Note on Conditional Constructivity

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Abstract. In this note we provide a straightforward translation $\mathcal{C}_p^{\Gamma}(T)$ for sets of formulas T and $H_{\Gamma}(\exists x A(x))$ for existential formulas $\exists x A(x)$ s.t. $\mathcal{C}_p^{\Gamma}(T) \vdash H_{\Gamma}(\exists x A(x))$ expresses "$\exists x A(x)$ is derivable constructively from T iff it is derivable at all".

1 Preliminaries

The strength of mathematical logic lies often in its ability to express metamathematical statements on a mathematical level. In this note we deal with conditional constructive provability, i.e. with statements "if $\exists x A(x)$ is provable at all from T then it is constructively provable". In addition, the signature of the witness can be specified arbitrarily.

The idea is to add a variable position x to every atomic formula, let $B^*(x)$ be the transform of B and let $A^*(y, x)$ be the transform of $\exists y A(y)$ after deletion of the outermost existential quantifier. Let T^* be the $\forall$-closure of $\{B^*(x) \mid B \text{ in } T\}$. Then $T^* \vdash \exists x \forall y (A^*(y, x) \supset A^*(x, x))$ expresses the desired statement.

The reasons to express constructive provability and conditional constructive provability using a translation within classical logic are mainly the following:

– It is not possible to characterize classical constructive provability by a so-called constructive logic $\mathcal{L}$ weaker than classical logic: There will be always T and $\exists x A(x)$, such that $T \vdash \exists x A(x)$ non constructively in $\mathcal{L}$ but $T \vdash A(t)$ for some t in classical logic.[1]
– Usual classical models can be used to study classical non-provability in the constructive sense in the presence of provability.
– "$\exists x A(x)$ is constructively provable" can be used as assumption/axiom without extending the framework of classical logic, in case of conditional constructive provability as meaningful assumption/axiom for all extensions of a given theory.

2 $\mathcal{C}_p^{\Gamma} / \mathcal{C}_c^{\Gamma}$

Let $\vdash$ denote classical deduction and let $\mathrm{cl}_{\exists}(A)$ ($\mathrm{cl}_{\forall}(A)$) be the existential (universal) closure of A.

[1] $T \equiv ((Q \vee \neg Q) \supset P(0)) \wedge (P(0) \vee P(1))$ and $\exists x P(x)$ provide an example for any intermediate first-order logic $\mathcal{L}$.

S. Aguzzoli et al.(Eds.): Algebraic and Proof-theoretic Aspects, LNAI 4460, pp. 25–29, 2007.

Let T be a theory, Γ be assigned to $A \equiv \exists y A'(y)$, and assume that all bound variables are different to x.[2]

Define

$$\psi^x_\Gamma(z) \equiv z \quad (z \text{ a bound or free variable})$$
$$\psi^x_\Gamma(c) \equiv c^*(x)$$
$$(c \text{ constant}, c \notin \Gamma, c^* \text{ is a new function symbol of arity } 1)$$
$$\psi^x_\Gamma(c) \equiv c \quad (c \text{ constant}, c \in \Gamma)$$
$$\psi^x_\Gamma(f(t_1, \ldots, t_r)) \equiv f^*(\psi^x_\Gamma(t_1), \ldots, \psi^x_\Gamma(t_r), x)$$
$$(f \notin \Gamma, f^* \text{ a new function symbol of arity } r+1)$$
$$\psi^x_\Gamma(f(t_1, \ldots, t_r)) \equiv f(\psi^x_\Gamma(t_1), \ldots, \psi^x_\Gamma(t_r)) \quad (f \in \Gamma)$$
$$\psi^x_\Gamma(P(v_1, \ldots, v_n)) \equiv P^*(\psi^x_\Gamma(v_1), \ldots, \psi^x_\Gamma(v_n), x)$$
$$(P^* \text{ a new predicate symbol of arity } n+1)$$
$$\psi^x_\Gamma(\neg A) \equiv \neg \psi^x_\Gamma(A)$$
$$\psi^x_\Gamma(A \vee B) \equiv \psi^x_\Gamma(A) \vee \psi^x_\Gamma(B)$$
$$\psi^x_\Gamma(A \wedge B) \equiv \psi^x_\Gamma(A) \wedge \psi^x_\Gamma(B)$$
$$\psi^x_\Gamma(A \supset B) \equiv \psi^x_\Gamma(A) \supset \psi^x_\Gamma(B)$$
$$\psi^x_\Gamma(\exists y A) \equiv \exists y \psi^x_\Gamma(A)$$
$$\psi^x_\Gamma(\forall y A) \equiv \forall y \psi^x_\Gamma(A).$$

