A Tight Lower Bound for the Bit Error Ratio of Space-Time Block Codes

Gerhard Gritsch, Hans Weinrichter, Markus Rupp
Institute of Communications and Radio Frequency Engineering, Vienna University of Technology
Gusshausstrasse 25/389, A-1040 Vienna, Austria (ggritsch,jweinri,mrupp)@nt.tuwien.ac.at

***Abstract*— In this contribution a tight lower bound for the Bit Error Ratio (BER) of Space-Time Block Codes (STBCs) over uncorrelated wireless Multiple-Input Multiple-Output channels using Maximum Likelihood receiver is presented. We show that the performance of an STBC essentially depends on a few so-called Error Types (ETs), each ET being associated with a specific Euclidean distance governing the error probability. In our performance evaluation, only the smallest distance out of these distinct distances (one for each ET), is considered, thus leading to a Minimum Distance Lower Bound (*MDLB*). The resulting *MDLB* is compared to the *union bound* and to the simulated BER performance, showing a tight approximation for all STBCs considered.**

I. Introduction

Recent research shows that space-time codes are a powerful mean to increase the quality of data transmission over wireless Multiple-Input Multiple-Output (MIMO) channels. The performance evaluation of such codes requires extremely time consuming simulations. In order to save computation time and to get more insight into the error behavior, it is very important to find an analytic expression to approximate the resulting Bit Error Ratio (BER) performance. In this contribution, we focus on the performance of Space-Time **Block** Codes (STBCs). One way to estimate the code performance is to use a *union bound*. The disadvantage of most union bounds is, that they are very loose at low values of Signal to Noise Ratio (SNR). The SNR threshold for which the bound becomes tight depends on the STBC and on the applied modulation format. Our aim is to find a tight bound for the error performance of STBCs. To this end the ideas first presented in [1] are extended and generalized. For reasons of simplicity, all explanations in this paper are based on the cyclic STBC with BPSK modulation, as an illustrative example. Results for other STBCs and higher modulation formats (4PSK, 16QAM) are also shown. The rest of the paper is organized as follows. The system- and the channel model are defined in Sec. II. In Sec. III the term Error Type (ET) is defined and some properties of ETs are discussed. A tight *union bound* is derived in Sec. IV. A systematic way to analytically calculate the novel *MDLB* is presented in Sec. V. The *MDLB* and the simulated BER performance for some STBCs and several modulation formats are compared in Sec. VI. Conclusions close the paper.

II. System- and Channel Model

The transmission system for flat fading MIMO channels is defined as:

$$\mathbf{Y} = \mathbf{H}\mathbf{C} + \mathbf{N}\,, \tag{1}$$

where $\mathbf{Y}$ denotes the $(n_R \times n_B)$ receive signal matrix, $\mathbf{H}$ is the $(n_R \times n_T)$ MIMO channel matrix, $\mathbf{C}$ is the transmitted code word of size $(n_T \times n_B)$ and $\mathbf{N}$ is the additive noise matrix of size $(n_R \times n_B)$ where n_R denotes the number of receive antennas, n_T the number of transmit antennas and n_B the block length.

The noise $\mathbf{N}$ is an independent identically complex Gaussian distributed random matrix consisting of entries with zero mean and variance σ_n^2.

The temporal behavior of the channel is assumed to be quasi-static or block fading, i.e., the channel is constant for the duration of n_B time slots and changes arbitrarily after each block. Such a type of fading is observed in systems with interleavers in slowly fading channels. The channel $\mathbf{H}$ is an independent identically complex Gaussian distributed random matrix consisting of entries with zero mean and unit variance. Throughout the paper perfect channel knowledge at the receiver is assumed.

For the rest of the paper, the mean SNR is defined as the total signal power at the receiver side divided by the total noise power at the receiver (summed up over all receive antennas). This leads to:

$$\text{SNR} = \frac{n_T\, P_s}{\sigma_n^2}\,, \tag{2}$$

where the mean transmit power on each transmit antenna is equal and denoted by P_s.

