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# Multiple scales analysis of the steady-state Korteweg-de Vries equation perturbed by a damping term

**H. Steinrück***

Institute of Fluid Mechanics and Heat Transfer, Vienna University of Technology, Resselgasse 3, 1040 Vienna, Austria

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The stationary Korteweg-de Vries equation perturbed by two different damping terms will be discussed. The solution is oscillatory with slowly varying amplitude, wave length and mean value. The multiple scales method is applied and equations for the slowly varying quantities are be derived.



## 1 Introduction

In this paper we study (non trivial) solutions of a perturbed steady-state Korteweg-de Vries (KdV) equation

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}H + (H-1)\frac{\mathrm{d}}{\mathrm{d}x}H = \beta f(H, H'') \tag{1}$$

subject to the asymptotic boundary condition

$$\lim_{x\to-\infty} H(x) = 0\,, \tag{2}$$

for small values of the damping parameter $\beta$ using the multiple scales method. We will consider two different damping terms. In both cases eq. (1) describes the perturbation of an open shallow channel flow close to critical flow conditions (cf. [4,7]).

(i) For laminar flow (1) has been derived in [7] for $f = -H''$. A first analysis of that case is in [6]. Moreover this equation is also considered in the context of shock profiles in plasmas [2].
(ii) For turbulent flow in a certain distinguished limit concerning the Reynolds number and the Froude number eq. (1) has been derived in [3,4] with $f = H$. A multiple scales analysis of the full flow problem can be found in [11]. That equation has been previously mentioned by [10] in context of a resonant flow of a stratified fluid over a topography.

Modulated wave trains as solutions of the perturbed (time dependent) KdV-equation have been considered in [8] for both damping terms. Here we are interested in stationary solutions not discussed in [8]. However, the multiple scales analysis here is closely related to that in [8].

In both cases the (nontrivial) solution of (1), (2) is unique up to translation with respect to $x$. In the laminar case (1) can be integrated once. Linearizing around $H = 0$ and making the ansatz $H = \exp(\lambda x)$ yields the two eigenvalues $\lambda_1 = 1 + O(\beta)$, $\lambda_2 = -1 + O(\beta)$. In the turbulent case we linearize (1) at $H = 0$ make the ansatz $H = \exp(\lambda x)$ and get the three eigenvalues $\lambda_1 = 1 + O(\beta)$, $\lambda_2 = -1 + O(\beta)$, $\lambda_3 = -\beta + O(\beta^2)$. Thus we have in both cases a one parametric manifold of solutions satisfying (1), (2).

Setting formally the damping parameter $\beta$ equal to zero in (1) the resulting equation can be integrated twice to give

$$3\left(\frac{\mathrm{d}}{\mathrm{d}x}H\right)^2 = p(H; R, S) := -H^3 + 3H^2 + 6R\,H + 6S\,, \tag{3}$$

* e-mail: herbert.steinrueck@tuwien.ac.at

where $R$ and $S$ are constants of integration. Let $p(H,R,S)$ be the third order polynomial defined in (3) with the three (real) roots $h_1 \le h_2 \le h_3$. Then we can express the solution in terms of Jacobian elliptical functions [1]:

$$H = \begin{cases} h_2 + (h_3 - h_2)\,\mathrm{cn}^2\left(\frac{x\sqrt{h_3-h_1}}{2\sqrt{3}}|\nu\right)\,, & \nu < 1\,,\\ h_2 + (h_3 - h_2)\,\mathrm{sech}^2\left(\frac{x\sqrt{h_3-h_1}}{2\sqrt{3}}\right)\,, & \nu = 1\,, \end{cases} \tag{4}$$

where $\nu = \frac{h_3-h_2}{h_3-h_1}$ is the parameter of the Jacobian elliptical function. If $\nu < 1$ these are periodic functions with the period

$$x_p = \frac{4\sqrt{3}}{\sqrt{h_3-h_1}}\;K(\nu)\,, \tag{5}$$

where $K(\nu) = \int_0^{\pi/2}(1-\nu\sin^2\phi)^{-1/2}\,\mathrm{d}\phi$ is the real quarter period of the Jacobian elliptical functions of the parameter $\nu$. In case of $\nu = 1$, i. e. $h_1 = h_2$ the solution is a solitary wave decaying to $h_2$ for both $X \to \pm\infty$.

However, none of the solutions (4) is uniformly valid: The periodic solutions ($\nu < 1$) do not satisfy the asymptotic boundary condition (2) while in Sect. 2 it is shown that the next order correction of the regular expansion starting with the solitary wave (for $\nu = 1$) tends to infinity for $x \to \infty$.

