Multiple Scales and Bifurcation Problems in Boundary Layer Theory

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Abstract. Boundary layer flow problems depending on a parameter which have non-unique self-similar or planar solutions are considered near a turning point. The behavior of non-self-similar, or non-planar time-dependent solutions is investigated by an asymptotic expansion with respect to the deviation of the parameter from its critical value. Considering solutions which vary slowly with respect to time and the additional space coordinate a simplified partial differential equation can be derived which describes the local behavior.

1 Introduction

We consider boundary layer flow problems depending on a parameter α near a regular turning point. That is: below a critical value α_c of the parameter α there are two steady state or similarity solutions which coincide at the critical value, and above the critical value there is no steady state or similarity solution, or vice versa. In Fig. 1a the similarity solution of mixed convection boundary-layer flow is represented by the dimensionless wall shear stress $f''(0)$ and in Fig. 1b the planar stationary solution of the marginal separation equation is represented by the wall shear stress at $x = 0$, $A(0)$. Both curves have a turning point.

The structure of the steady state or similarity solution, respectively, near the critical value can be analyzed by employing $\varepsilon^2 = |\alpha - \alpha_c|$ as an expansion parameter. Using usual methods of bifurcation theory and assuming that the linearization at the critical value has a one-dimension kernel the first order correction can only be found up to a constant C which has to be determined from a "bifurcation equation", which is a solvability condition for the equations of the second order correction. For the problem under consideration this is a quadratic equation.

Solutions of the original problem depend on time t and an additional space coordinate, say z. The latter can be eliminated by the assumption of planar or self-similar flow, but in this process the undetermined constant C of the first order correction can depend on the additional coordinates, i.e. $C = C(z, t)$. However, to be consistent with the expansion the additional independent coordinates have to be rescaled with ε, the distance to the critical conditions.

The bifurcation equation in that case is a partial differential equation for the undetermined "constant" as a function of the slow additional independent coordinates. Discussion of the bifurcation gives insight into exchange of stability and new types of solutions.

We will apply this approach to two different flow problems arising in boundary layer theory:

- Mixed convection boundary layer flow: The bifurcation equation is a kinematic wave equation with characteristics pointing upstream. Thus existence of solutions, connecting both original similarity solutions and upstream propagating perturbations can be

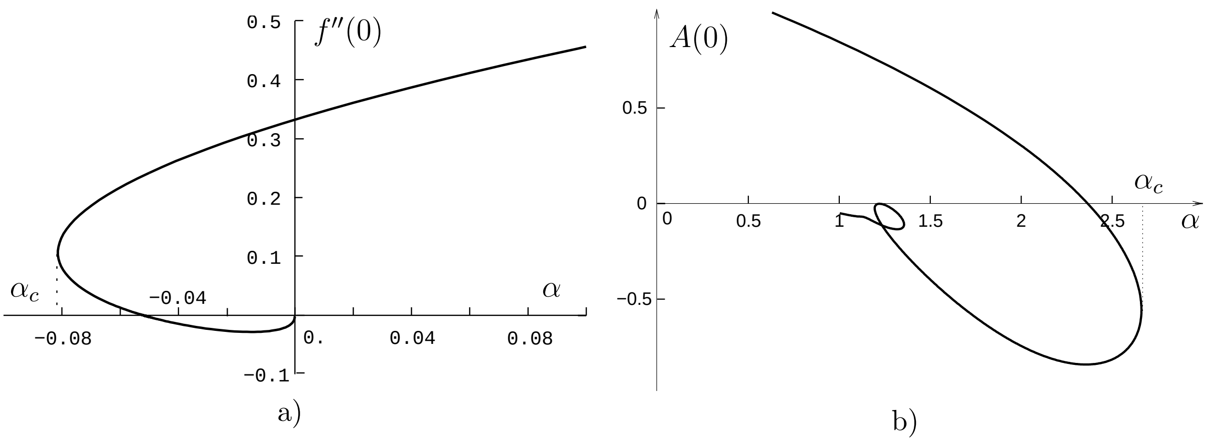

Fig. 1. Non-uniqueness of the solution. **a**) Similarity solution of the mixed convection boundary-layer equations represented here by the dimensionless wall shear stress as a function of the buoyancy parameter α, $Pr = 1$, **b**) solution of the planar steady marginal separation flow problem represented by the wall shear at the location $x = 0$ as a function of the deviation of the angle of attack from its critical value

demonstrated. However, these are very unusual properties of a boundary layer flow with no reverse flow regions or interaction with the outer flow field.