III. Distances of Code Matrices

All concepts are explained by a simple example, namely the (4×4) cyclic STBC with BPSK modulation, which is defined by:

$$\mathbf{C} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_4 & s_1 & s_2 & s_3 \\ s_3 & s_4 & s_1 & s_2 \\ s_2 & s_3 & s_4 & s_1 \end{pmatrix}. \tag{3}$$

Using BPSK modulation ($s_i \in \pm 1$), there are $2^4 = 16$ different code matrices $\mathbf{C}_i$. Therefore, there are $16 \cdot 16 = 256$ possible crossover events, i.e., the code word $\mathbf{C}_i$ is transmitted and the receiver decides in favor of $\mathbf{C}_j$. Thus, 256 difference matrices $\mathbf{B} = \mathbf{C}_i - \mathbf{C}_j$ and 256 distance matrices $\mathbf{A} = \mathbf{B}^H\mathbf{B}$ can be calculated[1]. For each distance matrix a set of four eigenvalues $\lambda_j^{(k)}$ $(j = 1, \ldots, 4)$ can be calculated. The surprising result is, that there are only eight different sets of eigenvalues. This

[1]We have dropped the index i, j to simplify the notation

0-7803-8256-0/04/$20.00 (C) 2004 IEEE

results from the symmetries in the modulation alphabet and from the specific structure of the STBC under investigation.

Definition: *A so-called Error Type (ET k) is defined as the set of all crossover events which have the same set of eigenvalues $\lambda_j^{(k)}$ and the same number n_{bk} of different information bits (k=1, ... n_{ET}). The number of distinct ETs is denoted by n_{ET}. The number of crossover events which have the same set of eigenvalues is denoted by n_{Pk}.*

The eigenvalues $\lambda_j^{(k)}$, the number n_{Pk} of crossover events leading to a certain ETk and the number n_{bk} of different information bits for each ET utilizing the cyclic code are listed in Table I. The values in Table I have been found by an exhaustive computer search over all distance matrices.

TABLE I

SETS OF EIGENVALUES, THE NUMBER n_{Pk} OF CROSSOVER EVENTS LEADING TO A CERTAIN ET AND THE CORRESPONDING NUMBER n_{bk} OF DIFFERENT INFORMATION BITS FOR EACH ET UTILIZING THE CYCLIC CODE.

ETk	$\lambda_1^{(k)}$	$\lambda_2^{(k)}$	$\lambda_3^{(k)}$	$\lambda_4^{(k)}$	n_{Pk}	n_{bk}
k	0	0	0	0	16	0
1	4	4	4	4	64	1
2	16	8	8	0	64	2
3	16	16	0	0	32	2
4	36	4	4	4	32	3
5	20	20	4	4	32	3
6	16	16	16	16	8	4
7	32	32	0	0	4	4
8	64	0	0	0	4	4

Extremely important parameters of any coded data transmission are the squared Euclidean distances d_k^2 between any two different signal matrices obtained at the receiver output. The distance d_k^2 for one specific difference matrix $\mathbf{B}_k$ can be calculated as:

$$\begin{aligned} d_k^2 &= \|\mathbf{H}\,\mathbf{B}_k\|^2 = \sum_{i=1}^{n_R} \mathbf{h}_i \underbrace{\mathbf{B}_k\mathbf{B}_k{}^H}_{\mathbf{U}_k\mathbf{D}_k\mathbf{U}_k{}^H} \mathbf{h}_i{}^H = \sum_{i=1}^{n_R} \mathbf{h}_i^{(k)}\,\mathbf{D}\,\mathbf{h}_i^{(k)H} \\ &= \sum_{i=1}^{n_R}\sum_{j=1}^{n_T} \lambda_j^{(k)} \left|h_{ij}^{(k)}\right|^2 = \sum_{j=1}^{n_T} \lambda_j^{(k)} \underbrace{\sum_{i=1}^{n_R} \left|h_{ij}^{(k)}\right|^2}_{\alpha_j^{(k)}}, \end{aligned} \tag{4}$$

The vector $\mathbf{h}_i$ is the i-th row of $\mathbf{H}$. Because of the eight distinct ETs of the cyclic code, there are eight distinct sets of eigenvalues and therefore we can distinguish between eight different squared Euclidean distances d_k^2; $(k = 1 \dots 8)$.

At this point we can partition the set of all STBCs into two subsets (this partitioning will be used in Sec. V): **Subset one** is the set of STBCs which have identical unitary matrices $\mathbf{U}_k$ for all ETs. STBCs with this property are e.g.: the Extended Alamouti code [4] ($\mathbf{U}_k$ is a Hadamard matrix), the cyclic STBC ($\mathbf{U}_k$ is the FFT-matrix) and STBCs from the generalized orthogonal design[2]. A consequence of having the same matrix $\mathbf{U}$ for all ETs is, that the realizations of $\alpha_j^{(k)}$ are the same for all ETs for a specific channel realization $\mathbf{H}$. This property is used further ahead.