Previous investigations [6] show that the constants of integration $R$ and $S$ are not constant but vary slowly with $x$. Thus a multiple scales approach is appropriate.

For the laminar case a multiple scales analysis has already been performed in [6], but at a certain step it has been assumed that the period does not depend on the slow scale. This is inconsistent, since the zeros $h_1$, $h_2$ and $h_3$ vary on the slow scale and thus the period as a function of the zeros $h_1$, $h_2$ and $h_3$ varies on the slow scale either.

In Sect. 3 the multiple scales analysis is performed and equations for the fast and slowly varying variables are derived. In Sect. 4 asymptotic formulas for the height of the first elevation and the first wave length are derived. In Sect. 5 the downstream behavior is discussed while in Sect. 6 a comparison of the asymptotic approach with a numerical solution is given.

## 2 Regular expansion

It can be verified that (1) is equivalent to the first order system

$$\tfrac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}H\right)^2 + \tfrac{1}{6}H^3 - \tfrac{1}{2}H^2 = R\,H + S\,, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}R = \beta\, f\left(H, R + H - \tfrac{1}{2}H^2\right)\,, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}S = -\beta\, H\, f\left(H, R + H - \tfrac{1}{2}H^2\right)\,. \tag{8}$$

Note that in the general case ($\beta > 0$) $R$ and $S$ are functions of the independent variable $x$. As a first attempt we introduce a regular expansion of the solution:

$$H = \hat{H}_0(x) + \beta\hat{H}_1(x) + \dots\,, \qquad R = \hat{S}_0(x) + \beta\hat{S}_1(x) + \dots\,, \qquad S = \hat{S}_0(x) + \beta\hat{S}_1(x) + \dots\quad. \tag{9}$$

Inserting into (6–8) and comparing equal powers of $\beta$ and using (2) yields for the leading order terms:

$$\hat{H}_0 = 3\,\mathrm{sech}^2\,\frac{x}{2}\,, \qquad \hat{R}_0 = \hat{S}_0 = 0\,. \tag{10}$$

For the first-order terms we obtain:

$$\hat{H}_{1,x}\hat{H}_{0,x} + \tfrac{1}{2}\hat{H}_0^2\hat{H}_1 - \hat{H}_0\hat{H}_1 = \hat{R}_1\hat{H}_0 + \hat{S}_1\,, \tag{11}$$

$$\hat{R}_{1,x} = \hat{f}_1\,, \qquad \hat{S}_{1,x} = -\hat{H}_0\hat{f}_1 \tag{12}$$

with $\hat{f}_1 = f(\hat{H}_0, \hat{H}_0 - \hat{H}_0^2/2)$. The general solution is given by:

$$\hat{H}_1(x) = \hat{H}_{0,x}\left(G_2(x)\int_{-\infty}^{x}\hat{f}_1\,\mathrm{d}x' - G_1(x)\int_{-\infty}^{x}\hat{H}_0\hat{f}_1\,\mathrm{d}x' - \int_{x_*}^{x} G_2\,\hat{f}_1\,\mathrm{d}x' + \int_{x_*}^{x} G_1\,\hat{H}_0\,\hat{f}_1\,\mathrm{d}x'\right) \tag{13}$$

with

$$G_1(x) = \int_{x_*}^{x} \frac{1}{\hat{H}_{0,x}^2}\,\mathrm{d}x' = \left.\frac{1 + 15\mathrm{e}^{x'} + 15\mathrm{e}^{4x'} + \mathrm{e}^{5x'} + 12\mathrm{e}^{3x'}(5x' - 12) - 4\mathrm{e}^{2x'}(4 + 15x')}{288\mathrm{e}^{2x'}(\mathrm{e}^{x'} - 1)}\right|_{x_*}^{x},$$

$$G_2(x) = \int_{x_*}^{x} \frac{\hat{H}_0}{\hat{H}_{0,x}^2}\,\mathrm{d}x' = \left.\frac{-1 + \mathrm{e}^{-x'} + \mathrm{e}^{2x'} - \mathrm{e}^{x'}(17 - 6x') - 6x'}{12(\mathrm{e}^{x'} - 1)}\right|_{x_*}^{x},$$

where the lower bound of the integral $x_*$ has to be chosen such that $\hat{H}_{1,x}(0) = 0$. Since $G_1$ and $G_2$ have poles of order 1 at $x = 0$ and $\hat{H}_{1,x}$ has a simple zero at $x = 0$ the value of $H_1(0)$ turns out to be independent of the constant of integration $x_*$. It is

$$\hat{H}_1(0) = \frac{2}{\sqrt{3}}(3a - b)\,, \tag{14}$$

where $a = \int_{-\infty}^{0} \hat{f}_1\,\mathrm{d}x/\sqrt{3}$ and $b = \int_{-\infty}^{0} \hat{H}_0\hat{f}_1\,\mathrm{d}x/\sqrt{3}$. However, one can verify that $\hat{H}_1(x) \sim \sqrt{3}b\,\exp(x)/24$ for $x \to \infty$. Therefore the expansion (9) is not uniformly valid. Therefore a multiple scales expansion will be introduced.