- Marginal separation at the leading edge of an airfoil at an angle of attack: In that case the 'bifurcation equation' is the Fisher equation, a non-linear diffusion equation. Besides the well known planar solutions it has solutions that are periodic in the lateral direction. Moreover the stability can be discussed.

The study of the much simpler 'bifurcation equations' allows insight into properties of the solution of the full problem and thus is an important tool of the analysis. As shown by these examples traditional methods of bifurcation and multiple scales analysis are combined.

2 Multiple scales analysis and bifurcations

In this section we present the main ideas of the analysis. Let us assume self-similar or planar stationary flow, and suppose that the equation for the corresponding flow field is given by a non-linear equation which depends on a parameter α,

$$N(u, \alpha) = 0, \tag{1}$$

such that the solution has a turning point bifurcation at some critical parameter α_c. We denote the corresponding solution by u_c. Note that the non-linear operator N acts on u only as a function of the similarity variable x. A necessary condition for a turning point is that the linearization N' of N at the turning point has a one dimensional kernel.

Furthermore we assume that the governing equations for non-self-similar (non-planar) flows can be written in the form

$$N(u, \alpha) = -K \frac{\partial}{\partial t} u + J \frac{\partial^\beta}{\partial z^\beta} u \tag{2}$$

where β is some positive integer and the linear operators K and J commute with the differential operators $\partial/\partial t$ and $\partial/\partial z$, respectively.

Our goal is to describe time and z-dependent solutions near the turning point. Since the similarity solutions behave like $\sqrt{|\alpha-\alpha_c|}$ near the critical point (c.f. Fig. 1a and Fig. 1b) we introduce the perturbation parameter $\varepsilon^2 = |\alpha-\alpha_c|$. The main idea is to rescale the time t and the additional space coordinate such that the t and z-dependent operators enter the asymptotic expansion with respect to ε in the second order terms only. This can be achieved by introducing the following slow variables,

$$Z = \varepsilon^{1/\beta} z, \quad T = \varepsilon t, \tag{3}$$

and expanding u in terms of ε,

$$u = u_c(x) + \varepsilon u_1(x, Z, T) + \varepsilon^2 u_2 + \ldots \quad . \tag{4}$$

Inserting (4) into (2) and comparing terms of order ε we obtain the linear equation

$$N' u_1 = 0. \tag{5}$$

Since the operator N' acts on u_1 as a function of x only, the solution of (5) is given by $u_1 = C(Z,T)e$, where $e = e(x)$ is an eigenfunction of N' and C is a yet unknown function of the slow variables Z and T.

Comparing terms of order ε^2 we obtain

$$N' u_2 = -\frac{C^2}{2} N''(e,e) - N_\alpha - \frac{\partial C}{\partial T} K e + \frac{\partial^\beta C}{\partial Z^\beta} J e, \tag{6}$$

where N_α is the derivative of the operator N with respect to the parameter α. Note that the second derivative $N''(.,.)$ of the nonlinear operator N with respect to u is a bilinear operator.

Since the operator N' is not regular the right-hand side of (6) has to satisfy a solvability condition. Now let f be an eigenfunction of the adjoint operator of N'. The 'left' eigenfunction f can be interpreted as a linear functional and we use the usual notation for applying a linear functional f onto a function u: $\langle f, u \rangle$. In the examples given in that paper $\langle f, u \rangle = \int f\, u dx$ denotes the usual L_2 scalar product. Applying the linear functional f onto (6) we obtain the solvability condition

$$-\langle f, Ke \rangle \frac{\partial C}{\partial T} + \langle f, Je \rangle \frac{\partial^\beta C}{\partial Z^\beta} - \frac{1}{2} \langle f, N''(e,e) \rangle C^2 - \langle f, N_\alpha \rangle = 0, \tag{7}$$

which is in general a partial differential equation for C. If $\beta = 1$ (mixed convection) the nonlinear kinematic wave equation

$$C_T - \nu C_Z + \mu C^2 - \delta = 0 \tag{8}$$

is obtained. If $\beta = 2$ (marginal separation) (7) is the non-linear diffusion equation

$$C_T - \nu C_{ZZ} + \mu C^2 - \delta = 0 \tag{9}$$

which can be transformed to the well-known Fisher equation [5]. Implicitly we have assumed that the coefficients μ, ν and δ are non-vanishing and finite. In the following we will apply these ideas to the above mentioned examples from boundary-layer theory and discuss the resulting bifurcation equations.