$$\mathcal{C}^\Gamma_p(T) \equiv \{\mathrm{cl}_\forall(\psi^x_\Gamma(B)) \mid B \in T\}$$
$$\mathcal{C}^\Gamma_c(\exists y A(y)) \equiv \exists x \psi^x_\Gamma(A(x)).$$ [3]

Example 1. Consider $P(0) \vee P(1)$ and $\exists y P(y)$.

$$\Gamma = \{0, 1\}$$
$$\psi^x_\Gamma(P(0) \vee P(1)) \equiv P^*(0, x) \vee P^*(1, x) \qquad \psi^x_\Gamma(\exists y P(y)) \equiv \exists y P^*(y, x)$$
$$\mathcal{C}^\Gamma_p(P(0) \vee P(1)) \equiv \forall x (P^*(0, x) \vee P^*(1, x))$$
$$\mathcal{C}^\Gamma_c(\exists y P(y)) \equiv \exists x P^*(x, x)$$

$$\Gamma = \{1\}$$
$$\psi^x_\Gamma(P(0) \vee P(1)) \equiv P^*(0^*(x), x) \vee P^*(1, x) \qquad \psi^x_\Gamma(\exists y P(y)) \equiv \exists y P^*(y, x)$$
$$\mathcal{C}^\Gamma_p(P(0) \vee P(1)) \equiv \forall x (P^*(0^*(x), x) \vee P^*(1, x))$$
$$\mathcal{C}^\Gamma_c(\exists y P(y)) \equiv \exists x P^*(x, x)$$

[2] Γ is a signature for the specification of terms which are admitted as witnesses.

[3] Note that by duality the translation $\mathcal{C}^\Gamma_p / \mathcal{C}^\Gamma_c$ can be used to control resources, i.e. to a priori limit the number of instances of universal axioms to be used in the proof.

Theorem 1. *$T \vdash A(t)$ for some closed term of the signature of Γ iff $\mathcal{C}_p^\Gamma(T) \vdash \mathcal{C}_c^\Gamma(\exists x A(x))$.*

Proof. See [1] or [2], for a more general setting ($\mathcal{C}_p^\Gamma/\mathcal{C}_c^\Gamma$ correspond to $\mathcal{C}_p^\chi$ / $\mathcal{C}_c^\chi$ with $\chi = \left\langle \begin{matrix} 1 \\ \Gamma \end{matrix} \right\rangle$). The proof uses the following property of resolution refutations. For sets of clauses $\{(\neg)L_{i1}(\bar{y}_{i1}, x), \ldots, (\neg)L_{ik_i}(\bar{y}_{ik_i}, x)\}$ ground substitutions of refutations must always coincide at the x-position: x stores the term t making the argument constructive. Function symbols not in Γ cannot occur in t as their translation depends on x.

Example 2. $P(0) \vee P(1)$ does not prove $\exists x P(x)$ constructively as

$$\mathcal{C}_p^{\{0,1\}}(P(0) \vee P(1)) \equiv \forall x(P^*(0, x) \vee P^*(1, x)),$$
$$\mathcal{C}_c^{\{0,1\}}(\exists x P(x)) \equiv \exists x P^*(x, x),$$
and $\forall x(P^*(0, x) \vee P^*(1, x)) \nvdash \exists x P^*(x, x)$.

