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LINEAR METHODS FOR TFARMA PARAMETER ESTIMATION AND SYSTEM APPROXIMATION

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ABSTRACT

Time-frequency autoregressive moving-average (TFARMA) models have recently been introduced as parsimonious parametric models for underspread nonstationary random processes. In this paper, we propose linear TFARMA and TFMA parameter estimators based on a high-order TFAR model. These estimators extend the Graupe–Krause–Moore and Durbin methods for time-invariant parameter estimation to underspread nonstationary processes. We also derive linear methods for approximating an underspread time-varying linear system by a TFARMA-type system. The linear equations obtained have Toeplitz/block-Toeplitz structure and thus can be solved efficiently by the Wax-Kailath algorithm. Simulation results demonstrate the performance of the proposed methods.

1. INTRODUCTION

This paper proposes linear estimators for *time-frequency autoregressive moving-average (TFARMA)* models. TFARMA models have been introduced in [1–3] as parsimonious models for underspread [4] nonstationary random processes. They are special time-varying ARMA (TVARMA) models [5] that are physically intuitive because of their formulation in terms of time shifts (delays) and frequency (Doppler) shifts.

The TFARMA Model. A TFARMA($M_{\rm A}, L_{\rm A}; M_{\rm B}, L_{\rm B}$) process $x[n]$, $n = 0, \dots, N-1$ is defined by the input-output relation

$$x[n] = -\sum_{(m,l)\in\mathcal{A}_1} a_{m,l}(\mathbb{S}_{m,l}\, x)[n] + \sum_{(m,l)\in\mathcal{B}} b_{m,l}(\mathbb{S}_{m,l}\, e)[n]\,. \quad (1)$$

Here, $a_{m,l}$ and $b_{m,l}$ are the TFAR and TFMA parameters, respectively; $\mathbb{S}_{m,l}$ is the cyclic time-frequency (TF) shift operator defined by $(\mathbb{S}_{m,l}\, x)[n] = e^{j\frac{2\pi}{N}ln} x[(n-m) \bmod N]$; $e[n]$ is a stationary white innovations process with variance 1; and the delay-Doppler (DD) support regions $\mathcal{A}_1$ and $\mathcal{B}$ are given by $\mathcal{A}_1 \triangleq \{1, \dots, M_{\rm A}\} \times \{-L_{\rm A}, \dots, L_{\rm A}\}$ and $\mathcal{B} \triangleq \{0, \dots, M_{\rm B}\} \times \{-L_{\rm B}, \dots, L_{\rm B}\}$, with $M_{\rm A}$ and $M_{\rm B}$ the TFAR and TFMA delay order and $L_{\rm A}$ and $L_{\rm B}$ the TFAR and TFMA Doppler order, respectively. The input-output relation (1) is depicted in Fig. 1, using the elementary cyclic time shift $(\mathbb{T}x)[n] = x[(n-1) \bmod N]$ and the elementary frequency shift $(\mathbb{M}x)[n] = e^{j\frac{2\pi}{N}n} x[n]$ (note that $\mathbb{S}_{m,l} = \mathbb{M}^l\mathbb{T}^m$).

The TFARMA model is parsimonious if the number of TFARMA parameters, $M_{\rm A}(2L_{\rm A}+1)+(M_{\rm B}+1)(2L_{\rm B}+1)$, is much smaller than the signal length N. Two special cases of the TFARMA($M_{\rm A}, L_{\rm A}$; $M_{\rm B}, L_{\rm B}$) model are the TFAR(M, L) model obtained for $M_{\rm A} = M$, $L_{\rm A} = L_{\rm B} = L$, and $M_{\rm B} = 0$ and the TFMA(M, L) model obtained for $M_{\rm B} = M$, $L_{\rm B} = L$, and $M_{\rm A} = L_{\rm A} = 0$.

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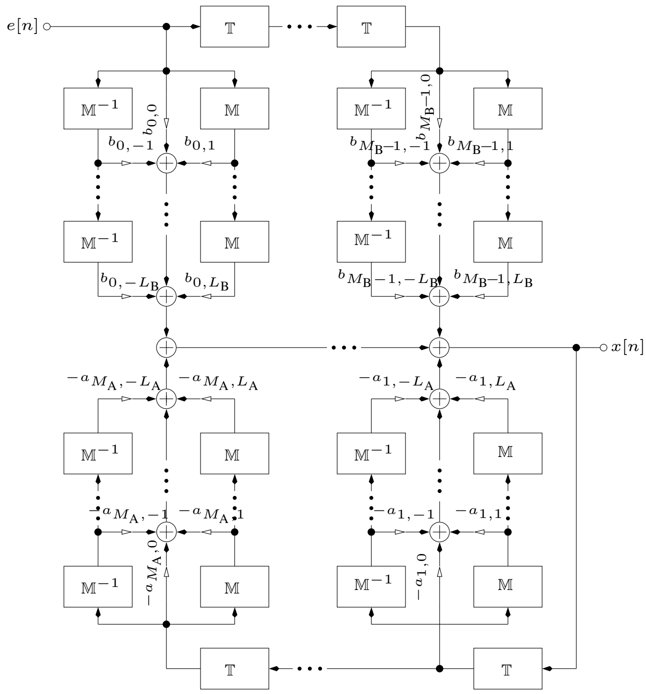

Figure 1: *Block diagram of the TFARMA(*$M_{\rm A}, L_{\rm A}; M_{\rm B}, L_{\rm B}$*) model.*

State of the Art. The standard approach to linear TVARMA parameter estimation is to estimate the TVAR part using extended Yule-Walker equations, estimate the input (intermediate TVMA) process of the TVAR part through inverse filtering, fit a high-order TVAR model to the intermediate TVMA process, estimate the innovations signal $e[n]$ through another inverse filtering, and finally use linear system identification methods to estimate the TVMA part [5]. This complicated procedure is used because classical linear methods for time-invariant ARMA and MA parameter estimation [6–8] cannot be straightforwardly extended to general TVARMA models (i.e., non-TFARMA models). A linear TFAR parameter estimator based on "TF-Yule-Walker" (TFYW) equations and a nonlinear TFMA parameter estimator based on the TF cepstrum have been proposed in [1] and [2], respectively. Methods for TFARMA order estimation and stabilization have been presented in [3].