## 3 Multiple scales expansion

We define the slow variable

$$\mathcal{X} = \beta(x - x_0)\,, \tag{15}$$

where the phase shift $x_0 = x_0(\beta) = o(1/\beta)$ is chosen appropriately. Following [8] it is useful to introduce the new independent variable $\xi$:

$$\xi = \frac{1}{\beta}\Omega(\mathcal{X})\,, \quad \text{with} \quad \mathrm{d}\xi = \omega\,\mathrm{d}x\,, \qquad \omega = \frac{\mathrm{d}\Omega}{\mathrm{d}\mathcal{X}} \tag{16}$$

such that $H$ is 'periodic' with wave length 1 as a function of $\xi$. The variable $\Omega$ serves as the corresponding slow variable. Its derivative $\omega$ with respect to $\mathcal{X}$ is the wave number with respect to the original variable $x$.

For the dependent variables we introduce the multiple scales expansion

$$\begin{aligned} H(x,\beta) &= H_0(\xi,\Omega) + \beta H_1(\xi,\Omega) + \dots\,, \\ R(x,\beta) &= R_0(\xi,\Omega) + \beta R_1(\xi,\Omega) + \dots\,, \\ S(x,\beta) &= S_0(\xi,\Omega) + \beta S_1(\xi,\Omega) + \dots\,, \end{aligned} \tag{17}$$

where the functions $H_0$, $H_1$, $R_0$, $R_1$, $S_0$, $S_1$ are assumed to be periodic with period one as functions of the first argument. Inserting into (8) and comparing equal powers of $\beta$ yields that $R_0$ and $S_0$ are functions of the slow variable $\Omega$ only. Furthermore we obtain:

$$\omega\frac{\partial}{\partial\xi}R_1 + \omega\frac{\partial}{\partial\Omega}R_0 = f(H_0, R_0 + H_0 - H_0^2/2)\,, \tag{18}$$

$$\omega\frac{\partial}{\partial\xi}S_1 + \omega\frac{\partial}{\partial\Omega}S_0 = -H_0 f(H_0, R_0 + H_0 - H_0^2/2)\,. \tag{19}$$

Integrating (18) and (19) with respect to $\xi$ over an interval of length 1 and using that $R_1$ and $S_1$ are periodic with respect to $\xi$ we obtain the differential equations for the slowly varying quantities

$$\frac{\partial}{\partial\Omega}R_0 = \frac{1}{\omega}\int_0^1 f(H_0, R_0 + H_0 - H_0^2/2)\,\mathrm{d}\xi = 2\sqrt{3}I_1(R_0, S_0)\,, \tag{20}$$

$$\frac{\partial}{\partial\Omega}S_0 = -\frac{1}{\omega}\int_0^1 H_0\,f(H_0, R_0 + H_0 - H_0^2/2)\,\mathrm{d}\xi = -2\sqrt{3}I_2(R_0, S_0)\,, \tag{21}$$

with

$$I_j(R,S) = \int_{h_2}^{h_3} \frac{h^{j-1} f(h, R + h - \frac{h^2}{2})\, \mathrm{d}h}{\sqrt{(h-h_1)(h-h_2)(h_3-h)}}\,, \tag{22}$$

where we have made use of

$$\omega^2 \left(\frac{\partial}{\partial \xi} H_0\right)^2 = p(H_0; R_0, S_0) := -\frac{1}{3} H_0^3 + H_0^2 + 2R_0\, H_0 + 2S_0\,. \tag{23}$$

We can express the solution in terms of the Jacobian elliptic function [1]:

$$H_0 = h_2 + (h_3 - h_2)\, \mathrm{cn}^2\, (2\xi K(\nu)|\nu) \tag{24}$$

and the wave number $\omega$ is given by

$$\omega = \frac{\sqrt{h_3 - h_1}}{4\sqrt{3} K(\nu)}\,. \tag{25}$$

From the asymptotic boundary condition (2) initial conditions for $R_0$ and $S_0$ follow:

$$R_0 = 0\,, \qquad S_0 = 0 \quad \text{for} \quad \Omega = 0\,. \tag{26}$$

The original variables are given by

$$\mathcal{X} = \int_0^{\Omega} \frac{\mathrm{d}\Omega}{\omega}\,, \qquad \xi = \frac{\Omega}{\beta}\,, \tag{27}$$

with

$$x(\xi,\beta) = \frac{1}{\beta}\mathcal{X} + x_0(\beta)\,, \qquad x_0(\beta) = -\int_0^{1/2} \frac{1}{\omega(\xi)} \mathrm{d}\xi\,. \tag{28}$$