Subset two consists of the entire set of STBC not contained in subset one. A well known representative of subset two is the D-STTD [5]. For STBCs in this subset the realizations of $\alpha_j^{(k)}$ are not the same for all ETs for a specific channel realization. Although we are aware of the fact that the unitary matrices $\mathbf{U}_k$ are not the same for all ETs, we assumed that the realizations of $\alpha_j^{(k)}$ are equal for all ETs for one specific channel realization. The results shown further ahead prove this assumption to be practicable.

In order to obtain a tight approximation of the BER we need the Probability Density Functions (PDFs) of the squared distances d_k^2. In the following, we roughly show the way of calculating these PDFs. In principle, in our case of the cyclic STBC there are two different types of PDFs. If the non-zero eigenvalues of the ETs are all equal (as they are in case of ET1, ET3, ET6, ET7, ET8), then we get PDFs of the first type, namely χ^2-distributed RVs with a certain degree of freedom (n_{DF_k}) and the parameter $(\lambda^{(k)})$[3]:

$$p_{d_k^2}(\xi) = \frac{\xi^{n_{DF_k}-1}}{(\lambda^{(k)})^{n_{DF_k}}\Gamma(n_{DF_k})} e^{-\frac{\xi}{\lambda^{(k)}}} \quad k=1,3,6,7,8 \tag{5}$$

From Table I we get $n_{DF_1} = 4n_R$ and $\lambda^{(1)} = 4$ in case of ET1 (k=1), $n_{DF_3} = 2n_R$ and $\lambda^{(3)} = 16$ in case of ET3 (k=3) and so on. On the other hand, in Table I, we find two different eigenvalues in case of ET2, ET4 and ET5 (k=2,4,5). In the following these two different eigenvalues are denoted by $\lambda_1^{'(k)}$ and $\lambda_2^{'(k)}$ and the multiplicities of these eigenvalues are denoted by $n_1^{(k)}$ and $n_2^{(k)}$, respectively. With these definitions we can rewrite Eqn. (4):

$$d_k^2 = \sum_{l=1}^{n_1^{(k)}n_R} \lambda_1^{'(k)} |h_l|^2 + \sum_{m=1}^{n_2^{(k)}n_R} \lambda_2^{'(k)} |h_m|^2 . \tag{6}$$

h_l (l=1...$n_1^{(k)}n_R$) consists of the elements $h_{ij}^{(k)}$ for all i and j=1...$n_1^{(k)}$ and h_m (m=1...$n_2^{(k)}n_R$) consists of the elements $h_{ij}^{(k)}$ for all i and j=$n_1^{(k)}+1 ... n_1^{(k)}+n_2^{(k)}$. Thus, the Characteristic Function (CF) of the squared distance d_k^2 results in:

$$\Psi_{d_k^2}(j\,\omega) = \frac{1}{\left(1 - j\,\omega\,\lambda_1^{'(k)}\right)^{n_1^{(k)}n_R}\left(1 - j\,\omega\,\lambda_2^{'(k)}\right)^{n_2^{(k)}n_R}} .$$

Knowing the CF we can calculate the PDFs of the squared distance for ET2, ET4 and ET5 by applying the inverse Fourier transform to $\Psi_{d_k^2}(-j\,\omega)$. The inverse Fourier transformation can be performed easily after applying the partial fraction expansion of $\Psi_{d_k^2}(-j\,\omega)$. Then, the PDF of the distance d_k^2 results in:

$$p_{d_k^2}(\xi) = e^{-\frac{\xi}{\lambda_1^{'(k)}}} \sum_{p=1}^{n_1^{(k)}n_R} \frac{K_{p1}\,\xi^{p-1}}{(\lambda_1^{'(k)})^p (p-1)!} + e^{-\frac{\xi}{\lambda_2^{'(k)}}} \sum_{p=1}^{n_2^{(k)}n_R} \frac{K_{p2}\,\xi^{p-1}}{(\lambda_2^{'(k)})^p (p-1)!}$$

[2]For such STBCs the distance matrix is always diagonal ($\mathbf{A} = \mathbf{B}^H\mathbf{B} = \mathbf{D} = \mathbf{I}\mathbf{D}\mathbf{I}^H$ for all crossover events.) and $\mathbf{U} = \mathbf{I}$ for all error events.