Contribution and Paper Structure. In this paper, we propose linear methods for TFARMA and TFMA parameter estimation and system approximation. (TFAR parameter estimation is not discussed because a linear method—the TFYW method—was proposed in [1].) In Section 2, we consider TFARMA parameter estimation based on an intermediate high-order TFAR model. Our approach, which extends the (time-invariant) ARMA and MA methods of Graupe–Krause–Moore [7] and Durbin [6], is to formulate TFARMA parameter estimation as the approximation of a linear time-varying (LTV) system by a TFARMA system. In Section 3, therefore, we present linear methods for this system approximation problem. We obtain linear Toeplitz/block-Toeplitz equations that can be solved efficiently by the Wax-Kailath algorithm [9]. In Section 4, we apply our system approximation methods to TFARMA and TFMA parameter estimation. Finally, simulation results assessing the performance of our methods are provided in Section 5.

Some Fundamentals. The TFARMA($M_{\rm A}, L_{\rm A}; M_{\rm B}, L_{\rm B}$) process $x[n]$ in (1) is closely related to the causal LTV systems (operators)

$$\mathbb{A} \triangleq \sum_{(m,l)\in\mathcal{A}} a_{m,l}\, \mathbb{S}_{m,l}\,, \qquad \mathbb{B} \triangleq \sum_{(m,l)\in\mathcal{B}} b_{m,l}\, \mathbb{S}_{m,l}\,, \qquad (2)$$

where $\mathcal{A} \triangleq \{0,\ldots,M_{\rm A}\} \times \{-L_{\rm A},\ldots,L_{\rm A}\}$ and $\mathcal{B}$ was defined before. The operator $\mathbb{A}$ is monic, i.e., $a_{0,l} \triangleq \delta[l]$. The input-output relation (1) can be written in terms of the operators $\mathbb{A}$ and $\mathbb{B}$ as $(\mathbb{A}x)[n] = (\mathbb{B}e)[n]$ or, equivalently, as

$$x[n] = (\mathbb{H}_{\rm TFARMA}\, e)[n]\,, \quad \text{with } \mathbb{H}_{\rm TFARMA} \triangleq \mathbb{A}^{-1}\mathbb{B}\,. \qquad (3)$$

The causal LTV operator $\mathbb{H}_{\rm TFARMA}$ is an *innovations system* for the TFARMA process $x[n]$.

In what follows, we will use the *spreading function* (SF) of a causal LTV operator $\mathbb{H}$ that is defined as [10]

$$S_{\mathbb{H}}[m,l] \triangleq \langle \mathbb{H}, \mathbb{S}_{m,l}\rangle = \sum_{n=0}^{N-1} h[n,m]\, e^{-j\frac{2\pi}{N}ln}, \qquad (4)$$

where $h[n,m]$ is the time-varying impulse response of $\mathbb{H}$ and $\langle \mathbb{H}_1, \mathbb{H}_2\rangle \triangleq \sum_{n=0}^{N-1}\sum_{m=0}^{N/2-1} h_1[n,m]\, h_2^*[n,m]$. The SF is the coefficient function in an expansion of $\mathbb{H}$ into TF shift operators $\mathbb{S}_{m,l}$:

$$\mathbb{H} = \frac{1}{N}\sum_{m=0}^{N/2-1}\sum_{l=-N/2}^{N/2-1} S_{\mathbb{H}}[m,l]\,\mathbb{S}_{m,l}\,. \qquad (5)$$

An operator $\mathbb{H}$ whose SF is highly concentrated about the origin of the DD plane is called *underspread* [10].

Comparing (5) with (2), we see that

$$S_{\mathbb{A}}[m,l] = \begin{cases} N\, a_{m,l}\,, & (m,l)\in\mathcal{A} \\ 0\,, & \text{elsewhere}\,, \end{cases} \qquad (6)$$

and similarly for $S_{\mathbb{B}}[m,l]$. That is, the nonzero SF values of $\mathbb{A}$ and $\mathbb{B}$ are equal (up to a factor of N) to the TFAR and TFMA parameters $a_{m,l}$ and $b_{m,l}$, respectively.

2. SYSTEM APPROXIMATION APPROACH TO TF(AR)MA PARAMETER ESTIMATION

We consider a nonstationary process $x[n]$ that is *underspread*, i.e., its temporal and spectral correlations are negligible for larger time lags and frequency lags, respectively [4]. This assumption is justified in many applications. For an underspread process, it is always possible to find an underspread innovations system.

Our methods for TFARMA and TFMA estimation are based on an *intermediate high-order TFAR model* for $x[n]$. This TFAR model is assumed to have been previously estimated from one or several observed realizations of the process $x[n]$ (e.g., by means of the TFYW method proposed in [1]). It is given by (cf. (3))

$$\mathbb{H}_{\rm TFAR} = \mathbb{C}^{-1}\mathbb{D}_0\,, \qquad (7)$$

with the "pure TFAR part" $\mathbb{C} \triangleq \sum_{(m,l)\in\mathcal{C}} c_{m,l}\,\mathbb{S}_{m,l}$ where $\mathcal{C} \triangleq \{0,\ldots,M_{\rm C}\}\times\{-L_{\rm C},\ldots,L_{\rm C}\}$ and $c_{0,l} = \delta[l]$ and the degenerate TFMA part $\mathbb{D}_0 \triangleq \sum_{l=-L_{\rm C}}^{L_{\rm C}} d_{0,l}\,\mathbb{M}^l$. Here, $\mathbb{D}_0$ is included to model a time-varying variance of the white input process. The orders $M_{\rm C}, L_{\rm C}$ are chosen sufficiently high for good modeling accuracy but we assume that there is still $M_{\rm C}L_{\rm C} \ll N$ (i.e., $\mathbb{C}$ is an underspread operator). Furthermore, because $x[n]$ is assumed underspread, its (estimated) TFAR innovations operator $\mathbb{H}_{\rm TFAR} = \mathbb{C}^{-1}\mathbb{D}_0$ will be assumed to be underspread as well; this can be achieved through stabilization of the poles of $\mathbb{C}^{-1}$ [3].

The TFARMA parameter estimates are obtained by fitting a low-order TFARMA system $\mathbb{H}_{\rm TFARMA} = \mathbb{A}^{-1}\mathbb{B}$ (cf. (3)) to the intermediate high-order TFAR system $\mathbb{H}_{\rm TFAR}$ in (7), i.e., by determining $\mathbb{A}$ and $\mathbb{B}$ such that

$$\mathbb{H}_{\rm TFARMA} \approx \mathbb{H}_{\rm TFAR}\,.$$

This amounts to minimizing the error $\|\mathbb{H}_{\rm TFARMA} - \mathbb{H}_{\rm TFAR}\|^2$ where $\|\mathbb{H}\|^2 \triangleq \langle\mathbb{H},\mathbb{H}\rangle = \sum_{n=0}^{N-1}\sum_{m=0}^{N/2-1}|h[n,m]|^2$. Therefore, in the next section, we will introduce linear methods for approximating a general LTV system $\mathbb{H}$ by a TFARMA system $\mathbb{H}_{\rm TFARMA}$. In Section 4, these methods will be used to formulate computationally efficient TFARMA and TFMA parameter estimators.