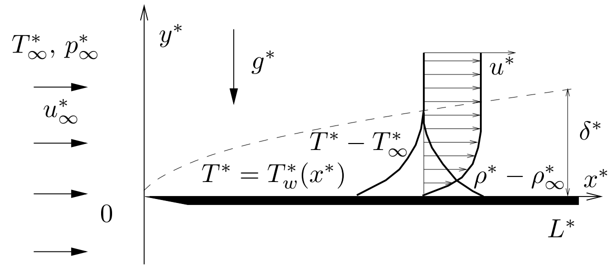

Fig. 2. Mixed convection boundary layer flow over a horizontal plate

3 Mixed convection - non-linear wave equation

3.1 Problem formulation

As a first example we consider mixed convection boundary-layer flow along a horizontal semi-infinite flat plate aligned parallel to a uniform free stream (Fig. 2).

It has been shown by Schneider [10] that a similarity solution exists if the plate temperature is proportional to the inverse of the square root of the distance from the leading edge $T_w^*(x) - T_\infty^* \sim \sqrt{x^*}$. The influence of buoyancy is measured by the dimensionless buoyancy parameter $\alpha = Gr/Re^{5/2} = g^*\beta^*(T_w^*(L^*) - T_\infty^*)(L^*\nu^*)^{1/2}(U_\infty^*)^{-5/2}$, where U_∞^*, T_∞^*, T_w^*, β^*, ν^*, g^* are the unperturbed velocity and temperature of the oncoming fluid, the plate temperature, the isobaric expansion coefficient, the kinematic viscosity and the gravity acceleration, respectively. The definition of α depends on the plate temperature at some distance L from the leading edge. However, due to the assumption that the plate temperature is proportional to $x^{-1/2}$ the buoyancy parameter α is independent of L^*. Dimensional quantities are denoted by a star.

The modified boundary layer equations are formulated in terms of the 'similarity' variable $\eta = y/\sqrt{x} = y^*\sqrt{U^*/\nu^* x^*}$, but retain the dependence on the coordinate $x = x^*/L^*$ parallel to the plate and time t. Instead of the lateral Cartesian coordinate y we use $\eta = y/\sqrt{x}$. As dependent variables $f = \psi/\sqrt{x}$, $\vartheta = \sqrt{x}\theta$ and $h = xp_x$ are used where ψ, θ and p_x are the dimensionless streamfunction, temperature distribution and pressure gradient parallel to the plate, respectively. Derivatives with respect to η are denoted with a prime, [11]. Thus the transformed boundary-layer equations read

$$2f''' + f\,f'' - \alpha h = 2x(f'f'_x - f''f_x + f'_t), \tag{10}$$

$$\frac{2}{\mathrm{Pr}}\vartheta'' + (f\vartheta)' = 2x(f'\vartheta_x - \vartheta' f_x + \vartheta_t), \tag{11}$$

$$h' + (\eta\vartheta)' = 2x\vartheta_x. \tag{12}$$

At the wall $\eta = 0$ the no slip boundary condition for the flow field and the plate temperature are prescribed:

$$f(x,0,t) = f'(x,0,t) = 0, \quad \vartheta(x,0,t) = 1 \quad . \tag{13}$$

At $\eta = \infty$ the matching conditions to the outer flow field

$$f'(x,\infty,t) = 1, \quad \vartheta(x,\infty,t) = 0, \quad h(x,\infty,t) = 0 \tag{14}$$

have to be satisfied.