[3]The index j at $\lambda_j^{(k)}$ is omitted, because all eigenvalues for these ETs are equal.

with

$$K_{pq}=\frac{\left.\left\{\frac{d^{n_q^{(k)}n_R-1}}{dx^{n_q^{(k)}n_R-1}}\left(1+x\lambda_q^{\prime(k)}\right)^{n_q^{(k)}n_R}\prod_{q=1}^{2}\left(\frac{1}{1+x\lambda_q^{\prime(k)}}\right)^{n_q^{(k)}n_R}\right\}\right|_{x=-\frac{1}{\lambda_q^{\prime(k)}}}}{(n_q^{(k)}n_R-1)!(\lambda_q^{\prime(k)})^{n_q^{(k)}n_R-1}}$$

In the following we will use these PDFs to get a tight approximation of the overall BER of a specific STBC.

IV. Union Bound of the BER

With the knowledge of all ETs and their distances d_k^2, we can calculate a *union bound* of the BER by summing up the weighted Pairwise Error Probabilities (PEPs) for all ETs. The weights w_k are built from the code parameters listed in Table I:

$$\text{BER} \leq \sum_{k}^{n_{ET}} w_k\,\text{PEP}_k \qquad w_k = \frac{n_{Pk}}{|\mathcal{A}|^{n_T}}\,\frac{n_{bk}}{\log_2(|\mathcal{A}|)\,n_T}\,. \quad (7)$$

$|\mathcal{A}|$ denotes the size of the modulation alphabet and PEP_k is defined as:

$$\text{PEP}_k = \text{E}_{\mathbf{H}}\left\{\text{Q}\left(\sqrt{\frac{d_k^2}{2\sigma_n^2}}\right)\right\} \quad (8)$$

PEP_k is calculated by averaging over the channel statistics. Q(.) denotes the Gaussian Q-function. Because of the two different types of distance distributions (calculated in Sec. III), we again distinguish two different PEP formulas. For ET1, ET3, ET6, ET7, ET8, where the distances d_k^2 are χ^2 distributed, the true PEP results in:

$$\begin{aligned}\text{PEP}(\lambda^{(k)},r_k) &= \int_0^\infty \text{Q}\left(\sqrt{\frac{\xi}{2\sigma_n^2}}\right)\frac{\xi^{r_k n_R-1}}{(\lambda^{(k)})^{r_k n_R}\,\Gamma(r_k n_R)}e^{-\frac{\xi}{\lambda^{(k)}}}d\xi \\ &= \left(\frac{1-\mu}{2}\right)^{r_k n_R}\sum_{l=0}^{r_k n_R-1}\binom{r_k n_R-1+l}{l}\left(\frac{1+\mu_k}{2}\right)^l, \\ &\text{with}\quad \mu_k = \sqrt{\frac{\lambda^{(k)}}{4\sigma_n^2+\lambda^{(k)}}}\,. \end{aligned} \quad (9)$$

Here r_k denotes the number of non-zero eigenvalues for the k-th ET. The solution of the integral is taken from [3]. For the remaining ETs (ET2, ET4, ET5), where the eigenvalues are not equal, the resulting PEP results in:

$$\begin{aligned}&\text{PEP}(\lambda_1^{\prime(k)},n_1^{(k)};\lambda_2^{\prime(k)},n_2^{(k)}) \\ &= \int_0^\infty \text{Q}\left(\sqrt{\frac{\xi}{2\sigma_n^2}}\right)p_{d_k^2}(\xi)d\xi \quad (10)\\ &= \sum_{p=1}^{n_1^{(k)}n_R}K_{p1}\left(\frac{1-\mu_1^{(k)}}{2}\right)^p\sum_{l=0}^{p-1}\binom{p-1+l}{l}\left(\frac{1+\mu_1^{(k)}}{2}\right)^l + \\ &\quad\sum_{p=1}^{n_2^{(k)}n_R}K_{p2}\left(\frac{1-\mu_2^{(k)}}{2}\right)^p\sum_{l=0}^{p-1}\binom{p-1+l}{l}\left(\frac{1+\mu_2^{(k)}}{2}\right)^l \\ &\quad\text{with}\quad \mu_1^{(k)}=\sqrt{\frac{\lambda_1^{\prime(k)}}{4\sigma_n^2+\lambda_1^{\prime(k)}}}\quad \mu_2^{(k)}=\sqrt{\frac{\lambda_2^{\prime(k)}}{4\sigma_n^2+\lambda_2^{\prime(k)}}}\end{aligned}$$