The shift $x_0$ of the fast variable is introduced such that the first maximum occurs at $x = 0$. We will see later that $\mathcal{X}(0) = 0$ and $x_0$ is finite, for $\beta > 0$. Thus the multiple scales approximation (27) fails for $x \to -\infty$. But for $x < 0$ we can use the regular expansion from Sect. 2.

### 3.1 Laminar case

In case of laminar flow ($f = -H''$) the functions $I_1(R,S)$ and $I_2(R,S)$ are:

$$\omega \frac{\mathrm{d}R_0}{\mathrm{d}\Omega} = -\int_0^1 \left(R_0 + H_0 - H_0^2/2\right)\, \mathrm{d}\xi = -\int_0^1 \omega \frac{\mathrm{d}^2 H_0}{\mathrm{d}\xi^2}\, \mathrm{d}\xi = 0\,. \tag{29}$$

$$\begin{aligned} \omega \frac{\mathrm{d}S_0}{\mathrm{d}\Omega} &= \int_0^1 H_0\, \left(R_0 + H_0 - H_0^2/2\right)\, \mathrm{d}\xi = \int_0^1 \omega^2 H_0 \frac{\mathrm{d}^2 H_0}{\mathrm{d}\xi^2}\, \mathrm{d}\xi \\ &= -\int_0^1 \omega^2 \left(\frac{\mathrm{d}H_0}{\mathrm{d}\xi}\right)^2 \mathrm{d}\xi = -\frac{2\omega}{\sqrt{3}} \int_{h_2}^{h_3} \sqrt{p(h, R_0, S_0)}\, \mathrm{d}h\,, \end{aligned}$$

$$\frac{\mathrm{d}S_0}{\mathrm{d}\Omega} = -\frac{4}{\sqrt{3}} (h_3 - h_2)^2 (h_3 - h_1)^{1/2} \int_0^{\pi/2} \sin^2\phi \cos^2\phi \left(1 - \nu \sin^2\phi\right)^{1/2}\, \mathrm{d}\phi\,. \tag{30}$$

### 3.2 Turbulent case

In the case $f = H$ the functions $I_1(R,s)$ and $I_2(R,S)$ are:

$$I_1 = \frac{2}{\sqrt{h_3 - h_1}} \left(h_1 K(\nu) + (h_3 - h_1) E(\nu)\right) \tag{31}$$

$$I_2 = \frac{2}{\sqrt{h_3 - h_1}} \left(h_1^2 K(\nu) + (h_3^2 - h_1^2) E(\nu) - (h_3 - h_1)^2 \nu \int_0^{\pi/2} \sin^2\phi \sqrt{1 - \nu \sin^2\phi}\, \mathrm{d}\phi\right) \tag{32}$$

with $E(\nu) = \int_0^{\pi/2} (1 - \nu \sin\phi^2)^{1/2}\, \mathrm{d}\phi$.

## 4 First maximum and first wave length

To discuss the behavior of the solution in the first undulations we expand the slowly varying quantities around $R_0 = S_0 = 0$: Let $a = I_1(0,0)$ and $b = I_2(0,0)$, which is identical with the definition in Sect. 2. Therefore $R$ and $S$ behave regularly at $\Omega = 0$. Using

$$R_0 \sim R_{0,\Omega}(0)\Omega = 2\sqrt{3}a\,\Omega\,, \qquad S_0 \sim S_{0,\Omega}(0)\Omega = -2\sqrt{3}b\,\Omega \tag{33}$$

we obtain for the zeros of the polynomial $p(h)$:

$$h_1 \sim -2\cdot 3^{1/4}b^{1/2}\Omega^{1/2}\,, \qquad h_2 \sim 2\cdot 3^{1/4}b^{1/2}\Omega^{1/2}\,, \qquad h_3 \sim 3 + 4\cdot 3^{-1/2}(3a-b)\Omega\,. \tag{34}$$

Using $K(1-\epsilon) \sim -1/2\ln\epsilon + 2\ln 2$ for $\epsilon \ll 1$ (see [1]) we obtain for the wave number $\omega$

$$\frac{1}{\omega} = 4K\left(1 - \frac{h_2 - h_1}{h_3 - h_1}\right) \sim 2\ln\frac{h_3 - h_1}{h_2 - h_1} + 4\ln 4 \sim \ln\frac{1}{\Omega} + \ln\frac{16\cdot 3^{3/2}}{b}\,. \tag{35}$$

