1}{\rho_{\mathbf{H}}^2} q_{\mathbf{H}}(\Delta t)\, p_{\mathbf{H}}(\Delta f)$, with

$$p_{\mathbf{H}}(\Delta f) = \frac{\rho_{\mathbf{H}}^2}{1 + j2\pi\tau_0\Delta f}\,, \qquad q_{\mathbf{H}}(\Delta t) = \rho_{\mathbf{H}}^2\, J_0(2\pi\nu_{\max}\Delta t)\,,$$

where $J_0(\cdot)$ denotes the zero-order Bessel function of the first kind.

Statistical input-output relations for WSSUS channels. The scattering function and TF correlation function are useful for formulating input-output relations that show how the second-order statistics of the channel output signal $r(t)$ depend on the second-order statistics of the input signal $s(t)$. Let $s(t)$ be a nonstationary random process that is statistically independent of the random channel $\mathbf{H}$. The second-order statistics of a nonstationary random process $x(t)$ with correlation operator $\mathbf{R}_x$ can be described by the *generalized Wigner-Ville spectrum* $\overline{W}_x^{(\alpha)}(t,f) \triangleq L_{\mathbf{R}_x}^{(\alpha)}(t,f)$ or, alternatively, by the *generalized expected ambiguity function* $\bar{A}_x^{(\alpha)}(\nu,\tau) \triangleq S_{\mathbf{R}_x}^{(\alpha)}(\nu,\tau)$ which is the 2-D Fourier transform of $\overline{W}_x^{(\alpha)}(t,f)$ (see Article 9.4). It can then be shown that

$$\overline{W}_r^{(\alpha)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{\mathbf{H}}(\nu,\tau)\,\overline{W}_s^{(\alpha)}(t-\tau, f-\nu)\,d\tau\,d\nu\,, \tag{9.5.11}$$

$$\bar{A}_r^{(\alpha)}(\nu,\tau) = R_{\mathbf{H}}(\tau,\nu)\,\bar{A}_s^{(\alpha)}(\nu,\tau)\,. \tag{9.5.12}$$

For a (wide-sense) *stationary* transmit signal $s(t)$ with power spectral density $P_s(f)$ and autocorrelation $r_s(\tau)$, the received signal $r(t)$ is stationary as well and (9.5.11) and (9.5.12) reduce to

$$P_r(f) = \int_{-\infty}^{\infty} Q_{\mathbf{H}}(\nu)\,P_s(f-\nu)\,d\nu\,, \qquad r_r(\tau) = q_{\mathbf{H}}(\tau)\,r_s(\tau)\,.$$

Furthermore, $\mathrm{E}\{|r(t)|^2\} = \rho_{\mathbf{H}}^2\,\mathrm{E}\{|s(t)|^2\}$. Dual results are obtained for a nonstationary white transmit signal $s(t)$.

Finally, if $s(t)$ is *cyclostationary* with period T, cyclic correlation function $r_s^{(k)}(\tau)$, and cyclic spectral density $P_s^{(k)}(f)$ [5], the received signal $r(t)$ is cyclostationary with the same period T and we have

$$P_r^{(k)}(f) = \int_{-\infty}^{\infty} Q_{\mathbf{H}}^{(k)}(\nu)\,P_s^{(k)}(f-\nu)\,d\nu\,, \qquad r_r^{(k)}(\tau) = q_{\mathbf{H}}^{(k)}(\tau)\,r_s^{(k)}(\tau)\,,$$

where $q_{\mathbf{H}}^{(k)}(\Delta t) = R_{\mathbf{H}}(\Delta t, k/T)$ and $Q_{\mathbf{H}}^{(k)}(\nu) = \int_{-\infty}^{\infty} q_{\mathbf{H}}^{(k)}(\Delta t)\,e^{-j2\pi\nu\Delta t}\,d\Delta t$.

9.5.3 Underspread WSSUS Channels

A fundamental classification of WSSUS channels is into *underspread* and *overspread* channels [6] [11] [14]. As we will show in this section, underspread WSSUS channels have some interesting properties. We note that the underspread property for WSSUS random channels is analogous to the underspread property for deterministic time-varying systems that was considered in Article 4.7.

Definition of underspread channels. A WSSUS channel is *underspread* [11] [14] if its scattering function is highly concentrated about the origin.[2] The underspread property is practically relevant as most mobile radio channels are underspread.