3. TFARMA SYSTEM APPROXIMATION

We consider the problem of approximating an underspread, causal LTV system $\mathbb{H}$ by a TFARMA($M_{\rm A}, L_{\rm A}; M_{\rm B}, L_{\rm B}$) system $\mathbb{H}_{\rm TFARMA} = \mathbb{A}^{-1}\mathbb{B}$ of given—comparatively low—orders $M_{\rm A}, L_{\rm A}, M_{\rm B}, L_{\rm B}$. Because minimization of $\|\mathbb{H}_{\rm TFARMA} - \mathbb{H}\|^2 = \|\mathbb{A}^{-1}\mathbb{B} - \mathbb{H}\|^2$ is too difficult, we instead minimize $\|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2$, using the reasoning that[1] "if $\mathbb{A}^{-1}\mathbb{B} \approx \mathbb{H}$, then also $\mathbb{B} \approx \mathbb{A}\mathbb{H}$ and vice versa." The system approximation problem is thus formulated as

$$(\mathbb{A}_{\rm opt}, \mathbb{B}_{\rm opt}) \triangleq \arg\min_{\substack{\mathbb{A}\in\mathcal{S}_{M_{\rm A},L_{\rm A}}\\ \mathbb{B}\in\mathcal{S}_{M_{\rm B},L_{\rm B}}}} \|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2, \qquad (8)$$

where $\mathcal{S}_{M,L}$ denotes the Hilbert space of all LTV systems of the form $\sum_{m=0}^{M}\sum_{l=-L}^{L} c_{m,l}\,\mathbb{S}_{m,l}$ with given orders M, L (i.e., all LTV systems whose SF is zero outside the DD support region $\{0,\ldots,M\}\times\{-L,\ldots,L\}$). Due to the unitarity of the SF and the fact that $S_{\mathbb{B}}[m,l] = 0$ for $(m,l)\in\overline{\mathcal{B}}$, where $\overline{\mathcal{B}}$ denotes the complement of $\mathcal{B}$, the cost function in (8) can be rewritten as

$$\|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2 = \frac{1}{N}\sum_{m=0}^{N/2-1}\sum_{l=-N/2}^{N/2-1}\big|S_{\mathbb{B}}[m,l] - S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2$$

$$= \frac{1}{N}\sum_{(m,l)\in\mathcal{B}}\big|S_{\mathbb{B}}[m,l] - S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2 + \frac{1}{N}\sum_{(m,l)\in\overline{\mathcal{B}}}\big|S_{\mathbb{A}\mathbb{H}}[m,l]\big|^2. \qquad (9)$$

[1]We note, however, that the cost functions $\|\mathbb{A}^{-1}\mathbb{B} - \mathbb{H}\|^2$ and $\|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2$ are not equivalent. We have $\|\mathbb{A}^{-1}\mathbb{B} - \mathbb{H}\|^2 \geq \|\mathbb{B} - \mathbb{A}\mathbb{H}\|^2/\|\mathbb{A}\|_\infty^2$ where $\|\mathbb{A}\|_\infty = \sup_{\|x\|=1}\langle\mathbb{A}x, x\rangle$, i.e., $\|\mathbb{B}-\mathbb{A}\mathbb{H}\|^2$ normalized by $\|\mathbb{A}\|_\infty^2$ provides a lower bound on $\|\mathbb{H}_{\rm TFARMA} - \mathbb{H}\|^2 = \|\mathbb{A}^{-1}\mathbb{B} - \mathbb{H}\|^2$.

3.1. Optimization of $\mathbb{B}$

We first consider the minimization problem (8) for $\mathbb{B}$ with $\mathbb{A}$ fixed. This problem amounts to finding the best subspace approximation $\mathbb{B} \in \mathcal{S}_{M_\mathrm{B},L_\mathrm{B}}$ to the operator $\mathbb{AH}$. From the SF-domain formulation (9), it is seen that the SF of the optimum $\mathbb{B}$ satisfies the condition

$$S_\mathbb{B}[m,l] = S_{\mathbb{AH}}[m,l], \qquad (m,l) \in \mathcal{B}. \tag{10}$$

Note that this condition is consistent with the subspace constraint $\mathbb{B} \in \mathcal{S}_{M_\mathrm{B},L_\mathrm{B}}$. In fact, the same result is obtained by using the projection theorem [11] which requires that the approximation error $\mathbb{B}$–$\mathbb{AH}$ is orthogonal to the subspace $\mathcal{S}_{M_\mathrm{B},L_\mathrm{B}}$, i.e., $\langle \mathbb{B} - \mathbb{AH}, \mathbb{B}'\rangle = 0$ for all $\mathbb{B}' \in \mathcal{S}_{M_\mathrm{B},L_\mathrm{B}}$. Because $\{\mathbb{S}_{m,l}\}_{(m,l)\in\mathcal{B}}$ is a basis of the space $\mathcal{S}_{M_\mathrm{B},L_\mathrm{B}}$, this can be rewritten as $\langle \mathbb{B} - \mathbb{AH}, \mathbb{S}_{m,l}\rangle = 0$ for $(m,l) \in \mathcal{B}$, or equivalently $\langle \mathbb{B}, \mathbb{S}_{m,l}\rangle = \langle \mathbb{AH}, \mathbb{S}_{m,l}\rangle$ for $(m,l) \in \mathcal{B}$. Comparing with (4), we see that this is indeed equivalent to (10).

Condition (10) can be rewritten as

$$b_{m,l} = \frac{1}{N} \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_\mathbb{H}[m-m', l-l']\, e^{-j\frac{2\pi}{N}m'(l-l')}, \qquad (m,l) \in \mathcal{B}, \tag{11}$$

where we used the fact that the nonzero values of $S_\mathbb{B}[m,l]$ and $S_\mathbb{A}[m,l]$ are given by $N b_{m,l}$ and $N a_{m,l}$, respectively (see (6)). The right-hand side of (11) is the *twisted convolution* [12] of $S_\mathbb{A}[m,l]$ and $S_\mathbb{H}[m,l]$, which differs from the ordinary 2-D convolution by the phase factor $e^{-j\frac{2\pi}{N}m'(l-l')}$. Although not explicitly indicated by our notation, all convolutions and twisted convolutions are cyclic with period N. The relation (11) allows us to calculate the TFMA parameters $b_{m,l}$ (i.e., $\mathbb{B}$) from the TFAR parameters $a_{m,l}$ (i.e., $\mathbb{A}$) and the SF of $\mathbb{H}$.