Using the notation $w = (f, \vartheta, h)^T$ we write (10)-(14) in the compact form

$$N(w, \alpha) = x(J(w)w_x - K(w)w_t) \tag{15}$$

where N is the nonlinear differential operator on the left side of (10)-(12) and J and K are the differential operators on the right side of (10)-(12), respectively. Note that both J and K are non-linear operators. To be in accordance with the analysis of Sect. 2 one can replace $J(w)$, $K(w)$ by $J(w_c)$, $K(w_c)$, respectively, and note that the differences $J(w) - J(w_c)$, $J(w) - K(w_c)$ do not enter the solvability condition.

In contrast to the general procedure outlined in the previous section the operators on the right-hand side are multiplied by x. Therefore a coordinate transformation is necessary which will be discussed later.

For solutions which depend only on the similarity variable η the boundary layer-equation reduces to

$$N(w, \alpha) = 0. \tag{16}$$

This is a set of nonlinear ordinary differential equations. The solutions have been discussed by several authors [10], [1], [6].

For $\alpha > 0$ a unique solution has been reported, while for $\alpha < 0$ there are two solution branches connected at a turning point $\alpha = \alpha_c$. For buoyancy parameters below α_c there are no steady solutions at all , cf. Fig. 1a.

3.2 Analysis near the turning point

At the turning point α_c with unique solution u_c the linearization of the similarity equations:

$$N'(w_c, \alpha_c)\Delta w = 0 \tag{17}$$

has a nontrivial solution. In the following it will be denoted by $W = (F, D, H)^T$. Now analyzing the behavior near the turning point we introduce the expansion with respect to the perturbation parameter $\varepsilon^2 = |\alpha - \alpha_c|$ defined as the square root of the deviation of the buoyancy parameter from its critical value

$$w = w_c + \varepsilon w_1 + \varepsilon^2 w_2 + ... \tag{18}$$

Inserting it into the similarity equations we obtain for the first order correction terms

$$N' w_1 = 0, \tag{19}$$

which has the general solution $w_1 = CW$ where C is a yet undetermined function of x and t. Following the ideas of Sect. 2 we assume that the solution depends on large scale independent variables only such that the derivatives with respect to these variables are small. However, in the present case the term $x\partial/\partial x$ remains of the same order of magnitude if we rescale x by any power of ε. Therefore it is useful to introduce the new coordinate $\chi = \ln x$ instead. However, we cannot expand the terms containing $x\partial/\partial t$ uniformly. Thus we replace t by $\tau = \ln(1 + t/x)$. Using this new independent variables we have

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial \chi} - (1 - \mathrm{e}^{-\tau})\frac{\partial}{\partial \tau}, \qquad x\frac{\partial}{\partial t} = \mathrm{e}^{-\tau}\frac{\partial}{\partial \tau}. \tag{20}$$

Now we can introduce the slow variables,

$$\mathcal{X} = \varepsilon\chi, \quad \mathcal{T} = \varepsilon\tau, \tag{21}$$

and consider w as a function of the fast and slow variables

$$w \sim w_c + \varepsilon C(\mathcal{X},\mathcal{T}) W + \varepsilon^2 w_2(\mathcal{X},\mathcal{T},\tau) + \ldots \quad . \tag{22}$$

Inserting this expansion into the second order correction equation we obtain

$$\begin{aligned} N' w_2 + C^2 N''(W,W) + N_\alpha = {} & C_{\mathcal{X}} J(w_c)\, W - C_{\mathcal{T}}(1-\mathrm{e}^{-\tau}) J(w_c)\, W + \mathrm{e}^{-\tau} C_{\mathcal{T}} K(w_c) W \\ & + (1-\mathrm{e}^{-\tau}) J(w_c)\, w_{2,\tau} + \mathrm{e}^{-\tau} K(w_c) w_{2,\tau}, \end{aligned} \tag{23}$$

where N_α denotes the derivative of N with respect to α and N'' is the second derivative of N with respect to w. However, the fast time scale remains in the second order equation. After some initial transient behavior on the '*time scale*' τ the solution w_2 will depend on $\mathcal{T}$ and $\mathcal{X}$ only. The linear ordinary differential equation for w_2 is only solvable if the inhomogeneity satisfies a solvability condition. In order to formulate this condition we introduce the adjoint operator N'^{+} of N' and, by W^+, denote a basis vector of its one-dimensional kernel. It is given as the non-trivial solution of the system of ODEs

