Survey on an Asymptotic Description of (Marginally) Separating Turbulent Boundary Layers

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Basic assumptions and analysis

- nominally steady incompressible 2D TBL along solid surface $y = 0$
- no free-stream turbulence
- globally defined Reynolds number $Re = \frac{\bar{U} L}{\bar{v}} \rightarrow \infty$
- **Hypothesis**
  - layer of thickness $\Delta$, assume local turbulent velocity scale $U_t \ll 1$:
  \[
  \frac{\partial u}{\partial y} = O\left(\frac{U_t}{\Delta}\right) \Rightarrow -\langle u'^2 \rangle, -\langle v'^2 \rangle, -\langle u'v' \rangle = O\left(U_t^2\right)
  \]
  \[
  \Rightarrow 1 - \frac{u}{u_e} \ll 1
  \]
  - overlap: log-law
  \[
  \frac{\partial u}{\partial y} \sim \frac{U_t}{\kappa y}, \quad U_t = u_{\tau} = \sqrt{\tau_w}
  \]
  - near-wall scaling $u = O(u_{\tau})$ from match with fully turbulent region!
Axiomatic reformulation of **classical** asymptotic theory

... by adopting a minimum of assumptions:

- outer main regime: velocity defect $1 - u/u_e \ll 1$
- viscous wall layer:
  
  order-of-magnitude analysis $\Rightarrow$ velocity scale $u_\tau \Rightarrow$ log-law
  
  $$u^+ = u/u_\tau \sim \kappa^{-1} \ln y^+ + C^+, \quad y^+ = yu_\tau/\tilde{\nu} \to \infty$$
  
- common overlap $\Rightarrow$ **two-tiered** TBL:
  
  $$\gamma = u_\tau/u_e \sim \kappa/\ln Re, \quad \delta = O(\gamma)$$
  
  $$1 - u/u_e \sim \gamma(x) F'(x, \eta), \quad -\langle u'v' \rangle \sim \gamma(x)^2 T(x, \eta), \quad \eta = y/\delta(x)$$
Extensions/shortcomings of small-defect approach

• 3D TBL, flow near plane of symmetry
  Degani, Smith & Walker 1992, 1993

• compressible TBL  Walker 1995

• weak viscous/inviscid interaction:
  ◇ flow past trailing edge of aligned flat plate
  ◇ TBL over humps  Sykes 1980
    shock wave impingement
    Adamson & Feo 1975, Melnik & Grossman 1976,
    compression corner flow  Agrawal & Messiter 1984
    small-defect assumption prevents separation
  ◇ bluff-body separation  Neish & Smith 1992
    separation at rear stagnation point
  ◇ TBL exposed to smooth APG  Neish & Smith 1992, Scheichl 2001
    separation not provoked at all

• Wall layer dynamics for attached time-mean flow  Walker 1989
Extended small-defect theory – multi-valued solutions

- **viscous wall layer:** \( \delta_v \sim (Re \, u_\tau)^{-1} \), \( u_\tau^2 = Re^{-1} \partial u / \partial y|_{y=0} \) \( U_t = u_\tau \)

- **defect region:** \( D = 1 - u/u_e \ll 1 \)
  
  (i) **classical theory:** two-tiered \( U_t = u_\tau \) \( D = O(U_t) \)

  (ii) **extended theory:** three-tiered (Scheichl & Kluwick 2004)

  \[ \begin{align*}
  & \text{intermediate layer: } U_t = u_\tau \\
  & \text{outer layer: } U_t \gg u_\tau \quad D = O(U_t) \quad Re^{-1} = 0 ? \\
  & \text{quasi-equilibrium flow: } u_e \propto x^m, \quad m = -1/3 + \mu, \quad (U_t, \mu) = O(u_\tau^{2/3})
  \end{align*} \]

- \( D = O(1) \Rightarrow \) slenderness parameter \( 0 < \alpha \ll 1 \) as \( Re^{-1} = 0 \)
Serious shortcoming of small-defect approach

... with respect to (marginal/internal) separation:

- \( \frac{d p_e}{d x} = -u_e \frac{d u_e}{d x} = O(1) \), continuous

- separation \( \Rightarrow \) need for region where \( D = 1 - \frac{u}{u_e} = O(1) \)

- assumption of small defect \( D = O(\epsilon) \ll 1 \) in outermost region...

\[
1 - \frac{u}{u_e} \sim \epsilon F'(x, \eta) + \cdots, \quad \langle u'v' \rangle \sim \epsilon^2 T(x, \eta) + \cdots, \quad \delta \sim \epsilon \Delta(x) + \cdots, \quad \eta = \frac{y}{\delta},
\]

- \( \cdots \) yields \( \frac{u_\tau}{u_e(x)} \sim \frac{\kappa}{\ln \text{Re}} \) due to log-law,

- \( \cdots \) does not allow for necessary growth of its order of magnitude!

- physical interpretation:

  insufficient vorticity transfer from wall layer via log-law \( \Rightarrow \)
  outer region governed by Bernoulli's law \( \Rightarrow \) no flow reversal!