[2]For simplicity, we assume that the scattering function is centered about $\tau = 0$, which means that an overall delay $\tau_0 > 0$ has been split off from the channel.

A simple method for quantifying the concentration of the scattering function $C_{\mathbf{H}}(\nu,\tau)$ is based on the assumption that the support of $C_{\mathbf{H}}(\nu,\tau)$ is contained within a rectangle $[-\nu_{\max},\nu_{\max}]\times[-\tau_{\max},\tau_{\max}]$ about the origin of the (ν,τ) plane. (This implies that, with probability one, the GSF $S_{\mathbf{H}}^{(\alpha)}(\nu,\tau)$ is supported within this rectangle as well.) The channel's *delay-Doppler spread* is then defined as the area of this rectangle, $\sigma_{\mathbf{H}}\triangleq 4\nu_{\max}\tau_{\max}$, and the channel is said to be underspread if $\sigma_{\mathbf{H}}\leq 1$ and overspread if $\sigma_{\mathbf{H}}>1$ [11] [14].

An alternative characterization of scattering function concentration that avoids the assumption of compact support uses normalized *weighted integrals* of the form

$$\overline{m}_{\mathbf{H}}^{(\phi)} \triangleq \frac{1}{\rho_{\mathbf{H}}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\nu,\tau)\,C_{\mathbf{H}}(\nu,\tau)\,d\nu\,d\tau \;=\; \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\nu,\tau)\,C_{\mathbf{H}}(\nu,\tau)\,d\nu\,d\tau}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_{\mathbf{H}}(\nu,\tau)\,d\nu\,d\tau}\,,$$

where $\phi(\nu,\tau)\geq 0$ is a weighting function that satisfies $\phi(\nu,\tau)\geq\phi(0,0)=0$ and penalizes scattering function components lying away from the origin. Special cases are the *moments* $\overline{m}_{\mathbf{H}}^{(k,l)}\triangleq\overline{m}_{\mathbf{H}}^{(\phi_{k,l})}$ obtained with the weighting functions $\phi_{k,l}(\nu,\tau)=|\nu|^l|\tau|^k$ with $k,l\in\mathbb{N}_0$. Of particular importance are the *delay spread* $\tau_{\mathbf{H}}$ and *Doppler spread* $\nu_{\mathbf{H}}$ that are obtained with $k=2$, $l=0$ and $k=0$, $l=2$, respectively:

$$\tau_{\mathbf{H}}^2 \triangleq \overline{m}_{\mathbf{H}}^{(2,0)} = \frac{1}{\rho_{\mathbf{H}}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau^2\,C_{\mathbf{H}}(\nu,\tau)\,d\nu\,d\tau\,, \tag{9.5.13}$$

$$\nu_{\mathbf{H}}^2 \triangleq \overline{m}_{\mathbf{H}}^{(0,2)} = \frac{1}{\rho_{\mathbf{H}}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu^2\,C_{\mathbf{H}}(\nu,\tau)\,d\nu\,d\tau\,. \tag{9.5.14}$$

Within this framework, a WSSUS channel is called underspread if specific weighted integrals and moments of the scattering function are small.

Approximate eigenfunctions and eigenvalues of underspread channels. It is known [6] [11] [14] that signals with good time and/or frequency concentration can pass an underspread WSSUS channel almost undistorted, i.e., merely multiplied by a random complex factor. We will analyze this effect using the approach in [7]. We note that similar results in a deterministic context are reported in Articles 4.7 and 13.3.

A normalized transmit signal $s(t)$ that remains undistorted, i.e., $(\mathbf{H}s)(t)=\lambda\,s(t)$, is an eigenfunction of the system $\mathbf{H}$; the associated eigenvalue is given by $\lambda=\langle\mathbf{H}s,s\rangle$. Since $\mathbf{H}$ is random, the relation $(\mathbf{H}s)(t)=\langle\mathbf{H}s,s\rangle\,s(t)$ is more appropriately formulated in the mean-square sense, i.e.,

$$\mathrm{E}\big\{\big\|\mathbf{H}s-\langle\mathbf{H}s,s\rangle\,s\big\|^2\big\} \;=\; 0\,.$$

The eigenfunctions of a WSSUS channel $\mathbf{H}$ are random and generally do not possess a specific structure. However, in the *underspread* case, TF translates of a function

