MIMO paradox of non-orthogonal space–time block codes

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An astonishing difference in the performance of non-orthogonal space–time block codes (NOSTBCs) on wireless multiple-input/multiple-output (MIMO) systems compared to one-dimensional block codes operating on single-input/single-output (SISO) channels is derived. Using NOSTBCs, single symbol errors are no longer the only dominating error types, but multiple symbol errors can dominate the error performance in the high SNR range.

Introduction: Space–time block codes (STBCs) are an efficient means to transmit data reliably over wireless MIMO channels. Recently we have found that the error performance of NOSTBCs is not only dominated by single symbol errors but also in some SNR regions by multiple symbol errors. This error performance is in a sharp contrast to one-dimensional block codes used in memoryless SISO transmission systems where single symbol errors dominate the error performance in the entire SNR range. We call this peculiarity the MIMO paradox of NOSTBCs.

What is the reason for this MIMO paradox? Code-specific realisations of the MIMO fading channels can transform pairs of codewords (code matrices) with large distances at the transmitter (corresponding to multiple symbol errors) into pairs of signal matrices with extremely small distances at the receiver, and vice versa. This leads to the astonishing fact that the error performance of some NOSTBCs is not only dominated by single symbol errors, but also by multiple symbol errors, especially at high SNR. This is in sharp contrast to one-dimensional block codes used on SISO channels, and therefore we call this peculiarity the MIMO paradox. Interestingly, the above-mentioned error performance holds only for NOSTBCs; orthogonal STBCs behave like one-dimensional block codes on SISO channels, where only single symbol errors dominate the error performance in the entire SNR range. These single symbol errors correspond to codeword pairs with small distances.

Impact of MIMO paradox: First, in analytical calculations of the error performance of NOSTBCs a severe mistake is made if only single symbol errors are taken into account. Such an approach may underestimate the error performance at high SNR. The second consequence is the fact that, because of multiple symbol errors the maximum achievable diversity advantage (slope of the BER against SNR curve) of $n_T n_R$ at high SNR is not obtained. This is due to rank deficient distance matrices corresponding to multiple symbol errors in the code matrices, which lead to a flat error curve at high SNR.

BER performance of NOSTBCs: In the following we will explain the impact of single and multiple symbol errors on the BER against SNR curves by means of the EAC that is a rather popular example for an efficient NOSTBC. The EAC for four transmit antennas ($n_T = 4$) is defined by the code matrix

$$\boldsymbol{C} = \begin{pmatrix} s_1 & s_2^* & s_3^* & s_4 \\ s_2 & -s_1^* & s_4^* & -s_3 \\ s_3 & s_4^* & -s_1^* & -s_2 \\ s_4 & -s_3^* & -s_2^* & s_1 \end{pmatrix} \quad (1)$$

s_1–s_4 are information symbols taken from a certain signal alphabet $\mathcal{A}$. Applying a maximum likelihood receiver, the Euclidean distances of undisturbed signal pairs at the receiver determine the error performance.

These pairwise distances can be calculated as

$$d_{R,k}^2 = ||\boldsymbol{H}\boldsymbol{B}_k||_F^2 = \mathrm{trace}(\boldsymbol{H}\boldsymbol{A}_k\boldsymbol{H}^H) = \sum_{l=1}^{r_k} \alpha_l \lambda_l^{(k)} \quad (2)$$

where $\boldsymbol{B}_k = \boldsymbol{C} - \tilde{\boldsymbol{C}}$ is the difference matrix of a codeword pair $\boldsymbol{C}$ and $\tilde{\boldsymbol{C}}$ ($\tilde{\boldsymbol{C}}$ is a code matrix with the same structure as $\boldsymbol{C}$ but with at least partially different information symbols $\tilde{s}_1$–$\tilde{s}_4$), and $\boldsymbol{A}_k = \boldsymbol{B}_k\boldsymbol{B}_k^H$ is the corresponding distance matrix. The index k runs over all valid distinct difference matrices. The set of all nonzero eigenvalues of the distance matrix $\boldsymbol{A}_k$ is denoted by $\lambda_l^{(k)}$ and r_k is the rank of $\boldsymbol{A}_k$. The random coefficients α_l stem from the entries $h_{i,j}$ of the channel matrix $\boldsymbol{H}$. The channel coefficients $h_{i,j}$ are assumed to be independently and identically complex Gaussian distributed. Then the random variables α_l are χ^2 distributed with 2 n_R degrees of freedom. The number of receive antennas is n_R.