- consequence: small-defect TBL does not separate at all

case: bluff-body separation

experiments: break-away separation

transition

≈ 110°

free shear layer

locally \( D = O(1) \)

(Neish & Smith 1992)

attached bulk flow ⇒ Goldstein-type singularity at \( G \)

• fundamental question: velocity defect \( 1 - \frac{u}{u_e} = O(1) \): \( \alpha(Re) = ? \)
Shear layer approximation

- expansions for $\alpha \to 0$:

$$\psi \sim \alpha \Psi(x,Y), \quad -\langle u'v' \rangle \sim \alpha T(x,Y), \quad \delta \sim \alpha \Delta(x), \quad y = \alpha Y$$

- BL equations:

$$\frac{\partial \Psi}{\partial Y} \frac{\partial^2 \Psi}{\partial Y \partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial^2 \Psi}{\partial Y^2} = u_e \frac{du_e}{dx} + \frac{\partial T}{\partial Y}, \quad T = \ell^2 \frac{\partial^2 \Psi}{\partial Y^2} \bigg|_{\partial^2 \Psi}$$

- wake-type boundary conditions:

$$(\partial \Psi / \partial Y)[x, \Delta(x)] = u_e(x), \quad \Psi(x,0) = T[x, \Delta(x)] = 0$$
Large velocity defect: BL slenderness $\alpha$ depends on $Re$ ?

- **yes:** Sychev & Sychev (1980ies): mismatch encountered!
- **no:**
  - Schneider (1991): free shear layers
  - Scheichl & Kluwick (2006):
    \[ U_t = O(1) \Rightarrow \text{no small turbulent velocity scale} \]
    \[ \Rightarrow \text{most general structure of TBL, based on dynamical considerations (coherent motions)} \]
  
- note:
  
  \[ \varepsilon \sim Re^{-1} \left< \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_j}{\partial x_j} \right> = O(1) \]

  \[ k - \text{budget: } \varepsilon \sim -\langle u'v' \rangle \frac{\partial u}{\partial y} - \frac{\partial \langle (k + p')v' \rangle}{\partial y}, \quad k = \frac{u'^2 + v'^2 + w'^2}{2} \]
General asymptotic structure of APG-TBL flows

- order-of-magnitude analysis: rotational part of $u$ of $O(U_t)$, $U_t = u_\tau$
- viscous wall layer: $\delta_v \sim (Re \ u_\tau)^{-1}$
- intermediate layer: $\delta_\tau \sim \tau$, transition from log- to half-power-law:

$$
\eta \propto \frac{y}{u_\tau^2}, \quad \frac{-\langle u'v' \rangle}{u_\tau^2} \sim 1 + \eta, \quad u - U_s(x) \sim u_\tau \ U(x,\eta) \sim \frac{1}{\kappa} \begin{cases} 
\ln \eta, & \eta \to 0 \\
2\sqrt{\eta}, & \eta \to \infty 
\end{cases}
$$
**BL solutions: marginally separated flows**

**APG controlled by parameter:** $u_e(x; \beta)$

laminar (Ruban 1981)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\delta^*$</th>
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<td>0.25</td>
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<td>2.2</td>
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<tr>
<td>0.05</td>
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symmetric

marginal-separation singularity for $\beta = \beta_M$,
solution linearized around separating profile upstream and downstream

nonlinear downstream wake

weak Goldstein singularity for $\beta > \beta_M$

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turbulent (Scheichl & Kluwick)

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$U_s$</th>
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<td>9</td>
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<tr>
<td>7</td>
<td>0.06</td>
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<td>5</td>
<td>0.02</td>
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</tbody>
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asymmetric!
Canonical local representation: bifurcating flows

local variable  \( s = x - x_M \to 0 \)

upstream shift  \( \epsilon \propto \tilde{s} - s = O(-1/\ln |\beta - \beta_M|), \beta - \beta_M \to 0 \)

inner scaling  \( X = \tilde{s}/\epsilon^2, \bar{Y} = Y/\epsilon^{2/3}, \bar{U}_s \propto U_s/\epsilon, \bar{T} \propto T/\epsilon^{2/3} \)

exponential branching

\[ \bar{U}_s, \bar{T} - \bar{Y} \to \bar{A}, \bar{Y} \to \infty \]

- downstream: locally infinite acceleration (cf. Messiter 1970)
  \[ \bar{A} \sim 1.0386 \ldots X^{1/3}, \bar{U}_s \sim 1.1835 \ldots X^{1/2}, X \to \infty \]

- upstream: induced favorable pressure \( \Rightarrow \) decreasing \( \tilde{U}_s \)
Local interaction theory: marginal separation

- singularity of BL solutions: \( u_e = u_{e,M}(x) \Rightarrow U_s(x_M) = 0 \)
- \( \hat{s} = x - x_M + O(\epsilon), \quad \epsilon \propto -1/\ln |u_e(x_M) - u_{e,M}(x_M)| \to 0_+ \)

