Realization Theorems for Justification Logics: Full Modularity*

Annemarie Borg and Roman Kuznets**

Institut für Computersprachen, Technische Universität Wien, Austria
annem.borg@gmail.com, roman@logic.at

Abstract. Justification logics were introduced by Artemov in 1995 to provide intuitionistic logic with a classical provability semantics, a problem originally posed by Gödel. Justification logics are refinements of modal logics and formally connected to them by so-called realization theorems. A constructive proof of a realization theorem typically relies on a cut-free sequent-style proof system for the corresponding modal logic. A uniform realization theorem for all the modal logics of the so-called modal cube, i.e., for the extensions of the basic modal logic K with any subset of the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$, has been proven using nested sequents. However, the proof was not modular in that some realization theorems required postprocessing in the form of translation on the justification logic side. This translation relied on additional restrictions on the language of the justification logic in question, thus, narrowing the scope of realization theorems. We present a fully modular proof of the realization theorems for the modal cube that is based on modular nested sequents introduced by Marin and Straßburger.

1 Introduction

Justification logics can be seen as explicit counterparts of modal logics that replace one modality $\Box$, understood as *provable* or *known*, etc., by a family of justification terms representing the underlying reason for the provability or knowledge, etc. respectively. The formal connection between a modal logic and a justification logic is provided by a *realization theorem*, showing that each occurrence of modality in a valid modal formula can be *realized* by some justification term in such a way that the resulting justification formula is valid, and vice versa.

The first justification logic, the Logic of Proofs LP, was introduced by Artemov [Art01] as a solution to Gödel's problem of providing intuitionistic logic

---

* This is a preprint version of a paper to appear in LNAI, vol. 9323. It is different from the publisher's version in two ways: 1) it contains the Appendix with proofs omitted from the Proceedings of Tableaux 2015 due to space limitations; 2) it implements the author-year citation style, as requested by one of the reviewers and preferred by the authors but as prohibited by the LNAI instructions.

** Supported at different stages of research by Austrian Science Fund (FWF) grants: Lise Meitner M 1770-N25 and START Y 544-N23.

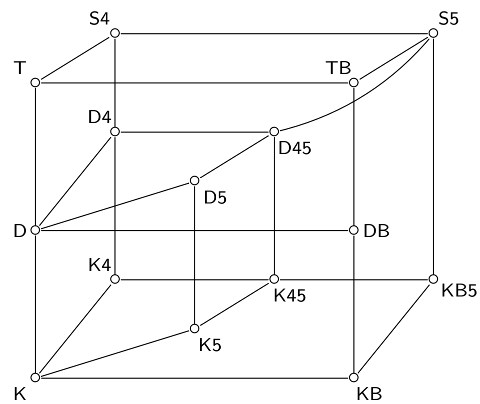

Fig. 1. *Modal cube*

with classical provability semantics. Artemov proved a realization theorem connecting LP with the modal logic of informal provability S4 by means of a cut-free sequent system for S4.

Justification language enables one to study whether self-referential proofs of the type $t : A(t)$ are implicitly present in a particular kind of modality [Kuz10] or in intuitionistic reasoning [Yu14]. Justification logic has also been used in the epistemic setting to provide the missing formal treatment of *justified* in Plato's celebrated definition of knowledge as *justified true belief*. In particular, justification language enables one to analyze in the object language, i.e., on the logical rather than metalogical level, famous epistemic paradoxes such as Gettier examples showing the deficiencies of Plato's definition of knowledge (see an extended discussion of this and other examples [AF12]).

For logics lacking a cut-free sequent calculus, constructive proofs of realization theorems can be achieved by using more complex sequent-style formalisms, e.g., hypersequents and nested sequents. In this paper, we focus on realizations of the 15 modal logics from the *modal cube*, visualized in Fig. 1 (see [Gar14] for a detailed explanation of this diagram), i.e., for all extensions of the basic normal modal logic K with any subset of the axioms d, t, b, 4, and 5. Realization theorems for several logics weaker than S4, including the realization of K into the basic justification logic J, was achieved by Brezhnev [Bre00] using appropriate sequent calculi. The strongest logic in the cube, S5, which lacks a cut-free sequent representation, was realized by Artemov et al. [AKS99] using Mints's cut-free hypersequent calculus from [Min92]. However, several logics from the cube lack even a cut-free hypersequent representation, which prompted Goetschi and Kuznets [GK12] to use cut-free nested sequent calculi introduced by Brünnler [Brü09] to prove realization for all these 15 modal logics in a uniform way.