The instantaneous pairwise error probability (PEP) of two codewords $\boldsymbol{C}$ and $\tilde{\boldsymbol{C}}$ with the difference matrix $\boldsymbol{B}_k$ can be calculated [1]

$$\mathrm{PEP}_k = Q\left(\sqrt{\frac{d_{R,k}^2}{2\sigma_n^2}}\right) \quad (3)$$

where σ_n^2 is the variance of the additive complex Gaussian noise at each receiver input. Obviously, small distances lead to high instantaneous PEPs and large distances to small instantaneous PEPs. An important property of the distance metric given in (2) is the fact that the distance $d_{R,k}^2$ can be zero if $\boldsymbol{H}\boldsymbol{B}_k = 0$, even if both matrices $\boldsymbol{H}$ and $\boldsymbol{B}_k$ are non-zero! A necessary condition for $d_{R,k}^2 = 0$ is that both matrices $\boldsymbol{H}$ and $\boldsymbol{B}_k$ are rank deficient. In the following we will discuss code properties that can lead to rank deficient difference matrices $\boldsymbol{B}_k$. To illustrate the impact of these error types on the resulting error performance we will focus on two specific distance matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ of the EAC given in (1). Let us assume that QPSK modulation is used with the symbol alphabet: $(1, j, -1, -j)$. The distance matrix $\boldsymbol{A}_1$ corresponds to a single symbol error in a pair of code matrices with different symbols $s_1 = 1$ and $\tilde{s}_1 = j$. The remaining symbols of the two competing codewords $\boldsymbol{C}$ and $\tilde{\boldsymbol{C}}$ are assumed to be the same, i.e. $s_2 = \tilde{s}_2$, $s_3 = \tilde{s}_3$ and $s_4 = \tilde{s}_4$. The corresponding distance matrix A_1 has full rank $r = 4$ and four eigenvalues that are equal and have the value: $\lambda_1^{(1)} = \lambda_2^{(1)} = \lambda_3^{(1)} = \lambda_4^{(1)} = 2$. Let $\boldsymbol{A}_2$ correspond to a multiple (double) symbol error in a pair of codewords assuming $s_1 = 1$ but $\tilde{s}_1 = j$, $s_2 = \tilde{s}_2$, $s_3 = \tilde{s}_3$ and $s_4 = 1$ but $\tilde{s}_4$–j. The corresponding distance matrix $\boldsymbol{A}_2$ has only rank $r = 2$ and eigenvalues: $\lambda_1^{(2)} = \lambda_2^{(2)} = 8$ and $\lambda_3^{(2)} = \lambda_4^{(2)} = 0$. NOSTBCs are usually designed such that single symbol errors always lead to full-rank difference matrices. Then the corresponding distances cannot vanish, except for the trivial case $\boldsymbol{H} = 0$. On the other hand, multiple symbol errors often lead to rank deficient distance matrices, as for example in the case of $\boldsymbol{A}_2$ just discussed. In this case $\boldsymbol{H}\boldsymbol{B}_k$ can be zero for code-specific non-trivial realisations of the channel matrix $\boldsymbol{H}$.

Another important property of NOSTBCs is the fact that the probability of very small distances is higher for multiple symbol errors than for single symbol errors. This fact can be seen by comparing the probability density functions (pdf's) of distances $d_{R,k}^2$ for multiple symbol errors and for single symbol errors. Note that the distance $d_{R,k}^2$ is a random variable due to the random channel matrix $\boldsymbol{H}$. Fig. 1 shows the pdf's of the distances corresponding to the two distance matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ for a Rayleigh fading MIMO channel. As can be seen, the probability of very small distances $0 \le d_{R,k}^2 \le 8$ is higher for the double symbol error case (see the zoomed version in the upper right corner of Fig. 1).