Example 3. $P(0), P(0) \vee P(1)$ do prove $\exists x P(x)$ constructively. This is not the case if the signature is restricted to 1 as

$$\mathcal{C}_p^{\{1\}}(P(0) \vee P(1)) \equiv \forall x(P^*(0^x(x), x) \vee P^*(1, x)),$$
$$\mathcal{C}_p^{\{1\}}(P(0)) \equiv \forall x P^*(0^x(x), x),$$
$$\mathcal{C}_c^{\{1\}}(\exists x P(x)) \equiv \exists x P^*(x, x),$$
and $\forall x(P^*(0^x(x), x)), \forall x(P^*(0^x(x), x) \vee P^*(1, x)) \nvdash \exists x P^*(x, x)$.

Example 4. $\exists x P(x)$ does not prove $\exists x P(x)$ constructively ($\Gamma = \{0\}$) as

$$\mathcal{C}_c^{\{0\}}(\exists x P(x)) \equiv \forall x \exists y P^*(x, y),$$
$$\mathcal{C}_c^{\{1\}}(\exists x P(x)) \equiv \exists x P^*(x, x),$$
and $\forall x \exists y P^*(x, y) \nvdash \exists x P^*(x, x)$.

Example 5. We use a well known example of non-constructivity (cf. [3]). There are irrational numbers a, b such that a^b is rational: consider $\sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is rational let $a = b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, where

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2.$$

We formalize this argument using the predicate $R(x)$, the constant $\sqrt{2}$, and the function $\exp(x, y)$ (also written as x^y).

Let $T = \neg R(\sqrt{2}), R\left((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}\right)$ and $\Gamma = \left\{\sqrt{2}, \exp(x, y)\right\}$.

$$T \vdash \exists x \exists y(\neg R(x) \wedge \neg R(y) \wedge R(x^y))$$

$$\mathcal{C}_p^\Gamma(T) = \forall x \neg R^*(\sqrt{2}, x), \forall x R^* \left((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}, x \right)$$

$\mathcal{C}_c^\Gamma(\exists x \exists y(\neg R(x) \wedge \neg R(y) \wedge R(x^y))) = \exists x \exists y(\neg R^*(x,x) \wedge \neg R^*(y,x) \wedge R^*(x^y, x))$.
The following structure $\langle M, \bar{R}^*, \exp, \sqrt{2}\rangle$ is a model for

$$\forall x \neg R^*(\sqrt{2}, x), \forall x R^* \left((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}, x \right), \forall x \forall y (R^*(x,x) \vee R^*(y,x) \vee \neg R^*(x^y, x))$$

and consequently a counterexample to

$$\mathcal{C}_p^\Gamma(T) \vdash \mathcal{C}_c^\Gamma(\exists x \exists y(\neg R^*(x,x) \wedge \neg R^*(y,x) \wedge R^*(x^y, x))).$$

M is the set of terms constructed from $\sqrt{2}$ and $\exp(x,y)$,
$(\sqrt{2}, v) \notin \bar{R}^*$ for all v,
$(\sqrt{2}^{\sqrt{2}}, \sqrt{2}) \notin \bar{R}^*$,
$(u, v) \in \bar{R}^*$ otherwise.

Therefore there is no term t such that

$$T \vdash \exists y(\neg R(t) \wedge \neg R(y) \wedge R(t^y)).$$

3 Conditional Constructivity

Let $H_\Gamma(\exists y A(y)) \equiv \exists x \forall y(\psi_\Gamma^x(A(y)) \supset \psi_\Gamma^x(A(x)))$.

Example 6. $H_\Gamma(\exists x P(x)) \equiv \exists x \forall y(P^*(y,x) \supset P^*(x,x))$.

Proposition 1. *$T \vdash A \Leftrightarrow \{\forall x \psi_\Gamma^x(B) \mid B \in T\} \vdash \forall x \psi_\Gamma^x(A)$.*

Proof. By induction on the proof length.