Since we assumed the model orders $M_\mathrm{A}, L_\mathrm{A}, M_\mathrm{B}, L_\mathrm{B}$ to be small, $\mathcal{A} = \{0,\ldots,M_\mathrm{A}\} \times \{-L_\mathrm{A},\ldots,L_\mathrm{A}\}$ and $\mathcal{B} = \{0,\ldots,M_\mathrm{B}\} \times\{-L_\mathrm{B},\ldots,L_\mathrm{B}\}$ are small regions about the origin of the DD plane. Hence, we can set $e^{-j\frac{2\pi}{N}m'(l-l')} \approx 1$ in (11), whereby the twisted convolution is approximated by an ordinary 2-D convolution. We thus obtain

$$b_{m,l} \approx \frac{1}{N} \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_\mathbb{H}[m-m', l-l'], \qquad (m,l) \in \mathcal{B}. \tag{12}$$

This expression allows a simplified—though only approximate—calculation of the TFMA parameters $b_{m,l}$.

3.2. Optimization of $\mathbb{A}$

Next, we calculate the optimum TFAR operator $\mathbb{A}$. Because the optimum $\mathbb{B}$ satisfies (10), the cost function (9) becomes

$$\begin{aligned}\|\mathbb{B}-\mathbb{AH}\|^2 &= \frac{1}{N} \sum_{(m,l)\in\overline{\mathcal{B}}} \big|S_{\mathbb{AH}}[m,l]\big|^2 \\ &= \frac{1}{N} \sum_{(m,l)\in\overline{\mathcal{B}}} \bigg| \sum_{(m',l')\in\mathcal{A}} a_{m',l'}\, S_\mathbb{H}[m-m', l-l']\, e^{-j\frac{2\pi}{N}m'(l-l')}\bigg|^2 \\ &= \frac{1}{N} \sum_{(m,l)\in\overline{\mathcal{B}}} \bigg| S_\mathbb{H}[m,l] \\ &\qquad + \sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, S_\mathbb{H}[m-m', l-l']\, e^{-j\frac{2\pi}{N}m'(l-l')}\bigg|^2,\end{aligned} \tag{13}$$

where the last expression follows from $a_{0,l} = \delta[l]$. The minimization of this expression with respect to the parameters $a_{m,l}$, $(m,l) \in \mathcal{A}_1$ is a linear least-squares problem of the form

$$\boldsymbol{a}_\mathrm{opt} = \arg\min_{\boldsymbol{a}} \|\tilde{\boldsymbol{S}}\boldsymbol{a} - \boldsymbol{s}\|_F^2, \tag{14}$$

where $\|\cdot\|_F$ denotes the Frobenius norm and the vectors $\boldsymbol{a}$, $\boldsymbol{s}$ and matrix $\tilde{\boldsymbol{S}}$ are as follows. The parameter vector $\boldsymbol{a}$ of length $M_\mathrm{A}(2L_\mathrm{A}+1)$ is defined as

$$\boldsymbol{a} = \big[\boldsymbol{a}_1^T \cdots \boldsymbol{a}_{M_\mathrm{A}}^T\big]^T \quad \text{with} \quad \boldsymbol{a}_m = \big[a_{m,-L_\mathrm{A}} \cdots a_{m,L_\mathrm{A}}\big]^T. \tag{15}$$

The vector $\boldsymbol{s}$ of length $N^2/2 - (M_\mathrm{B}+1)(2L_\mathrm{B}+1)$ is given by

$$\boldsymbol{s} = \big[\boldsymbol{s}_0'^T \cdots \boldsymbol{s}_{M_\mathrm{B}}'^T\ \boldsymbol{s}_{M_\mathrm{B}+1}^T \cdots \boldsymbol{s}_{N/2-1}^T\big]^T$$

with the vectors

$$\begin{aligned}\boldsymbol{s}'_m &= \big[S_\mathbb{H}[m,-N/2] \cdots S_\mathbb{H}[m,-L_\mathrm{B}-1]\ \ S_\mathbb{H}[m,L_\mathrm{B}+1] \\ &\qquad \cdots S_\mathbb{H}[m,N/2-1]\big]^T \\ \boldsymbol{s}_m &= \big[S_\mathbb{H}[m,-N/2] \cdots S_\mathbb{H}[m,N/2-1]\big]^T\end{aligned}$$

of length $N - (2L_\mathrm{B}+1)$ and N, respectively. Finally, $\tilde{\boldsymbol{S}}$ is an $\big[N^2/2 - (M_\mathrm{B}+1)(2L_\mathrm{B}+1)\big] \times M_\mathrm{A}(2L_\mathrm{A}+1)$ matrix given by

$$\tilde{\boldsymbol{S}} = \big[\tilde{\boldsymbol{S}}'_0 \cdots \tilde{\boldsymbol{S}}'_{M_\mathrm{B}}\ \tilde{\boldsymbol{S}}_{M_\mathrm{B}+1} \cdots \tilde{\boldsymbol{S}}_{N/2-1}\big]^T$$

with the matrices

$$\begin{aligned}\tilde{\boldsymbol{S}}'_m &= \big[\tilde{\boldsymbol{s}}[m,-N/2] \cdots \tilde{\boldsymbol{s}}[m,-L_\mathrm{B}-1]\ \ \tilde{\boldsymbol{s}}[m,L_\mathrm{B}+1] \\ &\qquad \cdots \tilde{\boldsymbol{s}}[m,N/2-1]\big] \\ \tilde{\boldsymbol{S}}_m &= \big[\tilde{\boldsymbol{s}}[m,-N/2] \cdots \tilde{\boldsymbol{s}}[m,N/2-1]\big]\end{aligned}$$

of size $\big[N - (2L_\mathrm{B}+1)\big] \times M_\mathrm{A}(2L_\mathrm{A}+1)$ and $N \times M_\mathrm{A}(2L_\mathrm{A}+1)$, respectively, where $\tilde{\boldsymbol{s}}[m,l]$ is the length-$M_\mathrm{A}(2L_\mathrm{A}+1)$ vector obtained by stacking the columns of the $(2L_\mathrm{A}+1) \times M_\mathrm{A}$ matrix $\tilde{\boldsymbol{S}}[m,l]$ whose entries are

$$\big(\tilde{\boldsymbol{S}}[m,l]\big)_{l',m'} = S_\mathbb{H}[m-m', l-l'+L_\mathrm{A}+1]\, e^{-j\frac{2\pi}{N}m'(l-l'+L_\mathrm{A}+1)}, \qquad l' = 1,\ldots,2L_\mathrm{A}+1,\ m' = 1,\ldots,M_\mathrm{A}.$$

According to (14), the optimum TFAR parameters $\boldsymbol{a}_\mathrm{opt}$ are given by the solution of the system of $M_\mathrm{A}(2L_\mathrm{A}+1)$ linear equations [13]

$$\tilde{\boldsymbol{S}}^H\tilde{\boldsymbol{S}}\boldsymbol{a} = \tilde{\boldsymbol{S}}^H\boldsymbol{s}. \tag{16}$$

3.3. Efficient Suboptimum Calculation of $\mathbb{A}$

The equations (16) do not have a special structure that would allow an efficient solution. We now propose a more efficient but generally suboptimum method for calculating the TFAR system $\mathbb{A}$.