$$N'^{+}(W^+) = \begin{pmatrix} -2F^{+'''} + (f_b F^+)'' + f_b'' F^+ - \vartheta_b D^{+'} \\ \frac{2}{Pr}\vartheta^{+''} - f_b D^{+'} - \eta H^{+'} \\ -H^{+'} - K F^+ \end{pmatrix} = 0 \tag{24}$$

with the boundary conditions

$$F^{+'}(0) = F^{+'}(\infty) = D^+(0) = D'(\infty) = H^+(0) = 0. \tag{25}$$

Multiplying (23) by W^+ and integrating with respect to η from 0 to ∞ yields the solvability condition

$$C_{\mathcal{X}} - C_{\mathcal{T}} + \mu C^2 - \delta = 0, \tag{26}$$

with the constants δ and μ given by

$$\delta = \frac{\int_0^\infty W^+ N_\alpha \,\mathrm{d}\eta}{\int_0^\infty W^+ J W \,\mathrm{d}\eta}, \qquad \mu = -\frac{\int_0^\infty W^+ N''(W,W)\,\mathrm{d}\eta}{\int_0^\infty W^+ J W \,\mathrm{d}\eta}.$$

In terms of the original variable x and t the general solution of equation (26) is given by

$$C = \left(1 + \frac{2}{C_I(\varepsilon \ln(x+t))(1+t/x)^{2\varepsilon\sqrt{\delta\mu}} - 1}\right) c_s, \tag{27}$$

where $C_I(\mathcal{X})$ is an arbitrary function and $c_s = \sqrt{\delta/\mu}$.

3.3 Discussion of the results

Equation (26) is a wave equation with characteristics pointing upstream. Therefore perturbations can propagate upstream. Depending on the asymptotic behavior of C_I for $\mathcal{X} \to \infty$ the solution at a fixed location x converges either to the upper similarity solution ($C = c_s$), to the lower branch similarity $C = -c_s$ or to a stationary solution connecting both similarity solutions

$$C = c_s \frac{c - x^{\varepsilon\sqrt{\mu\delta}}}{c + x^{2\varepsilon\sqrt{\mu\delta}}}, \tag{28}$$

where c is an arbitrary constant. In Fig. 3 numerical solutions of the stationary modified boundary layer equations connecting both similarity solutions are shown and compared with the asymptotic result. The solution is represented by the wall shear stress $f''(x,0)$.

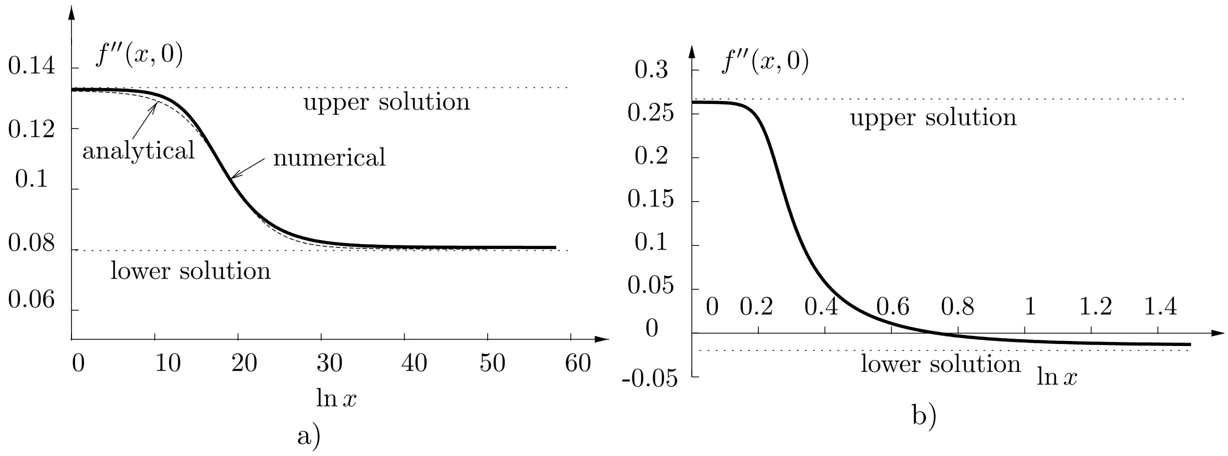

Fig. 3. Solution connecting two similarity solutions for **a**) $\alpha = -0.08$ ($\varepsilon = 0.1$), $\mathrm{Pr} = 1$ **b**) $\alpha = -0.04$, $\mathrm{Pr} = 1$

Moreover the connecting solutions exist not only for small values of ε. In Fig. 3b a numerical solution of the stationary boundary layer equation for $\alpha = -0.04$ is shown. An interesting fact is that this solution has a reverse flow region ($f'' < 0$).