Induced pressure feedback \( \sigma \tilde{P}(\hat{X}) \) acting along \( \hat{s} = \sigma \hat{X} \) ...

\[ y = \delta = O(\alpha) \]

\( \alpha^{3/2} \)

\( \alpha \sigma^{1/3} \)

\[ \sigma \]

\( 0 \)

\( \hat{s} \)

\( 1 \geq \Gamma \geq 0 \)

\( \epsilon^2 \leq \sigma = (\epsilon/\Gamma)^2 \leq \alpha^{3/5} \)

- upper deck UD: \( p = P_M + \sigma [P_M' \hat{X} + \Lambda(\Gamma) \tilde{P}(\hat{X})] + \cdots \)
- main deck MD: \( \psi/\alpha = \Psi_M(Y) + \sigma^{1/3} \ell_{0,M}^{2/3} \hat{A}(\hat{X}) \Psi_M'(Y) + \cdots, \quad Y = y/\alpha \)
- lower deck LD: \( \hat{P} = -\mathcal{H}\{\hat{A}'\} \) enters resulting interaction problem
Outer wake: triple-deck solutions

- central FDs, algebraic eqs solved directly (Powell’s hybrid method)

  boundary condition for $\hat{X} \to -\infty$: $\hat{U}_s \to \Gamma$

$\Lambda = 3, \Gamma = 0.019$

- small second bubble upstream for $0.016 \ldots \leq \Gamma \leq 0.019 \ldots$

- $\Gamma \to 0$: exponentially bifurcating eigensolutions vanish, $L \to \infty$

- $\Lambda \to 0$: BL solution – and massively separated solution for $\beta > \beta_M$?
Triple-deck solutions, cont’d

- effect of sublayers near surface on separating solutions?

overall slip velocity \( u_s(x, \alpha) = \sigma^{1/2} \hat{U}_s(\hat{X}) + O(\alpha^{3/4}) \) ⇒ breakdown

\[ \text{LD, } \hat{\Psi} = \text{const}: \] \( \Lambda = 3, \Gamma = 0.019 \)

small bubble, \( L \to 0: \)

- inner wake, \( \hat{X} - \hat{X}_0 = O(L), \) \( L = O(\alpha^{9/40}), \) \( \Gamma \approx 0.205: \)

\[ -\partial \langle u'v' \rangle / \partial y \sim P_0, \] \( \partial p/\partial x \sim P_0 = P_M' + \Lambda(\Gamma) \hat{P}'(\hat{X}_0) \) governs reverse-flow,

- smoothly separating solution, \( \text{log-law} \leftrightarrow \text{half-power law!} \)
Gradual transition from log-law to half-power law

- Viscous wall layer \( u_\tau \gg Re^{-1/3} \)

\[
u / u_\tau \sim u(x,y^+) + \cdots, \quad -\langle u'v' \rangle / u_\tau^2 \sim \tau^+(x,y^+) + \cdots, \quad y^+ = y u_\tau Re
\]

Log-law:

\[
u \sim A^+(x) \ln y^+ + B^+, \quad y^+ \to \infty
\]

\( \text{sgn}(A^+) = \text{sgn}(\tau_w), \quad \tau_w > 0 : \quad A^+ = \kappa \)

Skin friction law:

\[
u_\tau / u_s \sim A^+ / \ln(u_\tau^3 Re)
\]

- Viscous wall layer \( u_s, u_\tau = O(Re^{-1/3}) \)

\[
u / u_p \sim u(p^x, y^x) + \cdots, \quad -\langle u'v' \rangle / u_p^2 \sim \tau^x(p^x, y^x) + \cdots, \quad y^x = y u_p Re,
\]

\[
p^x = \text{sgn}(\tau_w) (u_p / u_\tau)^3, \quad u_p = (P_0 / Re)^{1/3}
\]

Half-power law:

\[
u \sim A^x(p^x) \sqrt{y^x} + B^x(p^x), \quad y^x \to \infty
\]

Skin friction law:

\[
u_s / u_p \sim B^x(p^x) \ldots \text{separation!}
\]
Turbulent gross separation – preliminary results

\[ \psi > 0, \nabla^2 \psi = 0 \]
\[ \alpha, \ \Delta p = O(a)! \]
\[ u(x, y = 0) = u_e \]
reference: \[ u_0 = 1 \]

- **inviscid-flow** theory (including rotational backflow):

  \[ \frac{\partial p}{\partial x}(x, y = 0) = k (-x)^{-1/2} + \frac{16}{3} k^2 + O((-x)^{1/2}), \quad k \geq 0, \quad x \to 0_– \]

- **formal limit** \[ Re^{-1} = 0, \ \alpha \to 0 \]: Brillouin–Villat condition \( k = 0 \) not met!

\[ p(x, k) - p(0, k) \]
\[ U_s(x, k) \]
Further outlook

- basic hypothesis for TBL scaling derived from first principles: multiple-scales analysis of Navier-Stokes eqs as $\alpha \to 0$
- closed reverse-flow region: $A^+(x), B^+(x)$
- extended skin-friction formula: $B^\times(p^\times)$
- global separation