Unfortunately, this uniform realization method did not provide a way of realizing individual modal principles independently of each other. While for each of the modal axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$, there is the corresponding justification axiom and the corresponding nested sequent rule, there are subsets X of these axioms such that Brünnler's nested calculus formed from the rules corresponding to the axioms from X is not complete for the logic $\mathsf{K} + X$ and, hence, cannot be used to prove the realization theorem for $\mathsf{K} + X$. These remaining realization theorems were proved by Goetschi and Kuznets by using additional "postprocessing": namely, by translating operations between justification logics [GK12]. Thus, their realization method lacks the desired modularity and also requires to partition the set of justification constants into countably many strata, an additional level of complexity one might wish to avoid.

In this paper, we provide a modular and uniform proof of the realization theorem for all axiomatizations occurring in the modal cube. Our proof makes a crucial use of the modular nested sequent calculi by Marin and Straßburger [MS14], which are complete for each subset X of the five modal axioms. Thereby, no additional restrictions on the justification language are necessary.

The paper is structured as follows. Section 2 recalls the modal logics of the modal cube and justification logics realizing them. Section 3 gives a formal definition of realization. Section 4 introduces the modular nested sequent calculi from [MS14]. Section 5 supplies notions and auxiliary lemmas used in the proof of the modular realization theorem, which is presented in Sect. 6.

2 Modal Logic and Justification Logic

2.1 Modal Logic

Modal logic extends propositional logic by modal operators called *modalities*. Let Prop be a countable set of propositional variables. We use the modal language in the negation normal form, with negation is restricted to propositional variables:

$$A ::= p \mid \overline{p} \mid \bot \mid \top \mid (A \vee A) \mid (A \wedge A) \mid \Box A \mid \Diamond A \ ,$$

where $p \in \mathsf{Prop}$.[1] The negation operation $\overline{A}$ is extended from propositional variables to all formulas by using De Morgan dualities and double negation elimination. $A \supset B$ is defined as $\overline{A} \vee B$ and, by default, is right-associative, as far as the usual omission of parentheses is concerned.

Definition 1 (Axiom systems for modal logics). *The* axiom system for the basic modal logic K *is obtained from that for classical propositional logic by*

[1] The use of negation normal form here is inherited from [MS14]. Such calculi can be easily modified to work with not-atomic negations, see, e.g., [FK15], but there is a price to pay. Either one loses (the naive formulation of) the subformula property or the underlying sequents need to be two-sided as, e.g., in [Fit14].

adding the normality axiom k *and the necessitation rule* nec*:*

$$\mathsf{k}: \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \ , \qquad \mathsf{nec}: \quad \frac{\vdash A}{\vdash \Box A} \ .$$[2]

The axiom systems for modal logics of the modal cube *are obtained by adding to the axiom system for* K *a subset of the following axioms:*

$$\mathsf{d}: \quad \Box\bot \supset \bot \ , \qquad \mathsf{t}: \quad \Box A \supset A \ , \qquad \mathsf{b}: \quad \neg A \supset \Box\neg\Box A \ ,$$
$$\mathsf{4}: \quad \Box A \supset \Box\Box A \ , \qquad \mathsf{5}: \quad \neg\Box A \supset \Box\neg\Box A \ .$$

All the 15 modal logics of the modal cube are depicted in Fig. 1, see also [Gar14]. Since 5 axioms produce 32 possible axiomatizations, some axiomatizations define the same logic. The name of the logic is typically derived from one of its axiomatizations, with the exception of the logic S4, axiomatized, e.g., by t and 4, and the logic S5, obtained from S4 by adding the axiom 5. We denote an arbitrary logic from the modal cube by ML.

2.2 Justification Logic

Instead of the modality $\Box$, justification logic employs a family of justification terms built from justification constants $d_0, d_1, \dots$ and justification variables $x_0, x_1, \dots$ by means of several operations according to the following grammar:

$$t ::= x_i \mid d_i \mid (t \cdot t) \mid (t + t) \mid\, !t \mid ?t \mid \bar{?}t \ .$$

The language of justification logic is defined by the following grammar

$$A ::= p \mid \bot \mid (A \supset A) \mid t : A \ ,$$

where $p \in \mathsf{Prop}$. Formulas $t : A$ are read "term t justifies formula A."