Using the bifurcation method combined with the multiple scale analysis the physical relevance of the two similarity solutions has been discussed, showing that the upper solution is stable provided that the downstream perturbations are not too large.

4 Marginally separated boundary layer flows - non-linear diffusion equation

4.1 Formulation of the problem

At high Reynolds numbers the flow around an airfoil is governed by the boundary-layer equations. The solution of the boundary-layer equations can be obtained as long as the angle of attack is below a limiting value. For lager angles the solution terminates in the form of a Goldstein singularity. At the limiting angle the marginal separation singularity occurs which can be eliminated by taking into account the interaction pressure resulting from the increase of the boundary-layer displacement.

The equation governing the local flow behavior near the marginal separation can be formulated in terms of the negative correction of the displacement thickness A which is proportional to the local wall shear stress. The origin of the coordinate system is supposed to be in the point where the marginal separation singularity occurs. The stream-wise coordinate is denoted by x and the lateral coordinate by z.

For a planar stationary flow $A(x)$ satisfies the integro-differential equation [12], [7]

$$A^2 - x^2 + \alpha - J\,A_{xx} = 0, \tag{29}$$

where the integral operator J is given by:

$$JA(x,z,t) = \lambda \int_x^\infty \frac{A(\xi,z,t)}{\sqrt{\xi - x}}\,\mathrm{d}\xi \tag{30}$$

and α denotes the deviation of the angle of attack from its limiting value. It is well known that (29) has solutions only for α below a critical value $\alpha_c = 2.66$. Furthermore, for

$0 < \alpha < \alpha_c$ the solution is not unique. In Fig. 1b the solutions, represented by the value of the displacement thickness at $x = 0$, are plotted as a function of α. Both solution branches are connected at the turning point $\alpha = \alpha_c$. Similarly as in the previous example we want to study the behavior of time dependent three dimensional solutions near the turning point.

The interaction equation for three dimensional time dependent flow is [4]

$$A^2 - x^2 + \alpha - J\,A_{xx} = J\,A_{zz} + L\,A_{xx} + L\,A_{zz} - K\,A_t, \tag{31}$$

where K, L are integral operators defined by

$$LA(x,z,t) = -\frac{\lambda}{2\pi}\int_{-\infty}^{X}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(s-\xi)(A(\xi,\eta,t)-A((\xi,z,t))}{[(s-\xi)^2+(z-\eta)^2]^{3/2}[x-s]^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s, \tag{32}$$

$$KA(x,z,t) = \gamma\int_{-\infty}^{X}\frac{1}{(x-\xi)^{1/4}}A(\xi,z,t)\,\mathrm{d}\xi, \tag{33}$$

and λ and γ are some known constants. Note that we have written equation (31) in such a form that the right-hand side vanishes for strictly planar stationary flows and the operators on the left side are identical with the planar stationary problem (29).

4.2 Bifurcation analysis

Again we want to make use of the linearization of the planar problem

$$2A_c b - J b_{xx} = 0, \tag{34}$$

where $A_c(x)$ is the solution of the planar problem (29) at $\alpha = \alpha_c$. Note that equation (34) has a one dimensional kernel spanned by an eigenfunction $b(x)$.

The main idea is to introduce properly scaled slow time and lateral space variables, such that the terms on the right-hand side of (31) are sufficiently small. Following [3] we introduce the perturbation parameter $\varepsilon^2 = \alpha_c - \alpha$ and the slow variables $Z = \sqrt{\varepsilon}z$, $T = \varepsilon t$. Thus we expand A in terms of ε as

$$A = A_c(x) + \varepsilon a_1(x,Z,T) + \varepsilon^2\Big((\ln\varepsilon)a_2'(x,T) + a_2(x,Z,T)\Big) + \ldots. \quad . \tag{35}$$