A comparison of this *refined union bound* with the well known *Tarokh union bound* and the simulated BER performance is shown in Fig. 1. For the *Tarokh union bound* the PEPs derived in [2] are inserted in Eqn. (7):

$$\text{PEP}_k \leq \prod_{j=1}^{r_k}\left(1+\lambda_j^{(k)}\frac{\text{SNR}}{4\,n_T}\right)^{-n_R}. \quad (11)$$

As can be seen in Fig. 1 the *Tarokh union bound* only reflects the global trend of the BER performance, but differs from simulation results by roughly 2dB SNR. This gap is quite remarkable. Our *refined union bound* approximates the simulation result much better and becomes tight at a BER of 10^{-3}. Note that the simulations have only be performed down to a BER of about 10^{-5}.

In our *refined union bound* calculated above, two sources of small errors exist: 1.) The error regions of different ETs are overlapping and thus lead to overestimated error probabilities at low SNR values. 2.) The ETs are treated as independent, although they are in fact not independent.

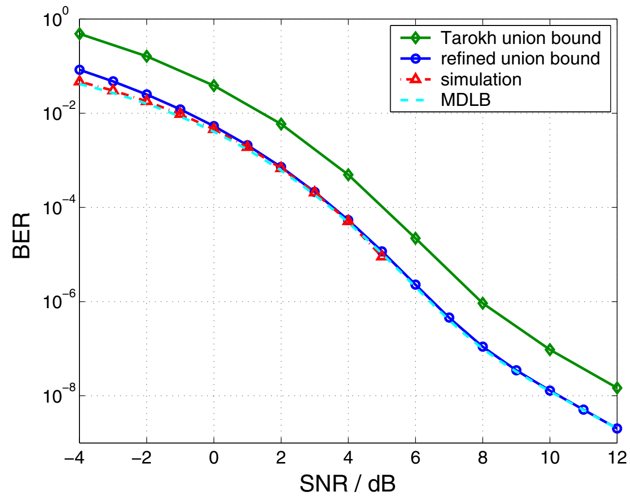

Fig. 1. The *Tarokh union bound* (solid green curve, ◇-marker), the *refined union bound* (solid blue curve, ○-marker), the simulation (dashed-dotted red curve, △-marker) and the *MDLB* (cyan dashed curve) of the BER vs. SNR performance for the cyclic code with BPSK modulation

V. Minimum Distance Lower Bound (MDLB)

A. Outline of the MDLB calculation

In this section we will give an overview how to calculate a tight *MDLB* for the BER of a specific STBC. Motivated by the nearest neighbor BER approximation [8] in Single-Input / Single-Output (SISO) systems, where only the nearest neighbored modulation signal points are taken into account, we now only take into account the ETk which leads to the minimum distance d_k^2. In fact, this contribution to the overall BER is weighted according to its probability of occurrence. Consider the above example of the cyclic STBC with BPSK modulation. In this case we have 8 distinct ETs. It depends on the channel realization which ET leads to the minimum distance. According to Table I and Eqn. (7) the corresponding weights w_k for the distinct each ETs are different. Therefore, we calculate first the error probability P_{ε_k} if ETk leads to the minimum distance $d^2_{\min}$=min$(d^2_1,\dots d^2_8)$. With the aid of the probability $P_{d^2_k=d^2_{\min}}$ that the distance d^2_k is $d^2_{\min}$ and the weights w_k the BER can be lower bounded by:

$$\mathrm{BER}_{MDLB} \geq \sum_{k=1}^{n_{ET}} w_k \, P_{\varepsilon_k} \, P_{d^2_k=d^2_{\min}} . \tag{12}$$