The optimum TFAR parameters minimize (14) or equivalently (13). They can hence be viewed as the least-squares solution to the overdetermined system of equations $\tilde{\boldsymbol{S}}\boldsymbol{a} = \boldsymbol{s}$ or equivalently

$$\sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, S_\mathbb{H}[m-m', l-l']\, e^{-j\frac{2\pi}{N}m'(l-l')} = -S_\mathbb{H}[m,l], \qquad (m,l) \in \overline{\mathcal{B}}.$$

These are $N^2/2-(M_\mathrm{B}+1)(2L_\mathrm{B}+1)$ equations in the $M_\mathrm{A}(2L_\mathrm{A}+1)$ unknowns $a_{m,l}$. Rather than solving this overdetermined system of equations in the least-squares sense, we now propose to calculate the *exact* solution of a *subset* of $M_\mathrm{A}(2L_\mathrm{A}+1)$ equations, corresponding to $M_\mathrm{A}(2L_\mathrm{A}+1)$ DD indices $(m,l) \in \widetilde{\mathcal{B}}$ where $\widetilde{\mathcal{B}} \subset \overline{\mathcal{B}}$.

By this approach, we force $S_{\mathbb{AH}}[m,l]$ to be zero on $\widetilde{\mathcal{B}}$ instead of minimizing the energy of $S_{\mathbb{AH}}[m,l]$ on $\overline{\mathcal{B}}$. We choose

$$\widetilde{\mathcal{B}} \triangleq \{M_{\mathrm{B}}+1,\ldots,M_{\mathrm{B}}+M_{\mathrm{A}}\} \times \{-L_{\mathrm{A}},\ldots,L_{\mathrm{A}}\}$$

because it is within $\overline{\mathcal{B}}$ but still close to the origin of the DD plane. Since $\mathbb{A}$ and $\mathbb{H}$ are underspread, i.e., their SFs are concentrated about the origin, this choice allows us (i) to force comparatively dominant components of $S_{\mathbb{AH}}[m,l]$ to be zero and (ii) to use the underspread approximation $e^{-j\frac{2\pi}{N}m'(l-l')} \approx 1$. With this approximation, we obtain the system of equations

$$\sum_{(m',l')\in\mathcal{A}_1} a_{m',l'}\, S_{\mathbb{H}}[m-m',l-l'] \,=\, -S_{\mathbb{H}}[m,l]\,, \qquad (m,l)\in\widetilde{\mathcal{B}}. \tag{17}$$

These $M_{\mathrm{A}}(2L_{\mathrm{A}}+1)$ linear equations in the $M_{\mathrm{A}}(2L_{\mathrm{A}}+1)$ unknowns $a_{m,l}$ involve an ordinary 2-D convolution and have the form of underspread extended TFYW equations (cf. [1]). They can be compactly written as

$$\bar{\boldsymbol{S}}\boldsymbol{a} \,=\, -\bar{\boldsymbol{s}}\,, \tag{18}$$

with $\boldsymbol{a}$ defined as in (15), $\bar{\boldsymbol{s}}$ defined by

$$\bar{\boldsymbol{s}} = \left[\bar{\boldsymbol{s}}_1^T \cdots \bar{\boldsymbol{s}}_{M_{\mathrm{A}}}^T\right]^T$$

where

$$\bar{\boldsymbol{s}}_m = \left[S_{\mathbb{H}}[M_{\mathrm{B}}+m,-L_{\mathrm{A}}] \,\cdots\, S_{\mathbb{H}}[M_{\mathrm{B}}+m,L_{\mathrm{A}}]\right]^T,$$

and the $M_{\mathrm{A}}(2L_{\mathrm{A}}+1)\times M_{\mathrm{A}}(2L_{\mathrm{A}}+1)$ Toeplitz/block-Toeplitz (TBT) matrix[2]

$$\bar{\boldsymbol{S}} = \mathrm{toep}\big\{\boldsymbol{S}_{M_{\mathrm{B}}+M_{\mathrm{A}}-1},\ldots,\boldsymbol{S}_{M_{\mathrm{B}}-M_{\mathrm{A}}+1}\big\}$$

containing the $(2L_{\mathrm{A}}+1)\times(2L_{\mathrm{A}}+1)$ Toeplitz blocks

$$\boldsymbol{S}_m = \mathrm{toep}\big\{S_{\mathbb{H}}[m,2L_{\mathrm{A}}],\ldots,S_{\mathbb{H}}[m,-2L_{\mathrm{A}}]\big\}\,.$$

Since the equations (18) have TBT structure, the *Wax-Kailath algorithm* [9] can be used for solving them with complexity $\mathcal{O}(M_{\mathrm{A}}^2 L_{\mathrm{A}}^3)$.

3.4. Special Cases: TFMA and TFAR System Approximation

The pure TFMA case, i.e., approximation of $\mathbb{H}$ by $\mathbb{H}_{\mathrm{TFMA}} = \mathbb{B}$, is a special case for which $a_{m,l} = \delta[m]\delta[l]$. The TFMA parameters are here obtained from (11) as

$$b_{m,l} \,=\, \frac{1}{N}\, S_{\mathbb{H}}[m,l]\,, \qquad (m,l)\in\mathcal{B}\,.$$

The TFAR case, i.e., approximation of $\mathbb{H}$ by $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{A}^{-1}\mathbb{B}_0$ (cf. (7)), is another special case that is obtained for $b_{m,l} = b_{0,l}\,\delta[m]$. The TFAR parameters $a_{m,l}$ can be calculated, e.g., by the suboptimum method of Section 3.3 with $M_{\mathrm{B}} = 0$. The TFMA parameters $b_{0,l}$ are subsequently obtained by (11) or (12) evaluated for $m=0$. The entire calculation is analogous to the TFYW method for TFAR parameter estimation presented in [1], with the SF $S_{\mathbb{H}}[m,l]$ taking the place of the expected ambiguity function appearing in [1].

4. TFARMA AND TFMA PARAMETER ESTIMATORS

Let us now return to our original problem of developing TFARMA and TFMA parameter estimators for an underspread process $x[n]$.