Theorem 2.

(i) $T \vdash A(t)$ for some closed term of the signature of $\Gamma \Rightarrow \mathcal{C}_p^\Gamma(T) \vdash H_\Gamma(\exists x\, A(x))$.
(ii) $\mathcal{C}_p^\Gamma(T) \vdash H_\Gamma(\exists x A(x))$, $T \subseteq S$, $S \vdash \exists x A(x) \Rightarrow S \vdash A(t)$ for some closed term of the signature of Γ.

Proof.

(i) $T \vdash A(t)$ for some closed term of the signature of $\Gamma \Rightarrow \mathcal{C}_p^\Gamma(T) \vdash \exists x\, \psi_\Gamma^x(A(x))$ by Theorem 1. $\Rightarrow \mathcal{C}_p^\Gamma(T) \vdash \exists x \forall y(\psi_\Gamma^x(A(y)) \supset \psi_\Gamma^x(A(x))) \equiv H_\Gamma(\exists y A(y))$.
(ii) $\mathcal{C}_p^\Gamma(T) \vdash H_\Gamma(\exists y A(y))$, $T \subseteq S$, $S \vdash \exists x A(x) \Rightarrow \mathcal{C}_p^\Gamma(S) \vdash H_\Gamma(\exists y A(y))$, $\mathcal{C}_p^\Gamma(S) \vdash \forall x \exists y \psi_\Gamma^x(A(y)) \Rightarrow \mathcal{C}_p^\Gamma(S) \vdash \exists x \psi_\Gamma^x(A(x))$ as $\vdash H_\Gamma(\exists y A(y)) \supset (\forall x \exists y \psi_\Gamma^x(A(y))) \supset \exists x \psi_\Gamma^x(A(x))$ and $\mathcal{C}_p^\Gamma(S) \vdash \forall x\ \exists y\ \psi_\Gamma^x(A(y))$ by Proposition 1 $\Rightarrow S \vdash A(t)$ for some closed term t in the signature Γ by Theorem 1.
Note that t is already "known" to T as $H_\Gamma(\exists y A(y)) \equiv \mathcal{C}_c^\Gamma(\exists x \forall y(A(y) \supset A(x)))$ ($\vdash \exists x \forall y(A(y) \supset A(x))$!).

Example 7. $P(0) \vee P(1)$, $\exists x P(x) \supset P(0)$ prove the conditional constructivity for $\exists y P(y)$, prove $\exists x P(x)$, but do not prove $\exists x P(x)$ constructively.

$$\mathcal{C}_p^{\{0,1\}}(P(0) \vee P(1)) \equiv \forall x (P^*(0,x) \vee P^*(1,x))$$
$$\mathcal{C}_p^{\{0,1\}}(\exists x P(x) \supset P(0)) \equiv \forall x (\exists y P^*(y,x) \supset P^*(0,x))$$
$$H_{\{0,1\}}(\exists y P(y)) \equiv \exists x \forall y (P^*(y,x) \supset P^*(x,x)).$$

Obviously

$$\mathcal{C}_p^{\{0,1\}}(P(0) \vee P(1)), \mathcal{C}_p^{\{0,1\}}(\exists x P(x) \supset P(0)) \vdash H_{\{0,1\}}(\exists y P(y)).$$

4 Conclusion

The establishment of conditional (relative) constructivity is an essential feature of constructive mathematics, as relative constructivity proofs allow for the combination of constructive parts of non-constructive proofs with suitable constructive specializations. The translation presented in this paper makes it possible to use classical models to analyze, why relative constructivity fails.

References

1. Baaz, M.: Note on a translation to characterize constructivity. J. Proc. Steklov Inst. Math. 242, 125–129 (2003)
2. Baaz, M.: Controlling witnesses. Annals of Pure and Applied Logic 136, 22–29 (2005)
3. Troelstra, A., van Dalen, D.: Constructivism in Mathematics, An Introduction, vol. 1. North-Holland, Amsterdam (1988)