For the calculation of P_{ε_k} the PDF $p_{d^2_k|d^2_k=d^2_{\min}}$ of the squared distance d^2_k under the side constraint of $d^2_k = d^2_{\min}$ has to be evaluated. Then, the error probability P_{ε_k} can be calculated as:

$$\mathrm{P}_{\varepsilon_k} = \int_0^\infty \mathrm{Q}\left(\sqrt{\frac{\eta}{2\sigma^2_n}}\right) p_{d^2_k|d^2_k=d^2_{\min}}(\eta) d\eta \, . \tag{13}$$

The probability $P_{d^2_k=d^2_{\min}}$ of having $d^2_k = d^2_{\min}$ is:

$$P_{d^2_k=d^2_{\min}} = \int_0^\infty p_{d^2_k,d^2_k=d^2_{\min}}(\eta) \, d\eta \, , \tag{14}$$

where

$$p_{d^2_k|d^2_k=d^2_{\min}} = \frac{p_{d^2_k,d^2_k=d^2_{\min}}}{P_{d^2_k=d^2_{\min}}} \tag{15}$$

The most difficult task in calculating the *MDLB* is to find $p_{d^2_k,d^2_k=d^2_{\min}}$. The way of calculating this PDF is roughly explained in the following.

B. Dominating Error Types

For STBCs of subset one the unitary matrix $\mathbf{U}_k$ is the same for all ETs and therefore the realization of the random variables $\alpha_j^{(k)}$ are the same for all $k = 1, \dots 8$. For this reason for the cyclic code, d^2_4 corresponding to ET4 (see Table I) is always larger than d^2_1, due to the larger eigenvalues. (The eigenvalues of ET4 are equal or larger than the eigenvalues of ET1.) The same property holds for ET5 and ET6 which have also larger eigenvalues than ET1. Consequently, in fact only four ETs (ET1, ET2, ET3, ET8) contribute to the minimum distance.

Therefore, in the sum in Eqn. (12) only the ET1, ET2, ET3 and ET8 have to be considered to find $d^2_{\min}$.

C. Distance Calculation

In the following we roughly show how to calculate the PDF $p_{d^2_k,d^2_k=d^2_{\min}}$ of the distance d^2_k, under the constraint that it is $d^2_{\min}$. We focus on the PDF of the distance d^2_1 for ET1 for $d^2_1 = d^2_{\min}$ as an example. The random variables α_j [4] are uncorrelated (χ^2 distributed with $2n_R$ degrees of freedom) and thus the joint PDF of the four variables α_j $(j = 1 \dots 4)$ results in

$$p_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(\xi_1,\xi_2,\xi_3,\xi_4) = p_{\alpha_1}(\xi_1)p_{\alpha_2}(\xi_2)p_{\alpha_3}(\xi_3)p_{\alpha_4}(\xi_4) \, . \tag{16}$$

According to Eqn. (4) we have

$$d^2_1 = 4\alpha_1 + 4\alpha_2 + 4\alpha_3 + 4\alpha_4 \, , \tag{17}$$

[4]Here the superscript (1) is omitted.

and therefore we get the joint PDF for α_1, α_2, α_3 and d^2_1:

$$p_{\alpha_1,\alpha_2,\alpha_3,d^2_1}(\xi_1,\xi_2,\xi_3,\eta) = \\ p_{\alpha_1}(\xi_1)p_{\alpha_2}(\xi_2)p_{\alpha_3}(\xi_3)\frac{1}{4}p_{\alpha_4}\left(\frac{\eta}{4} - \xi_1 - \xi_2 - \xi_3\right) . \tag{18}$$

In order to get the marginal PDF of d^2_1, we have to integrate over α_1, α_2, α_3. Due to the condition $d^2_1 = d^2_{\min}$ the integration areas for α_1, α_2, α_3 are restricted to the region $\mathcal{R}$, where $d^2_1 = d^2_{\min}$ is valid. This region can be found taking into account the remaining relations for the distances:

$$\begin{aligned} d^2_2 &= 16\alpha_1 + 8\alpha_2 + 8\alpha_3 \\ d^2_3 &= 16\alpha_1 + 16\alpha_2 \\ d^2_8 &= 64\alpha_1 \, , \end{aligned} \tag{19}$$

and the constraints: $d^2_1 < d^2_2$, $d^2_1 < d^2_3$ and $d^2_1 < d^2_8$. Finally, the PDF can be calculated as:

$$p_{d^2_1,d^2_1=d^2_{\min}}(\eta) = \iiint_{\mathcal{R}} \Big[p_{\alpha_1}(\xi_1) \, p_{\alpha_2}(\xi_2) \, p_{\alpha_3}(\xi_3) \\ \cdot \frac{1}{4} p_{\alpha_4}\left(\frac{\eta}{4} - \xi_1 - \xi_2 - \xi_3\right) \Big] \, d\xi_1 \, d\xi_2 \, d\xi_3 \, . \tag{20}$$

D. Why is this BER approximation a lower bound?

As explained for the *union bound*, the overlapping error regions cause the error probability to be overestimated. Considering only the dominating minimum distance ET, only the most important BER contribution, is considered. For this reason, our method always underestimates the total error probability leading to a lower bound.