It becomes singular for $\Omega \to 0$. The slow original variable is given by

$$\mathcal{X}(\Omega) = \int_0^\Omega \frac{1}{\omega}\,\mathrm{d}\Omega \sim \Omega\left(\ln\frac{1}{\Omega} + 1 + \ln\frac{16\cdot 3^{3/2}}{b}\right)\,. \tag{36}$$

We get $\mathcal{X}(0) = 0$. The shift $x_0$ is determined such that the first maximum is located at $x = 0$

$$x_0(\beta) = -\int_0^{1/2} \frac{1}{\omega(\beta\xi)}\mathrm{d}\xi \sim \frac{1}{2}\ln\beta - \frac{1}{2} - \frac{1}{2}\ln\frac{32\cdot 3^{3/2}}{b}\,. \tag{37}$$

It's height is

$$h_{\max,0} = h_3(\beta\Omega/2) = 3 + \frac{2}{\sqrt{3}}(3a-b)\beta \tag{38}$$

which is identical with the value obtained by the regular expansion (14).

The location of the second maximum is at

$$x_1(\beta) = \int_{1/2}^{3/2} \frac{1}{\omega(\beta\xi)}\mathrm{d}\xi \sim \ln\frac{32}{b\,\beta} + 1\,. \tag{39}$$

Applying the above results to the two cases $f = H$ and $f = -H''$, respectively, we summarize the results in Table 1:

**Table 1** Asymptotic results for the first elevation.

| $f$ | a | b | $h_2$ | $x_1$ | $h_{\max,0}$ |
|---|---|---|---|---|---|
| $H$ | $2\sqrt{3}$ | $4\sqrt{3}$ | $4\sqrt{3\Omega}$ | $\ln 8/\sqrt{3}\beta + 1$ | $3 + 4\beta$ |
| $-H''$ | 0 | $\frac{4}{5}\sqrt{3}$ | $4\sqrt{0.6\,\Omega}$ | $\ln 40/\sqrt{3}\beta + 1$ | $3 - 1.6\beta$ |

However, a matching between the regular expansion and the multiple scales expansion in the sense of van Dyke [12] is not possible. Since both expansion agree at $x = 0$ to the order $O(\beta)$ we simply use the regular expansion for $x < 0$ and the multiple scales expansion for $x > 0$.

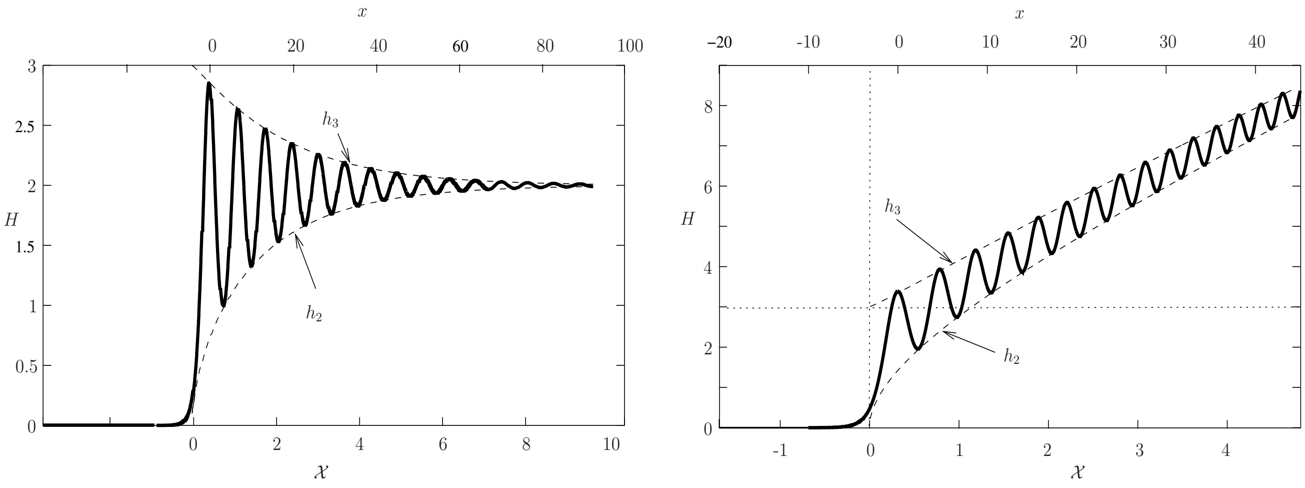
## 5 Asymptotic behavior far downstream ($\Omega$, $\mathcal{X} \to \infty$)