[2]The notation $\bar{\boldsymbol{S}} = \mathrm{toep}\big\{\boldsymbol{S}_{M_{\mathrm{B}}+M_{\mathrm{A}}-1},\ldots,\boldsymbol{S}_{M_{\mathrm{B}}-M_{\mathrm{A}}+1}\big\}$ means that the blocks of the block-diagonals of $\bar{\boldsymbol{S}}$ ordered from SW to NE are given by $\boldsymbol{S}_{M_{\mathrm{B}}+M_{\mathrm{A}}-1},\ldots,\boldsymbol{S}_{M_{\mathrm{B}}-M_{\mathrm{A}}+1}$.

As explained in Section 2, our approach is to approximate an intermediate high-order TFAR operator $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{C}^{-1}\mathbb{D}_0$ previously estimated from one or several observed realizations of $x[n]$ by a low-order TFARMA model $\mathbb{H}_{\mathrm{TFARMA}} = \mathbb{A}^{-1}\mathbb{B}$ or a low-order TFMA model $\mathbb{H}_{\mathrm{TFMA}} = \mathbb{B}$. As explained in Section 2, $\mathbb{C}$ is assumed to be underspread.

4.1. TFARMA Parameter Estimation

We consider estimation of a *monic* TFARMA$(M_{\mathrm{A}},L_{\mathrm{A}};M_{\mathrm{B}},L_{\mathrm{B}})$ model (i.e., $b_{0,l} = \delta[l]$) based on a monic intermediate high-order TFAR$(M_{\mathrm{C}},L_{\mathrm{C}})$ model (i.e., $d_{0,l} = \delta[l]$ or, equivalently, $\mathbb{D}_0 = \mathbb{I}$ and thus $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{C}^{-1}$ in (7)). Our method extends the Graupe–Krause–Moore method for time-invariant ARMA estimation [7] to the TFARMA case. Because the TFARMA model is monic and thus cannot model an input variance different from 1, we allow the variance σ_e^2 of the innovations process $e[n]$ to be different from 1. A simple estimator for σ_e^2 is the sample variance of the residuals $\hat{e}[n]$ that are obtained by inverse filtering based on the intermediate TFAR model $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{C}^{-1}$.

Our goal thus is to match a TFARMA system $\mathbb{H}_{\mathrm{TFARMA}} = \mathbb{A}^{-1}\mathbb{B}$ to the monic intermediate TFAR system $\mathbb{H}_{\mathrm{TFAR}} = \mathbb{C}^{-1}$. That is, we wish to calculate $\mathbb{A},\mathbb{B}$ such that $\mathbb{A}^{-1}\mathbb{B} \approx \mathbb{C}^{-1}$. Pre- and postmultiplying this relation by $\mathbb{A}$ and $\mathbb{C}$, respectively, we obtain $\mathbb{BC} \approx \mathbb{A}$. This is to be solved in the least-squares sense, i.e.,

$$(\mathbb{A}_{\mathrm{opt}},\mathbb{B}_{\mathrm{opt}}) \,\triangleq\, \arg\min_{\substack{\mathbb{A}\in\mathcal{S}_{M_{\mathrm{A}},L_{\mathrm{A}}}\\ \mathbb{B}\in\mathcal{S}_{M_{\mathrm{B}},L_{\mathrm{B}}}}} \|\mathbb{A}-\mathbb{BC}\|^2.$$

This is identical to (8) with $\mathbb{H}$ replaced by the (underspread) operator $\mathbb{C}$ and the roles of $\mathbb{A}$ and $\mathbb{B}$ interchanged. Hence, both the optimum method and the low-complexity, suboptimum method of Section 3 can immediately be applied with obvious modifications.

In what follows, we briefly discuss the application of the low-complexity method. According to (17), an approximation to the optimum TFMA parameters $b_{m,l}$ is given by the solution to the system of equations (note that $S_{\mathbb{C}}[m,l] = N\,c_{m,l}$ on $\mathcal{B}_1$)

$$\sum_{(m',l')\in\mathcal{B}_1} b_{m',l'}\, c_{m-m',l-l'} \,=\, -c_{m,l}\,, \qquad (m,l)\in\tilde{\mathcal{A}} \tag{19}$$

with $\mathcal{B}_1 \triangleq \{1,\ldots,M_{\mathrm{B}}\}\times\{-L_{\mathrm{B}},\ldots,L_{\mathrm{B}}\}$ and $\tilde{\mathcal{A}} \triangleq \{M_{\mathrm{A}}+1,\ldots,M_{\mathrm{A}}+M_{\mathrm{B}}\}\times\{-L_{\mathrm{B}},\ldots,L_{\mathrm{B}}\}$. These linear equations have the form of underspread extended TFYW equations. They can be written as (cf. (18))

$$\boldsymbol{C}\boldsymbol{b} \,=\, -\boldsymbol{c}\,, \tag{20}$$

with the $M_{\mathrm{B}}(2L_{\mathrm{B}}+1)\times 1$ vectors $\boldsymbol{b} = \left[\boldsymbol{b}_1^T \cdots \boldsymbol{b}_{M_{\mathrm{B}}}^T\right]^T$ and $\boldsymbol{c} = \left[\boldsymbol{c}_1^T \cdots \boldsymbol{c}_{M_{\mathrm{B}}}^T\right]^T$ containing the $(2L_{\mathrm{B}}+1)\times 1$ vectors $\boldsymbol{b}_m = \left[b_{m,-L_{\mathrm{B}}} \cdots b_{m,L_{\mathrm{B}}}\right]^T$ and $\boldsymbol{c}_m = \left[c_{M_{\mathrm{A}}+m,-L_{\mathrm{B}}} \cdots c_{M_{\mathrm{A}}+m,L_{\mathrm{B}}}\right]^T$, respectively and the $M_{\mathrm{B}}(2L_{\mathrm{B}}+1)\times M_{\mathrm{B}}(2L_{\mathrm{B}}+1)$ TBT matrix $\boldsymbol{C} = \mathrm{toep}\big\{\boldsymbol{C}_{M_{\mathrm{A}}+M_{\mathrm{B}}-1},\ldots,\boldsymbol{C}_{M_{\mathrm{A}}-M_{\mathrm{B}}+1}\big\}$ containing the $(2L_{\mathrm{B}}+1)\times(2L_{\mathrm{B}}+1)$ Toeplitz blocks $\boldsymbol{C}_m = \mathrm{toep}\big\{c_{m,2L_{\mathrm{B}}},\ldots,c_{m,-2L_{\mathrm{B}}}\big\}$. The TBT equation (20) can be solved with complexity $\mathcal{O}(M_{\mathrm{B}}^2 L_{\mathrm{B}}^3)$ by using the Wax-Kailath algorithm.