Inserting (35) into (31) and comparing terms of order ε^2 yields equation (34) for a_1. The general solution is given by $a_1 = C(Z,T)b(x)$ with a yet unknown function $C(Z,T)$ which has to be determined, as usually in multiple scales problems, from a solvability condition for higher order terms, namely the terms of $O(\varepsilon^2)$. The equation for a_2 reads

$$2A_c a_2 - J a_{2,xx} = 1 + C_{ZZ}Jb - C_T K b - C^2 b^2. \tag{36}$$

Since the integral operator on the left side is singular the inhomogeneity on the right side has to satisfy a solvability condition. It is obtained by taking the L_2-scalar product with respect to x of equation (36) with the eigenfunction $n(x)$ to the eigenvalue 0 of the adjoint operator of the linearization. Thus $n(x)$ is the solution of the integral equation

$$2A_c n - \frac{\mathrm{d}^2}{\mathrm{d}x^2}Jn = 0. \tag{37}$$

As a result we obtain the solvability condition

$$C_T - \nu C_{ZZ} + \mu C^2 - \delta = 0, \tag{38}$$

with

$$\nu = \frac{\langle n, Jb \rangle}{\langle n, Kb \rangle}, \quad \mu = \frac{\langle n, b^2 \rangle}{\langle n, Kb \rangle}, \quad \delta = \frac{\langle n, 1 \rangle}{\langle n, Kb \rangle}, \tag{39}$$

where $\langle n, q \rangle := \int_{-\infty}^{\infty} n(x) q(x)\, \mathrm{d}x$ denotes the scalar product in L_2. For details on the functions n and b the reader is referred to [3]. We have to note that the operator L does not influence the solvability equation. Insertion of $a_1 = C(Z,T) b(x)$ into L produces a term of higher order which has to be eliminated by a term of order $\varepsilon^2 (\ln \varepsilon) a_2'$ where a_2' is of the form $a_2' = d(T) b(x)$.

4.3 Discussion of the bifurcation equation

Equation (38) can be transformed to the Fisher equation [5] which is well known from the description of nonlinear wave propagation in gene populations and reaction diffusion processes.

In contrast to these studies, where meaningful solutions are limited to the interval between the two equilibrium states $(-c_s, c_s)$ with $c_s = \sqrt{\delta/\mu}$, in the present case no restriction on the magnitude and sign of C is imposed. In the following we will discuss some special types of solutions.

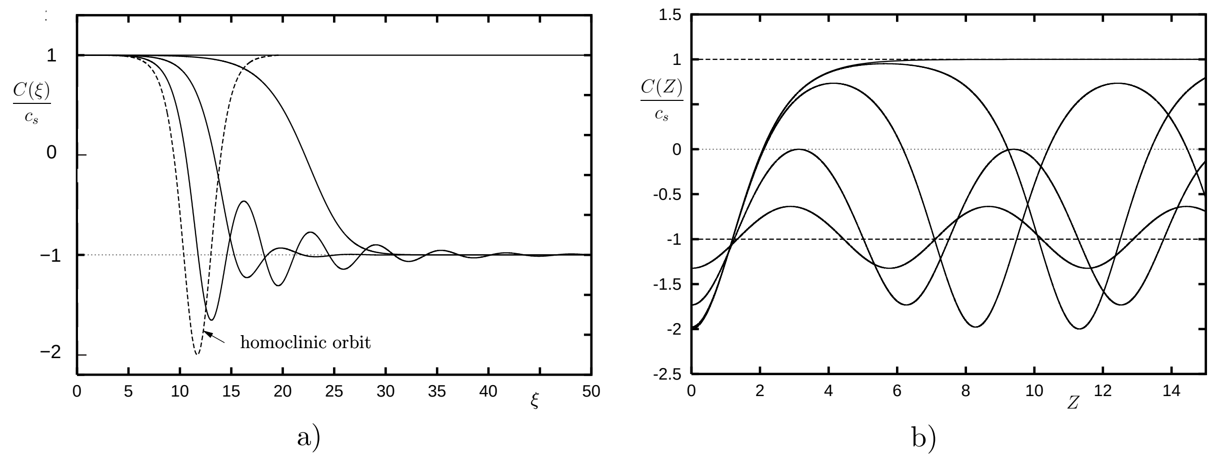

Fig. 4. Solutions of the Fisher equation **a**) Traveling wave solutions, **b**) periodic solutions. The solutions represent the variation of the wall shear stress in the span-wise direction

