A Classification of Spatial Processes Based on PDEs

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1. Introduction

Dynamic, or time dependent, spatial processes are becoming important subjects in the field of geographic information science and the development of geographic information systems (GIS). Examples for spatial processes are the spread of diseases (Newman 2002), human travel (Brockmann, Hufnagel et al. 2006), the migration of non-endemic species (Seppelt 2005), and the expansion of cities (Li 1999). Because of all the differences between processes in terms of spatial and temporal scales, involved parameters, suited models etc., the common grounds of processes are obscured. As part of our long term objective – extending GIS with elementary functionality for working on spatial processes – this paper presents our approach to identify the key elements of spatial processes. Our vision is that GIS can be used for communicating about processes on a qualitative level, when they are extended with functionality for working on processes. This can lead to an enhanced interoperability between GIS and specialized modeling packages.

Our approach to identify the common grounds of different spatial processes is an analysis of partial differential equations (PDEs) that can be used to model the processes. PDEs are generally used for describing dynamic processes mathematically. They are roughly classified into wave-like, diffusion-like, and equilibrium equations according to the type of phenomena they describe. This paper shows how we can build on this classification and which mathematical aspects should be included in a detailed analysis of PDEs.

2. Related Work

Geographic information systems (GIS) are generally used for managing, analyzing, and visualizing spatial data. They have been developed for managing extensive data collections in the 1970s. Currently proprietary GIS software is static, which means that the systems store one up-to-date state of the world (Worboys 2005). Worboys (2005) indicates the need for treating dynamic aspects of spatial phenomena when building models. He presents an approach that stresses the consideration of events in the common object/field models. Other works on time and dynamics in GIS include (Claramunt and Thériault 1996; Hornsby and Egenhofer 1997; Frank 1998; Galton 2004).

Martin Beckmann (1970) analyzed spatial diffusion processes in the context of location theory, which is concerned with the spatial location of economic activities. He presents basic models of innovation diffusion, expenditure diffusion, migration, and commodity flows and price waves. The outcome of his analysis is that all of those phenomena can be described with the diffusion equation or the wave equation. His intention is to build better models of economic processes rather than reproducing partial differential equations (PDEs). However, his analysis supports our assumption that processes have common grounds that can be identified by PDEs.

3. Partial Differential Equations

Our approach to identify the key elements of spatial processes is an investigation of partial differential equations (PDEs) that are used for modeling the processes. A PDE is "an equation that contains partial derivatives, expressing a process of change that depends on more than one independent variable. It can be read as a statement about how a process evolves without specifying the formula defining the process" (Encyclopædia Britannica 2006). PDEs are often used for computer-based analyses and simulations of continuous physical phenomena that occur in fields like mechanics, electrostatics, electrodynamics, and acoustics (Press, Flannery et al. 1986).

3.1 Describing a Process with a PDE

We discuss the advection equation as a concrete example for a PDE, based on explanations given in (Logan 2004). Advection is a term from biology for describing the bulk movement of particles in some transporting medium (e.g. pollutants carried downstream in a river, a swarm of insects that is moving from one place to another). The advection equation can be deduced from a conservation law. This law states that the change of some quantity in a region has to correspond to the amounts coming in, going out and being created or destroyed in that region. The flux F denotes the amounts of some particles coming in or going out, the sources *f* represent the amounts being added or removed, and the density *u* denotes the amount of particles existing in a region. Equation 1 is the expression of this conservation law:

$$u_t(x,t) + \phi_x(x,t) - \mathrm{f}(x,t)\, d\, x = 0 \qquad (1)$$

In the case of advection, the flux F is proportional to the density *u*. *C* is a constant (Equation 2). We omit the sources *f* and insert the statement for the flux F in Equation 1 to come to the advection equation without sources (Equation 3).

$$\phi = c\, u$$

$$u_t + cu_x = 0$$

3.2 Classification of PDEs

The advection equation presented in the previous section is a first order, linear, hyperbolic partial differential equation in one dimension. The order, linearity, and type are important for providing a classification of partial differential equations. The type of a PDE, hyperbolic, parabolic, and elliptic, indicates characteristics of the process that is modeled (Logan 2004):

– Hyperbolic equations like the wave equation or the advection equation are used for modeling wave-like phenomena. Waves occur, for example, in water, electromagnetism, and acoustics.
– Parabolic equations like the diffusion equation are used for modeling diffusion problems. Diffusion describes the random motion of particles.
– Elliptic equations like the Laplace's equation describe steady-state processes, like ground water flow.

For computational considerations the distinction into initial value and boundary value problems is more important than the classification stated above (Press, Flannery et al. 1986). The hyperbolic and parabolic equations fall into the class of initial value problems; they are also referred to as evolution equations showing how a process evolves over time. Elliptic equations are equilibrium equations that do not contain a time variable and fall into the class of boundary value problems.

4. Identifying Characteristics of Spatial Processes

The long-term objective of this work is to infer characteristics of processes from their mathematical description with PDEs. The general classification of PDEs presented in section 3.2 provides the basis for this investigation. The sensible issue is which mathematical aspects have to be included in the analysis of PDEs to reveal the characteristics of the modeled processes.

For studying the PDEs we propose to analyze the general classification aspects i.e. order, linearity, type, and if the equations include spatial reference and time. In addition we have to analyze if an equation can change its characteristics depending on boundary condition and the pre-or absence of sources. Concerning source terms, Logan (2004, p. 152, 153) states „a source term represents an outside influence in the system and leads to inhomogeneity in the PDE."

With this approach we can draw the following conclusions concerning the previously presented advection equation: the advection equation is a one dimensional equation and therefore a hyperbolic equation. Hyperbolic equations show a wave-like behavior which means that signals are propagated in a coherent way. It is an evolution equation belonging to the class of initial value problems. There are different forms of the advection equation including sources or not, and the equation can be combined with other equations like the diffusion equation.

5. Conclusions and Future Work

This work presented an approach for identifying the characteristics of spatial processes. The tools we are using are partial differential equations; by analyzing the PDEs, we can deduce properties of the processes that are modeled with the PDE. This approach was presented exemplarily for the advection equation.

The next step is to perform a detailed analysis of PDEs and to match them with spatial processes occurring in a GIS context. This proceeding will show if our classification is sensible enough for inferring characteristics of spatial processes from the equations.

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7. References

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