$g(t)$ with good TF concentration are approximate eigenfunctions. Specifically, consider the TF translates

$$g_{t_0,f_0}(t) = g(t-t_0)\, e^{j2\pi f_0 t}\,,$$

where $g(t)$ is a normalized function that is well concentrated about the origin of the TF plane. One can show

$$\mathrm{E}\big\{\big\|\mathbf{H}g_{t_0,f_0} - \langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle\, g_{t_0,f_0}\big\|^2\big\} \,\le\, \rho_{\mathbf{H}}^2\, \overline{m}_{\mathbf{H}}^{(\phi)}\,, \tag{9.5.15}$$

with $\phi(\nu,\tau) = 1 - |A_g^{(\alpha)}(\nu,\tau)|^2$ where $A_g^{(\alpha)}(\nu,\tau) = \int_{-\infty}^{\infty} g\big(t + (\frac{1}{2}-\alpha)\tau\big)\, g^*\big(t - (\frac{1}{2}+\alpha)\tau\big)\, e^{-j2\pi\nu t}\, dt$ denotes the generalized ambiguity function of $g(t)$. Therefore, if the channel is underspread, i.e., the channel's scattering function is concentrated about the origin (where $|A_g^{(\alpha)}(\nu,\tau)|^2 \approx |A_g^{(\alpha)}(0,0)|^2 = 1$ and thus $\phi(\nu,\tau) \approx 0$), the weighted integral $\overline{m}_{\mathbf{H}}^{(\phi)}$ will be small and one has the approximation (valid in the mean-square sense)

$$(\mathbf{H}g_{t_0,f_0})(t) \,\approx\, \langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle\, g_{t_0,f_0}(t)\,. \tag{9.5.16}$$

This shows that $g_{t_0,f_0}(t)$ is an approximate eigenfunction of $\mathbf{H}$. Furthermore, it can be shown that

$$\mathrm{E}\big\{\big|\langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle - L_{\mathbf{H}}^{(\alpha)}(t_0,f_0)\big|^2\big\} \,\le\, \rho_{\mathbf{H}}^2\, \overline{m}_{\mathbf{H}}^{(\phi')}\,, \tag{9.5.17}$$

with $\phi'(\nu,\tau) = |1 - A_g^{(\alpha)}(\nu,\tau)|^2$. Thus, under the same conditions as before, we have

$$\langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle \,\approx\, L_{\mathbf{H}}^{(\alpha)}(t_0,f_0) \tag{9.5.18}$$

(again valid in the mean-square sense), which shows that the approximate eigenvalue $\langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle$ is approximately equal to the GWS at the TF point (t_0,f_0).

In contrast to the exact eigenfunctions of $\mathbf{H}$, the approximate eigenfunctions $g_{t_0,f_0}(t)$ are TF translates of a single prototype function $g(t)$ and thus highly structured; they do not depend on the specific channel realization and their parameters t_0, f_0 have an immediate physical interpretation.

To illustrate the above eigenfunction/eigenvalue approximations, we simulated the transmission of a signal $g_{t_0,f_0}(t)$, with $g(t)$ a Hanning window of duration $T_g = 128\,\mu\mathrm{s}$, over a WSSUS channel. The channel's scattering function was $C_{\mathbf{H}}(\nu,\tau) = \frac{1}{\rho_{\mathbf{H}}^2}\, Q_{\mathbf{H}}(\nu)\, P_{\mathbf{H}}(\tau)$ with exponential $P_{\mathbf{H}}(\tau)$ (Eq. (9.5.9) with $\tau_0 = 1\,\mu\mathrm{s}$) and Jakes-type $Q_{\mathbf{H}}(\nu)$ (Eq. (9.5.10) with $\nu_{\max} = 305\,\mathrm{Hz}$). Fig. 9.1 illustrates the approximations (9.5.16) and (9.5.18) for a single channel realization. It is seen that the received signal $(\mathbf{H}g_{t_0,f_0})(t)$ and the approximation $L_{\mathbf{H}}^{(1/2)}(t_0,f_0)\, g_{t_0,f_0}(t)$ are practically identical. Furthermore, we used 500 realizations of $\mathbf{H}$ to estimate the normalized mean-square error $\mathrm{E}\big\{\big\|\mathbf{H}g_{t_0,f_0} - \langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle\, g_{t_0,f_0}\big\|^2\big\}/\rho_{\mathbf{H}}^2$ (see (9.5.15)). The result, $9 \cdot 10^{-4}$, confirms the validity of the eigenfunction/eigenvalue approximation. The associated upper bound $\overline{m}_{\mathbf{H}}^{(\phi)}$ (see (9.5.15)) was calculated as $5 \cdot 10^{-3}$. Finally,