### 5.1 Turbulent case

To discuss the behavior for $\mathcal{X} \to \infty$ it is appropriate to consider the slowly varying quantities

$$h_m = \tfrac{1}{2}(h_3 + h_2)\,, \qquad \Delta h = \tfrac{1}{2}(h_3 - h_2) \tag{40}$$

instead of $R$, $S$. Motivated by the numerical solution of (1) (cf. Fig. 1b) we assume that the amplitude $\Delta h$ decays to zero




a) laminar case ($f = -H''$) b) turbulent case ($f = H$)

**Fig. 1** Slowly varying quantities represented by $h_2$ and $h_3$ (dashed lines) in comparison with the numerical solution of eq. (1) for $\beta = 0.1$ (thick line). Note that the origin of the fast variable $x$ is shifted from the origin of the slow variable $\mathcal{X}$.

for $\mathcal{X} \to \infty$. Expanding the integrals $I_j$ for $\Delta h/h_m \ll 1$ we obtain the following differential equations for $\Delta h$ and $h_m$:

$$\Delta h_{\mathcal{X}} \sim -\frac{3\Delta h}{4h_m}\,, \qquad h_{m,\mathcal{X}} \sim \left(1 + \frac{1}{h_m}\right) \tag{41}$$

with the solution

$$h_m \sim \mathcal{X} + \ln(1+\mathcal{X})\,, \qquad \Delta h \sim \mathcal{X}^{-3/4}\,. \tag{42}$$

Thus the amplitude of the oscillation decays and the mean value over an oscillation period increases linearly. Inserting (42) into (5) we obtain the asymptotic behavior of the wave length $x_p$. It is given by

$$x_p \sim 2\pi/\sqrt{\mathcal{X}}\,. \tag{43}$$

To verify these results we expand the solution of (1) with $f = H$ for $x \to \infty$ and $\beta$ fixed. There are two possibilities. The solution can either decay to zero with the asymptotic behavior given by $H \sim \mathrm{e}^{\gamma x}$ with $\gamma$ one of the negative roots of $\gamma^3 - \gamma - \beta = 0$, which are $\gamma_{1,3} = \pm 1 + O(\beta)$ and $\gamma_2 = -\beta + O(\beta^2)$, respectively. However such a behavior is not in accordance with the previous analysis and the numerical results.

A second possibility for the downstream behavior is given by assuming that the terms $HH'$ and $\beta H$ dominate in eq. (1) for $x$ sufficiently large. This leads to the asymptotic expansion of the form

$$H \sim \beta x + \tilde{H}_1(x) + \tilde{H}_2(x) + \ldots\,, \quad \text{with} \quad \tilde{H}_1(x) = o(x)\,, \qquad \tilde{H}_2(x) = o\left(\tilde{H}_1(x)\right), \ldots \tag{44}$$

Inserting into (1) yields $\tilde{H}_1 = \ln(\beta x - 1)$ and for $\tilde{H}_2'(-x)$ we obtain the Airy differential equation. Thus we have

$$\tilde{H}_2(x) = C \int^{\beta^{1/3}x - \beta^{-2/3}} Ai(-z)\,\mathrm{d}z\,, \tag{45}$$

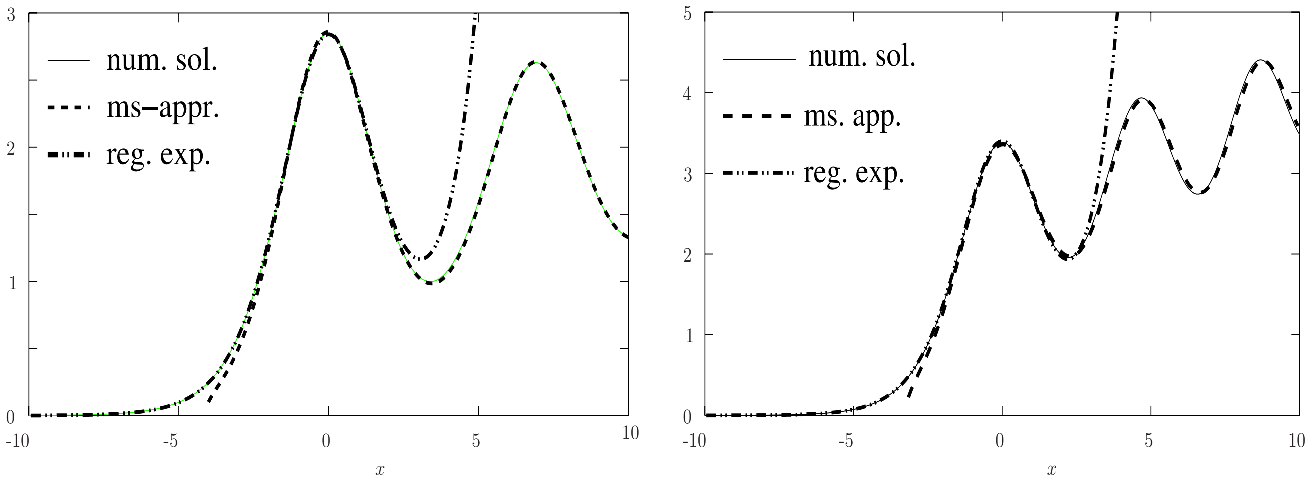
where $C$ is an arbitrary constant and $Ai$ is the Airy function. Using the asymptotic behavior of the Airy function for large negative arguments (cf. [1], 10.4.60) we obtain