For STBCs of subset two the corresponding BER approximation is still a lower bound. In our calculation we assume that the channel parameters $\alpha_j^{(k)}$ are equal for all ETs. In reality, the variables $\alpha_j^{(k)}$ are not equal for all ETs. For this reason, the distances d^2_k, which depend on the variables $\alpha_j^{(k)}$ are in reality not that strongly correlated as with our assumption. Therefore, in reality the probability of getting a small distance is larger than with our simplifying assumption and thus the error probability in reality is larger than our *MDLB*.

As can be seen in Fig. 1, this *MDLB* is tightly approaching the simulated BER-performance. Even the small difference between the *union bound* elaborated in Sec. IV and simulation results at low SNR does not show up any more in the *MDLB* approximation.

VI. Simulation Results

All curves in Fig. 1 are calculated analytically, except for the simulated BER curve. In the same way the curves in Fig. 2 could be calculated in principle. However, the analytic calculation of the *MDLB* is quite involved and therefore the curves shown in Fig. 2 have been calculated numerically. The numerical evaluation of the *MDLB* is called "hybrid method" in the following. The hybrid method follows the same way as the analytic calculation, the only difference is that all integrations are performed numerically. With this hybrid method a very fast and very accurate calculation of the *MDLB*

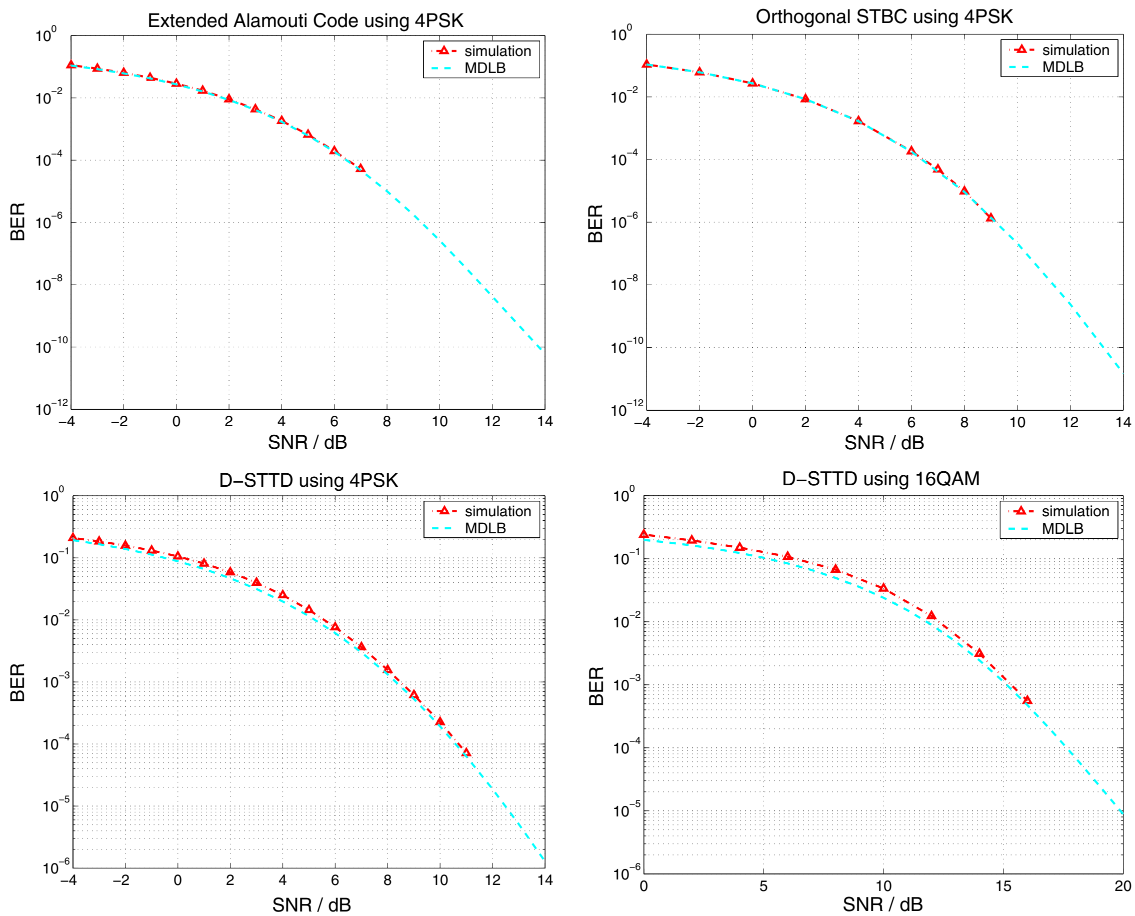

Fig. 2. Simulation results (dashed-dotted red curve, $\triangle$-marker) and *MDLB* (cyan dashed curve) of the BER vs. SNR curve for the Extended Alamouti code using 4PSK (upper left corner), an orthogonal STBC using 4PSK (upper right corner), the D-STTD using 4PSK (left lower corner) and the D-STTD using 16QAM (right lower corner).

is possible. The results obtained by the hybrid method are equal to the analytic calculation up to a BER of 10^{-12}, where the computing time of the MATLAB program is still quite short. As we can see in Fig. 2, the *MDLB* is a very tight lower bound for all examples shown here. Although there is a simpler way to calculate the BER performance of orthogonal STBCs [7], we also present our *MDLB* for such a code in order to show that our method estimates the BER-performance exactly. To this end, a well known representative of orthogonal