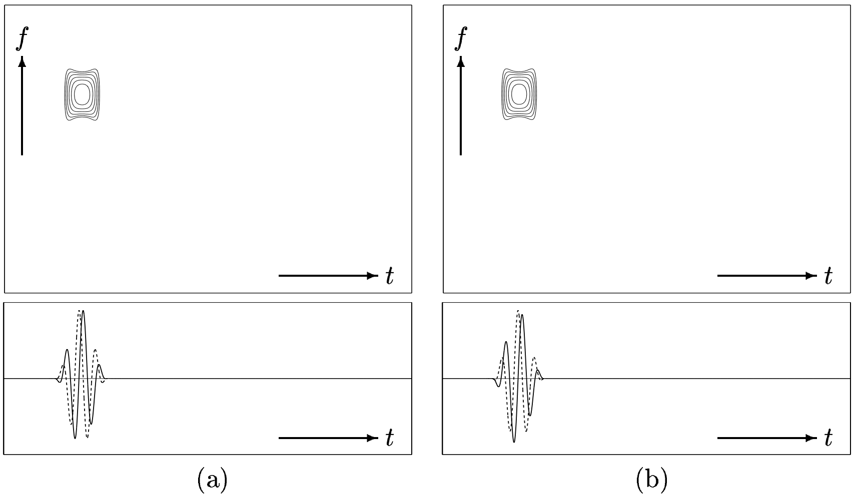


Figure 9.1: Eigenfunction/eigenvalue approximation for an underspread WSSUS channel: (a) Wigner distribution [9] (top) and real and imaginary parts (bottom) of received signal $r(t) = (\mathbf{H}g_{t_0,f_0})(t)$, (b) Wigner distribution (top) and real and imaginary parts (bottom) of approximation $L_{\mathbf{H}}^{(1/2)}(t_0,f_0)\, g_{t_0,f_0}(t)$.

the normalized mean-square error $\mathrm{E}\big\{\big|\langle \mathbf{H}g_{t_0,f_0}, g_{t_0,f_0}\rangle - L_{\mathbf{H}}^{(1/2)}(t_0,f_0)\big|^2\big\}/\rho_{\mathbf{H}}^2$ (see (9.5.17)) was estimated as $5 \cdot 10^{-6}$ and the associated upper bound $\overline{m}_{\mathbf{H}}^{(\phi')}$ was calculated as $2 \cdot 10^{-3}$.

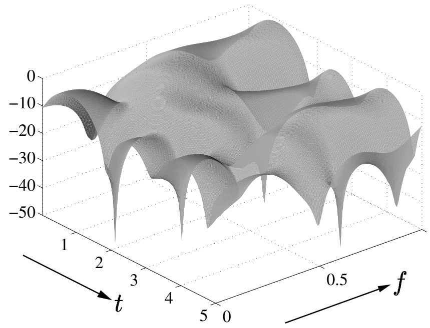
Sampling approximation for underspread channels. Next, we consider 2-D sampling of the channel's transfer function (GWS) $L_{\mathbf{H}}^{(\alpha)}(t,f)$. This is important for simplified channel representations that are used e.g. in the context of *orthogonal frequency division multiplexing (OFDM)* modulation [3] [8].

Consider the representation of a WSSUS channel $\mathbf{H}$ by the samples $L_{\mathbf{H}}^{(\alpha)}(kT,lF)$ of its GWS taken on the uniform sampling grid (kT,lF). The reconstructed (interpolated) GWS is given by

$$\hat{L}_{\mathbf{H}}^{(\alpha)}(t,f) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} L_{\mathbf{H}}^{(\alpha)}(kT,lF)\,\mathrm{sinc}\Big(\frac{\pi}{T}(t-kT)\Big)\,\mathrm{sinc}\Big(\frac{\pi}{F}(f-lF)\Big)\,,$$