$$H \sim (\beta x) + \ln((\beta x) - 1) + \tilde{C}\,(\beta x)^{-3/4} \cos\left(\tfrac{2}{3}\sqrt{\beta x^3} + \frac{\pi}{4}\right) + \ldots \tag{46}$$

with $\tilde{C}$ an arbitrary constant.

The wave length $x_p$ of the oscillating part of the asymptotic expression (46) in the limit $\beta \to 0$, $x \to \infty$ such that $\beta x = \mathcal{X} = \mathrm{const.}$ follows from

$$\lim_{\beta \to 0} \frac{2}{3}\sqrt{\beta}\left(\left(\frac{\mathcal{X}}{\beta} + x_p\right)^{3/2} - \left(\frac{\mathcal{X}}{\beta}\right)^{3/2}\right) = 2\pi \tag{47}$$




a) laminar case ($f = -H''$) b) turbulent case ($f = H$)

**Fig. 2** Comparison of the numerical solution of (1) with the multiple scales approximation and the regular expansion for $\beta = 0.1$

and is given by $x_p = 2\pi/\sqrt{\mathcal{X}}$ which agrees with (43). Thus we have shown that the asymptotic results (42) can be reproduced by an asymptotic expansion of (1) for $x \to \infty$ and $\beta$ small.

### 5.2 Laminar case

It can be verified that $S_0 = -2/3$ is a stationary point of (30). The corresponding zeros of the polynomial $p$ are $h_1 = -1$, $h_2 = h_3 = 2$. Linearizing (30) around that state we obtain

$$\Delta h_\Omega = -\pi \Delta\, h \tag{48}$$

or in the untransformed slow variable $X$:

$$\Delta h_{\mathcal{X}} = -\tfrac{1}{2}\Delta h\,. \tag{49}$$

Thus asymptotically the solution is a damped oscillation around $H = 2$ with wave length $1/2\pi$ and decay rate $\beta/2$. The same result can be obtained by linearizing (1) around $H = 2$.

## 6 Results and conclusions

The equations for the slowly varying variables (20), (21) have been solved with the fourth order Runge-Kutta method with the initial condition (26) for both cases. The roots of the polynomial $p(H; R_0, S_0)$ are determined by Newton's method and the integrals $I_j$ are evaluated by the trapezoidal rule after the substitution $h = h_3 - (h_3 - h_2)\sin\,\theta$.

In Figs. 1a,b the slowly varying quantities are represented by the zeros $h_2$ and $h_3$ of the polynomial $p(H; R_0, S_0)$ for both cases as functions of the original slow variable $\mathcal{X}$. Note that $R_0$ and $S_0$ can be obtained from $h_2$ and $h_3$ by :

$$S_0 = \tfrac{1}{6}(3 - h_2 - h_3)h_2 h_3\,, \qquad R_0 = \tfrac{1}{6}(h_2^2 + h_2 h_3 + h_3^2 - 3(h_2 + h_3))\,. \tag{50}$$