STBCs, first presented in [6], is used. Note, that in the case of orthogonal STBCs only one ET dominates the BER [1] and therefore the performance can be calculated straightforward with our method. Note, that for the D-STTD the assumption of equal realizations of $\alpha_j^{(k)}$ for all ETs for one channel realization is not valid. However, also in this case the simulated BER is well approximated by the *MDLB*.

VII. Conclusion

In this paper a tight Minimum Distance Lower Bound (*MDLB*) for the BER performance of STBCs for uncorrelated MIMO channels with ML receivers is presented. The *MDLB* is compared to a *refined union bound* and to simulation results. It turns out that the *MDLB* is much tighter than other bounds. In most cases the *MDLB* coincides with the simulation results. The *MDLB* is exact for a certain subset of STBC (subset one specified in Sec. V). For other STBCs a reasonable assumption is made to simplify calculations. The comparison of the simulated BER performance with the *MDLB* for a specific representative of subset 2 (D-STTD) confirms the validity of this assumption.

VIII. Acknowledgment

The authors would like to thank Prof. Ernst Bonek for support and encouragement.

References

[1] G.Gritsch, H.Weinrichter, M.Rupp, "Understanding the BER-Performance of Space-Time Block Codes", SPAWC, Rome, pp. 400-404, June 2003.

[2] V.Tarokh, N.Seshadri, A.R.Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction", IEEE Trans. Inform. Theory, vol. 44, no. 2, pp. 744-765, March 1998.

[3] Hsiao-feng Lu, Y.Wang, P.V.Kumar, K.M.Chuggi, "On the Performance of Space-Time Codes", Information Theory Workshop, pp. 49-52., 2002

[4] M.Rupp, C.F.Mecklenbräuker, "On Extended Alamouti Schemes for Space-Time Coding", WPMC, Honolulu, pp. 115-118, Oct. 2002.

[5] 3GPP RAN1 contribution of Texas Instruments, "Double-STTD scheme for HSDPA systems with four transmit antennas: Link Level Simulation Results", May 21-25, Busan Korea, 2001.

[6] O.Tirkkonen, A.Hottinen, "Complex Space-Time Block Codes for four TX antennas", Globecom 2000, vol. 2, pp. 1005-1009, Dec. 2000.

[7] Mohammad Gharavi-Alkhansari, Alex Gershman,"Exact Symbol Error Probability Analysis for Orthogonal Space-Time Block Codes with Two- and Multi-Dimensional Constellations", Asilomar Conference 2003

[8] John G. Proakis, "Digital Communications", 3rd ed. McGraw-Hill, Inc. 1995.