Considering planar solutions (independent of Z) we obtain the general solution

$$\frac{C(T)}{c_s} = \frac{1 - a\,\mathrm{e}^{-2\sqrt{\delta\mu}T}}{1 - a\,\mathrm{e}^{-2\sqrt{\delta\mu}T}}, \tag{40}$$

where a is an arbitrary constant, indicating the upper solution branch is stable when considering planar perturbations only. Considering laterally traveling waves we introduce the coordinate $\xi = Z - \lambda T$ and equation (38) reduces to the ODE

$$\lambda C_\xi + \nu C_{\xi\xi} - \mu C^2 + \delta = 0, \tag{41}$$

which can be discussed by phase plane analysis. For $\lambda > 0$ a hetero-clinic orbit exists connecting both steady state solutions $\pm c_s$. Solutions for different values of λ are shown in Fig. 4a. It should be noted that at a fixed location Z this solution tends to c_s corresponding to the upper solution branch. For $\lambda = 0$ a homo-clinic orbit exists.

Of special physical interest are stationary solutions which are periodic in the lateral direction. They can be expressed analytically using the Jacobian elliptical functions. The functions contain an arbitrary constant $\phi \in [0, \pi/3]$ which is a measure for the amplitude. Note that for $\phi \ll 1$ the amplitude is small and the solutions are close to harmonic functions. In the limiting case $\phi = \pi/3$ a homo-clinic orbit is obtained.

5 Conclusions

Using the bifurcation analysis combined with a multiple scales approach the complicated original (integro-)differential equations can be reduced to a single partial differential equation. This in turn leads to a significant simplification of the boundary-layer flow problems allowing the construction of new weakly non-similar and three-dimensional solutions and a discussion of their stability.

The presented examples concern laminar boundary-layer flows. However, as shown recently, a similar analysis applies to quasi equilibrium turbulent boundary-layer flows [9].
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References

1. F. R. deHoog, B. Lamminger, R. Weiss: A numerical study of similarity solutions for combined forced and free convection. Acta Mech. **51**, 139-149, (1984)
2. S. Braun, A. Kluwick: The effect of three-dimensional obstacles on marginally laminar boundary layer flows. J. Fluid Mech. **460**, 57-82, (2002)
3. S. Braun, A. Kluwick: Analysis of a bifurcation problem in marginally separated laminar wall jets and boundary layers. Acta Mech. **161**, 195-211, (2003)
4. S. Braun, A. Kluwick: Unsteady three-dimensional marginal separation caused by surface mounted obstacles and/or local suction, J. Fluid Mech. **514**, 121-152, (2004)
5. R. A. Fisher: The wave of advantageous genes. Ann. Eugenics **7**, 355-369, (1937)
6. J. H. Merkin, D. B. Ingham: Mixed convection similarity solutions on a horizontal surface, ZAMP **38**, 102-116, (1987)
7. A. Ruban Asymptotic theory of short separation regions on the leading edge of a slender airfoil. Izv. Akd. Nauk SSSR; Mekh. Zhidk. Gaza **1**, 42-51, (Engl. trans. Fluid dyn. **17**, 33-41), (1981)
8. A. Ruban: Stability of pre-separation boundary layer on the leading edge of a thin airfoil. Izv. Akd. Nauk SSSR; Mekh. Zhidk. Gaza **6**, 55-63, (Engl. trans. Fluid dyn. **17**, 860-867, (1983)
9. B. Scheichl, Non-Unique Quasi-Equilibrium Turbulent Boundary Layers, ICTAM 2004, FM2L-11083, Warsaw, Poland, August 18th, 2004
10. W. Schneider: A similarity solution for combined forced and free convection flow over a horizontal plate. Int. J. Heat Transfer **22**, 1401-1406 (1979)
11. H. Steinrück: Mixed convection over a horizontal plate: self-similar and connecting boundary-layer flows. Fluid Dyn. Res. **15**, 113-127, (1995)
12. K. Stewartson, F. T. Smith, K. Kaups: Marginal Separation. Stud. in Appl. Math. **67**, 45-61, (1982)