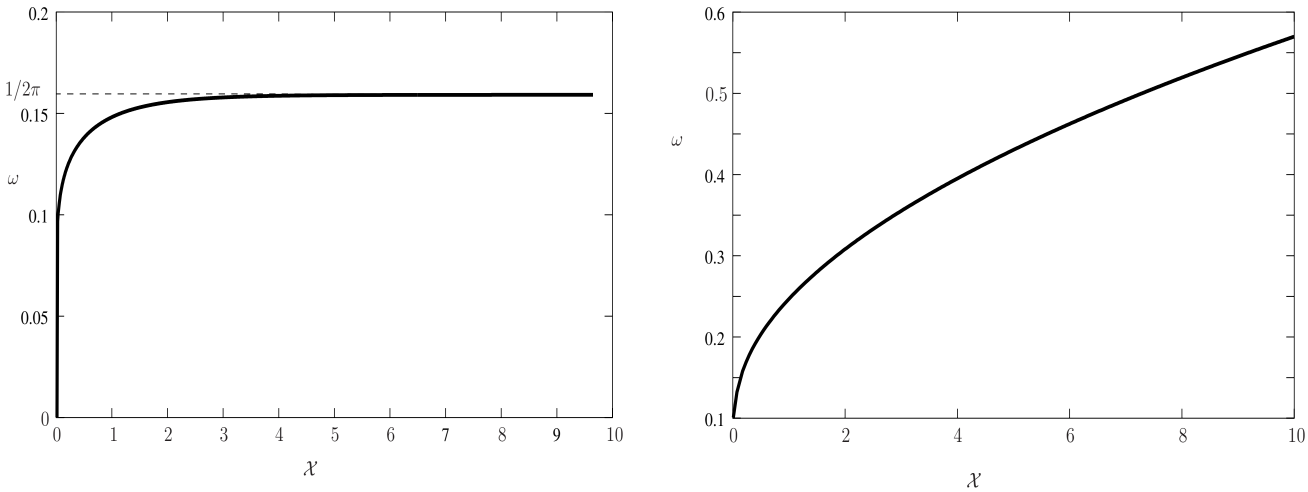
As expected near the origin $h_2$ behaves like $\sqrt{\mathcal{X}}$ in both cases. In the laminar case $h_2$ and $h_3$ tend to the asymptotic value 2 for $\mathcal{X} \to \infty$. In the turbulent case $h_3$ and $h_3$ grow linearly for $\mathcal{X} \to \infty$. This can be interpreted as a transition of the open channel flow to deep water. (However, the basic assumption of small perturbations ceases to be valid.)

Eq. (1) is integrated numerically by a fourth-order Runge-Kutta method for $\beta = 0.1$. The asymptotic boundary condition (2) is replaced by the initial condition $H(x_I) = \delta$, $\mathrm{d}H/\mathrm{d}x(x_I) = \delta\ \mathrm{d}^2H/\mathrm{d}x^2(x_I) = 0$ with $\delta = 10^{-8}$. The location $x_I$ is chosen such that the first maximum occurs at $x = 0$. Note that in Fig. 1 the axis of the slow $\mathcal{X}$ and fast $x$ variable are shifted.

In Figs. 2a,b comparisons of the approximation obtained by the multiple scales expansion and the regular expansion with a numerical solution of eq. (1) are given for both cases. In this plot only the first few oscillation are shown. However, we see there is reasonable agreement between the full numerical solution and the multiple scales approximation for $x > 0$ and the regular expansion for $x < 0$, respectively. In Table 2 the location of the second maximum of the numerical solution is compared with the analytic formula (39) for both cases and different values of $\beta$.

**Table 2** Comparison of the asymptotic result (39) for the distance between the first two maxima with a numerical solution of (1) with $f = H$ and $f = -H''$, respectively.

| $f$ | $\beta$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-3}$ | $10^{-5}$ | $10^{-6}$ |
|---|---|---|---|---|---|---|---|
| $H$ | $x_1$ (num.) | 4.68 | 7.05 | 9.39 | 11.69 | 14.00 | 16.30 |
| | $x_1$ (asymp.) | 4.83 | 7.14 | 9.44 | 11.74 | 14.04 | 16.35 |
| $-H''$ | $x_1$ (num.) | 6.91 | 8.78 | 11.01 | 13.31 | 15.61 | 17.91 |
| | $x_1$ (asymp.) | 6.44 | 8.74 | 11.05 | 13.35 | 15.65 | 17.96 |




a) laminar case ($f = -H''$) b) turbulent case ($f = H$)

**Fig. 3** Wave number $\omega$ as a function of the slow variable $\mathcal{X}$.

In Figs. 3a,b the wave number $\omega$ as a function of the slow variable $\mathcal{X}$ is shown for both cases, respectively. It is computed by inserting the numerical solution of (20), (21) into (25). It increases from 0 to the asymptotic value $1/2\pi$ in the laminar case. (In [6] eq. (14) $1/\omega$ has been assumed to be constant.) In the turbulent case $\omega$ increases apparently proportional to $\sqrt{\mathcal{X}}$ which is in accordance with (43).

Using the multiple scales method [9] an approximation for small values of $\beta$ to the solution of (1) is constructed. The solution is oscillatory with a slowly changing amplitude, period, and mean value of the oscillation. Ordinary differential equations for the slowly varying quantities have been derived, discussed and agreement with a full numerical solution has been obtained.

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