From the TFMA parameters $b_{m,l}$, an approximation to the optimum TFAR parameters can finally be obtained according to (12):

$$a_{m,l} \,=\, \sum_{(m',l')\in\mathcal{B}} b_{m',l'}\, c_{m-m',l-l'}\,, \qquad (m,l)\in\mathcal{A}_1\,. \tag{21}$$

4.2. TFMA Parameter Estimation

Next, we develop a linear method for nonmonic TFMA$(M_{\rm B}, L_{\rm B})$ parameter estimation that is again based on the intermediate high-order TFAR$(M_{\rm C}, L_{\rm C})$ model (7). Our method extends Durbin's method for time-invariant MA estimation [6,8] to the TFMA case.

We thus consider the approximation of the given TFAR model $\mathbb{H}_{\rm TFAR} = \mathbb{C}^{-1}\mathbb{D}_0$ by a low-order TFMA model $\mathbb{H}_{\rm TFMA} = \mathbb{B}$, i.e., we wish to calculate $\mathbb{B}$ such that $\mathbb{B} \approx \mathbb{C}^{-1}\mathbb{D}_0$. Multiplication by $\mathbb{C}$ yields $\mathbb{C}\mathbb{B} \approx \mathbb{D}_0$, which is to be solved in the least-squares sense:

$$\mathbb{B}_{\rm opt} \triangleq \arg\min_{\mathbb{B}\,\in\,\mathcal{S}_{M_{\rm B},L_{\rm B}}} \|\mathbb{D}_0 - \mathbb{B}\mathbb{C}\|^2. \tag{22}$$

This minimization problem is again similar to (8), with $\mathbb{H}$ replaced by $\mathbb{C}$, $\mathbb{B}$ replaced by $\mathbb{D}_0$, and $\mathbb{A}$ replaced by $\mathbb{B}$. However, $\mathbb{D}_0$ is known and thus the minimization is only with respect to $\mathbb{B}$.

The optimum and low-complexity, suboptimum solutions discussed in Sections 3.2 and 3.3 can again be used with suitable modifications. The minimization (22) is a linear least-squares problem in the TFMA parameters $b_{m,l}$ whose (optimum) solution is determined by linear equations of the form (16). To obtain a suboptimum but more efficient solution in the spirit of Section 3.3, we note that (22) can also be viewed as the least-squares solution to the overdetermined system of linear equations ($N^2/2$ equations in the $(M_{\rm B}+1)(2L_{\rm B}+1)$ unknowns $b_{m,l}$)

$$\sum_{(m',l')\in\mathcal{B}} b_{m',l'}\, c_{m-m',l-l'}\, e^{-j\frac{2\pi}{N}m'(l-l')} = d_{0,l}\,\delta[m]\,,$$
$$m = 0,\ldots,N/2-1,\; l = -N/2,\ldots,N/2-1\,.$$

Let us consider only the equations corresponding to $(m,l) \in \mathcal{B} = \{0,\ldots,M_{\rm B}\} \times \{-L_{\rm B},\ldots,L_{\rm B}\}$. We can then use the underspread approximation $e^{-j\frac{2\pi}{N}m'(l-l')} \approx 1$, which yields (cf. (17))

$$\sum_{(m',l')\in\mathcal{B}} b_{m',l'}\, c_{m-m',l-l'} = d_{0,l}\,\delta[m]\,, \qquad (m,l)\in\mathcal{B}\,. \tag{23}$$

These are $(M_{\rm B}+1)(2L_{\rm B}+1)$ equations in the $(M_{\rm B}+1)(2L_{\rm B}+1)$ unknowns $b_{m,l}$. They can be written as $\boldsymbol{C}\boldsymbol{b} = \boldsymbol{d}$ with the $(M_{\rm B}+1)(2L_{\rm B}+1)\times 1$ vector $\boldsymbol{b} = \big[\boldsymbol{b}_0^T \cdots \boldsymbol{b}_{M_{\rm B}}^T\big]^T$ where $\boldsymbol{b}_m = \big[b_{m,-L_{\rm B}} \cdots b_{m,L_{\rm B}}\big]^T$, the $(M_{\rm B}+1)(2L_{\rm B}+1)\times 1$ vector $\boldsymbol{d} = [d_{0,-L_{\rm B}} \cdots d_{0,L_{\rm B}}\; 0 \cdots 0]$, and the $(M_{\rm B}+1)(2L_{\rm B}+1) \times (M_{\rm B}+1)(2L_{\rm B}+1)$ lower-block-triangular TBT matrix $\boldsymbol{C} = \text{toep}\big\{\boldsymbol{C}_{M_{\rm B}},\ldots,\boldsymbol{C}_{-M_{\rm B}}\big\}$ where $\boldsymbol{C}_m = \text{toep}\big\{c_{m,2L_{\rm B}},\ldots,c_{m,-2L_{\rm B}}\big\}$ (note that $\boldsymbol{C}_0 = \mathbf{I}$ and $\boldsymbol{C}_m = \mathbf{0}$ for $m < 0$). These TBT equations can again be solved with complexity $\mathcal{O}(M_{\rm B}^2 L_{\rm B}^3)$ by means of the Wax-Kailath algorithm. The resulting TFMA estimator is much less complex than the nonlinear method of [2].

5. SIMULATION RESULTS

We now present simulation results to demonstrate the performance of our parameter estimation and system approximation methods.

5.1. Parameter Estimation

We generated 100 realizations of a TFARMA$(M, L; M-1, L)$ process of length N. We then estimated an intermediate TFAR model and, in turn, the TFARMA parameters from every single realization separately. The TFARMA parameters were estimated by means of the low-complexity, suboptimum method (eqs. (19) and (21)). Finally, we calculated the empirical normalized MSE, variance, and squared bias averaged over all parameter estimates and realizations. This experiment was performed for various combinations of N, M, and L as shown in Fig. 2. For determining the orders of the intermediate high-order TFAR$(M_{\rm C}, L_{\rm C})$ model, we compared the choices $M_{\rm C} = \alpha M$, $L_{\rm C} = \beta L$ for $\alpha \in \{1,\ldots,4\}$ and $\beta = \{2,\ldots,4\}$ and found the values $\alpha = \beta = 3$ to provide the best performance. The intermediate TFAR model was stabilized according to [3].