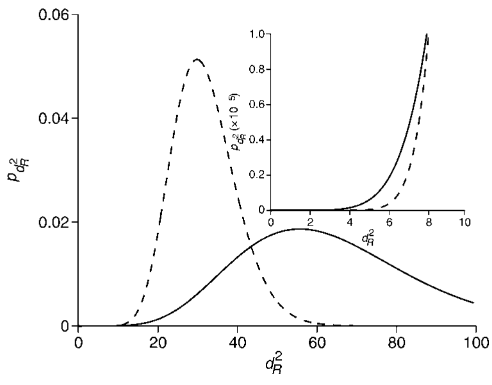

Fig. 1 *pdf of distances corresponding to* $\boldsymbol{A}_1$, $\boldsymbol{A}_2$.
------- $\boldsymbol{A}_1$ —— $\boldsymbol{A}_2$

A consequence of the higher probability of small distances is that at high SNR the error performance is dominated by this rank deficient distance matrix $\boldsymbol{A}_2$ corresponding to multiple symbol errors. In the following the mean PEP of single symbol errors corresponding to the distance matrix $\boldsymbol{A}_1$ is compared to the mean PEP of double symbol errors corresponding to $\boldsymbol{A}_2$. The mean PEPs can be calculated by averaging over all channel realisations or equivalently by averaging over all distances $d_{R,k}^2$. A simple upper bound on the mean PEP has been derived in [1]. For i.i.d. Rayleigh fading channel we have

$$\overline{\mathrm{PEP}}_k = \left(\prod_{l=1}^{rk}\left(1+\frac{\lambda_l^{(k)}}{4\sigma_n^2}\right)\right)^{-n_R} \quad (4)$$

Fig. 2 shows the two different mean PEPs corresponding to the full rank distance matrix $\boldsymbol{A}_1$ and the rank deficient distance matrix $\boldsymbol{A}_2$ against the mean SNR. The mean SNR is defined as $\mathrm{SNR} = n_T P_s/\sigma_n^2$, with $P_s = 1$. Obviously, the PEP for the single symbol error (distance matrix $\boldsymbol{A}_1$) dominates at low SNR and the multiple symbol error (distance matrix $\boldsymbol{A}_2$) dominates at high SNR.

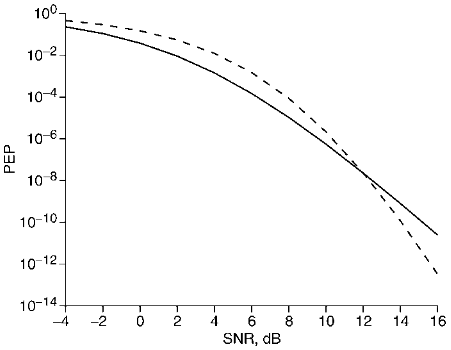

Fig. 2 *PEPs corresponding to distance matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$.*
------- $\boldsymbol{A}_1$ —— $\boldsymbol{A}_2$

Working out the example of the EAC in (1) for all possible distance matrices and their contribution Page 5 of 10 to the overall BER, it turns out that the two error events corresponding to $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ discussed above are indeed the dominating error events. Fig. 3 shows the overall error performance. A remarkable flattening out of the BER curve at high SNR can be observed, due to the double symbol error event. Note that the flattening out shows up at very small BER values. Therefore, we can use the union bound of the BER derived in [2, 3] to validate this effect, since simulations are too time consuming at such small BER values. Similar results have been obtained for the cyclic code discussed in [2, 3], where rank-one distance matrices corresponding to multiple errors dominate the BER at high SNR.

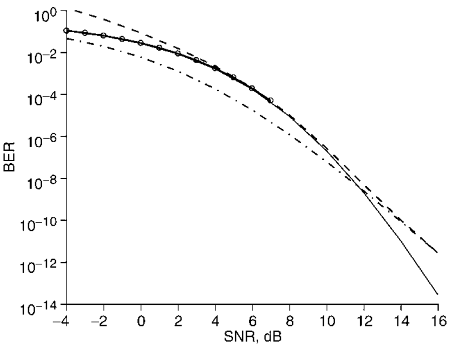

Fig. 3 *BER against SNR performance for EAC* ($n_T = n_R = 4$)
------- union bound ○ simulation
—— $\boldsymbol{A}_1$ – · – · $\boldsymbol{A}_2$

Conclusion: We have shown that the error performance of NOSTBCs is not only determined by single symbol errors in contrast to one-dimensional block codes designed for SISO channels. The impact of this MIMO paradox of NOSTBCs are flat BER against SNR curves at high SNR due to multiple symbol error events.

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