with $\mathrm{sinc}(x) = \sin(x)/x$. For WSSUS channels with scattering function $C_{\mathbf{H}}(\nu,\tau)$ compactly supported within a rectangular area $[-\nu_{\max},\nu_{\max}] \times [-\tau_{\max},\tau_{\max}]$ and for sampling grid constants satisfying $T \le 1/(2\nu_{\max})$ and $F \le 1/(2\tau_{\max})$, the above reconstruction can be shown to be exact in the sense that $\mathrm{E}\big\{\big|\hat{L}_{\mathbf{H}}^{(\alpha)}(t,f) - L_{\mathbf{H}}^{(\alpha)}(t,f)\big|^2\big\} = 0$. Note that a smaller channel spread $\sigma_{\mathbf{H}} = 4\nu_{\max}\tau_{\max}$ allows for a coarser sampling grid and thus for a more parsimonious channel representation.

If the above conditions are not satisfied, the reconstructed GWS $\hat{L}_{\mathbf{H}}^{(\alpha)}(t,f)$ will



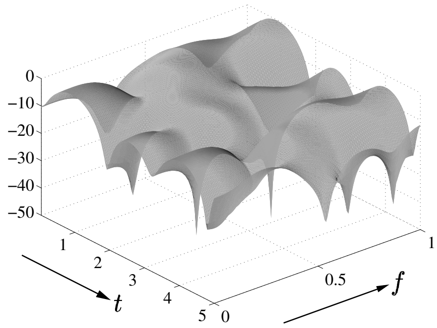


Figure 9.2: Sampling approximation for an underspread WSSUS channel: (a) $\big|L_{\mathbf{H}}^{(1/2)}(t,f)\big|^2$ and (b) $\big|\hat{L}_{\mathbf{H}}^{(1/2)}(t,f)\big|^2$ (in dB; time in ms, frequency in MHz).

contain errors due to aliasing. However, it can be shown that these errors are bounded as

$$\mathrm{E}\big\{\big|\hat{L}_{\mathbf{H}}^{(\alpha)}(t,f) - L_{\mathbf{H}}^{(\alpha)}(t,f)\big|^2\big\} \,\leq\, 2\,\rho_{\mathbf{H}}^2\,\big(\tau_{\mathbf{H}}^2F^2 + \nu_{\mathbf{H}}^2T^2\big)\,, \tag{9.5.19}$$

where $\tau_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ are the delay spread and Doppler spread as defined in (9.5.13), (9.5.14). Thus, for WSSUS channels with small $\tau_{\mathbf{H}}$ and/or small $\nu_{\mathbf{H}}$, i.e., for *underspread* channels, a sampling of the transfer function will result in negligible errors provided that the sampling periods T and F are chosen appropriately. Specifically, the upper error bound in (9.5.19) is minimized when $T/F = \tau_{\mathbf{H}}/\nu_{\mathbf{H}}$.

For the WSSUS channel with exponential/Jakes scattering function that was considered further above, and for sampling periods $T = 138\,\mu\text{s}$, $F = 136.72\,\text{kHz}$, the normalized mean-square error $\mathrm{E}\big\{\big|\hat{L}_{\mathbf{H}}^{(1/2)}(t,f) - L_{\mathbf{H}}^{(1/2)}(t,f)\big|^2\big\}/\rho_{\mathbf{H}}^2$ was estimated from 500 channel realizations as $6.4\cdot 10^{-3}$, and the upper bound $2\,\big(\tau_{\mathbf{H}}^2F^2 + \nu_{\mathbf{H}}^2T^2\big)$ was calculated as $3.2\cdot 10^{-2}$. Fig. 9.2 shows the squared magnitude of the true channel transfer function $L_{\mathbf{H}}^{(1/2)}(t,f)$ and of the reconstruction $\hat{L}_{\mathbf{H}}^{(1/2)}(t,f)$ for a specific channel realization.

9.5.4 Summary and Conclusions

In this article, we have considered time-frequency characterizations of (the second-order statistics of) random linear time-varying channels satisfying the assumption of wide-sense stationary uncorrelated scattering (WSSUS). We have shown that the practically important class of *underspread* WSSUS channels allows some interesting approximations. In particular, underspread WSSUS channels possess approximate eigenfunctions with time-frequency shift structure (which suggests the use of OFDM), and they can be discretized by means of a time-frequency sampling. Related considerations and results can be found in Articles 13.2 and 13.3.

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