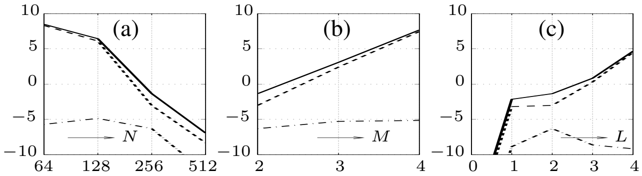

Figure 2: *Normalized MSE (solid lines), normalized variance (dashed lines), and normalized squared bias (dash-dotted lines) of the low-complexity TFARMA estimator for a TFARMA$(M, L; M-1, L)$ process: (a) variation with N for $M = L = 2$, (b) variation with M for $N = 256, L = 2$, (c) variation with L for $N = 256, M = 2$.*

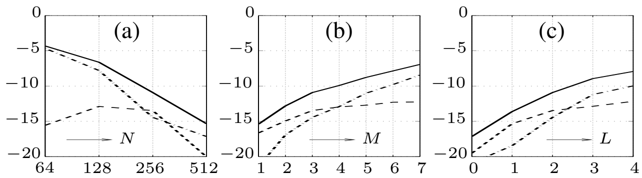

Figure 3: *Normalized MSE (solid lines), normalized variance (dashed lines), and normalized squared bias (dash-dotted lines) of the low-complexity TFMA estimator for a TFMA(M, L) process: (a) variation with N for $M = 3, L = 2$, (b) variation with M for $N = 256, L = 2$, (c) variation with L for $N = 256, M = 3$.*

Similar experiments were conducted to study the performance of the low-complexity TFMA parameter estimation method (23) for simulated TFMA(M, L) processes. The results are displayed in Fig. 3. The orders of the intermediate high-order TFAR model were chosen as $M_{\rm C} = 2M$, $L_{\rm C} = 2L$.

It is seen that the TFARMA parameter estimator is significantly less accurate than the TFMA estimator; for $N = 256$ its normalized MSE is above zero dB if the model has more than 20 parameters. Good results (normalized MSE lower than -5 dB) are obtained for ML/N below about $1/64$ for the TFARMA estimator and for ML/N below about $1/16$ for the TFMA estimator. In general, the performance of both estimators is better for lower model orders M, L and for higher signal length N.

Next, we fitted a TFARMA$(4, 1; 3, 1)$ model and a TFMA$(13, 1)$ model to a real process of length $N = 256$ that was measured by a pressure sensor in a combustion engine (cf. [14]). The total number of model parameters is 24 for the TFARMA model and 42 for the TFMA model. The model parameters were estimated from the single realization shown in Fig. 4(a),(b) by means of the suboptimum method. The (estimated) evolutionary spectra[3] calculated from the TFARMA and TFMA parameter estimates are depicted

[3] The evolutionary spectrum of a TFARMA process is defined as $P[n,k] \triangleq |B[n,k]|^2/|A[n,k]|^2$ with $B[n,k] = \sum_{(m,l)\in\mathcal{B}} b_{m,l}\, e^{j\frac{2\pi}{N}(nl-km)}$ and $A[n,k] = \sum_{(m,l)\in\mathcal{A}} a_{m,l}\, e^{j\frac{2\pi}{N}(nl-km)}$.

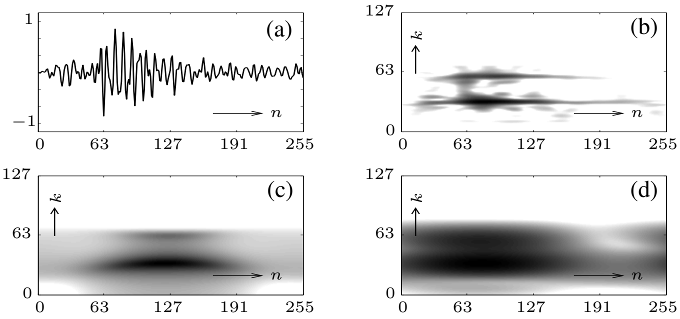

Figure 4: *TFARMA*$(4,1;3,1)$ *and TFMA*$(13,1)$ *modeling of a real (measured) process: (a) process realization* $x[n]$*, (b) its smoothed Rihaczek distribution [15], (c) estimated evolutionary spectrum of the TFARMA*$(4,1;3,1)$ *model, (d) estimated evolutionary spectrum of the TFMA*$(13,1)$ *model.*

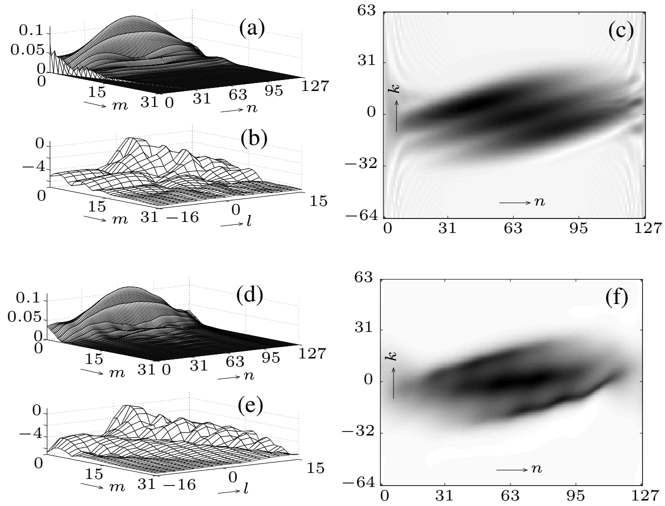

Figure 5: *TFARMA*$(3,7;2,7)$ *approximation of a non-TFARMA LTV system: (a)–(c) magnitude of the time-varying impulse response (first 32 delay taps), SF (in dB; first 32 delay taps), and time-varying transfer function of the original system, (d)–(f) the same for the TFARMA*$(3,7;2,7)$ *system approximation.*

in Figs. 4(c) and 4(d), respectively. It is seen that the TFARMA parameter estimator with low delay order $M = 4$ and only 24 parameters is better able to resolve the two signal components than the TFMA parameter estimator with high delay order $M = 13$ and 42 parameters.

5.2. System Approximation

We used the low-complexity TFARMA system approximation method (eqs. (17) and (12)) to approximate a nonparametric LTV system by a TFARMA$(3,7;2,7)$ system. The length of the time interval was $N = 128$. Fig. 5 compares the time-varying impulse response, SF, and time-varying transfer function[4] of the original system and its TFARMA approximation. Note that the TFARMA model uses 90 parameters whereas the system's time-varying impulse response comprises $N^2/2 = 8192$ samples.

6. CONCLUSIONS

We presented linear methods for estimating TFARMA model parameters for an underspread nonstationary random process, and for the related problem of calculating a TFARMA-type approximation to an underspread time-varying linear system. In both cases, we obtained a system of linear equations of the TF Yule-Walker type that has Toeplitz/block-Toeplitz structure and thus can be solved efficiently by the Wax-Kailath algorithm. The performance of the proposed methods was assessed through simulation results.

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