Definition 2 (Axiom systems for justification logics). *The* axiom system for the basic justification logic J *is obtained from that for classical propositional logic by adding the axioms* app *and* sum *and the axiom necessitation rule* AN*:*

$$\mathsf{sum}: \quad s : A \supset (s+t) : A \ , \qquad t : A \supset (s+t) : A \ ,$$
$$\mathsf{app}: \quad s : (A \supset B) \supset (t : A \supset (s \cdot t) : B) \ , \qquad \mathsf{AN}: \frac{A \textit{ is an axiom}}{c_n : \ldots : c_1 : A} \ .$$

The axiom systems for justification logics realizing modal logics of the modal cube *are formed by adding to the axiom system for* J *a subset of the following axioms:*

$$\mathsf{jd}: \quad t : \bot \supset \bot \ , \qquad \mathsf{jt}: \quad t : A \supset A \ , \qquad \mathsf{jb}: \quad A \supset \bar{?}t : \neg t : \neg A \ ,$$
$$\mathsf{j4}: \quad t : A \supset\, !t : t : A, \qquad \mathsf{j5}: \quad \neg t : A \supset ?t : \neg t : A \ .$$

[2] Note that such a rule may not be sound if applied to derivations with *local* assumptions, i.e., with assumptions contingently true rather than universally true. Since our setting does not require distinguishing *global* and *local* assumptions (see [FM98, Sect. 3.3] for details), we restrict the necessitation rule to derivations without assumptions. In particular, this guarantees the validity of the Deduction Theorem [HN12].

The intended meaning of the operations $\cdot$, $+$, $!$, $?$, and $\overline{?}$ can be read off these axioms. For instance, $\cdot$ is the *application* known from λ-calculus and combinatory logic and $+$ can be viewed as the monotone concatenation of proofs.

For each combination of axioms added to the axiom system for J, the name of the corresponding justification logic is formed by writing the capitalized axiom names and dropping all letters J except for the first one, e.g., the axiom system for $\mathsf{JT4}$ is that of J with the addition of the axioms jt and $\mathsf{j4}$ (this logic is better known as the Logic of Proofs LP). Note that the axiom jt subsumes the axiom jd, meaning that there are only 24 instead of 32 logics obtained this way. We denote any of these 24 logics by JL.

3 Realization Theorems

In this section, we define the formal connection between modal and justification logics by means of *realization theorems*. Intuitively, a realization theorem states that, for a given modal logic ML and justification logic JL, each valid fact about justifications in JL corresponds to a valid fact about modalities in ML and vice versa. In other words, JL describes the same kind of validity as ML but in the language refined with justification terms. Formally, the correspondence is formulated in terms of the *forgetful projection*.

Definition 3 (Forgetful projection). *The* forgetful projection $(\cdot)^\circ$ *is a function from the justification language to the modal language defined as follows:*

$$p^\circ := p \ , \qquad \bot^\circ := \bot \ , \qquad (B_1 \supset B_2)^\circ := B_1^\circ \supset B_2^\circ \ , \qquad (t : B)^\circ = \Box B^\circ \ .$$

The forgetful projection is extended to sets of justification formulas in the standard way, i.e., $X^\circ := \{A^\circ \mid A \in X\}$*.*

Definition 4 (Justification counterparts). *For a justification logic* JL *and a modal logic* ML*, we say that* JL *is a* justification counterpart *of* ML *if*

$$\mathsf{JL}^\circ = \mathsf{ML} \ .$$

We also say that ML *is the* forgetful projection *of* JL*, or that* JL realizes ML*, or that* JL *is a* realization *of* ML*.*

The first realization theorem was proved by Artemov [Art01]. He established that the Logic of Proofs LP is a justification counterpart of the modal logic $\mathsf{S4}$, known to be the modal description of intuitionistic provability.

Theorem 5 (Realization of $\mathsf{S4}$). $\mathsf{LP}^\circ = \mathsf{S4}$.

Example 6 ([Art01]). The theorem $\Box p \vee \Box q \supset \Box(\Box p \vee \Box q)$ of $\mathsf{S4}$ can be realized, for instance, by the theorem $x : p \vee y : q \supset (a \cdot !\, x + b \cdot !\, y) : (x : p \vee y : q)$ of LP. Note that this realization has an additional *normality* property: all negative occurrences of $\Box$ are realized by distinct justification variables. It is customary to prove realization theorems in the stronger formulation requiring that every modal theorem possess a *normal* realization derivable in the justification counterpart in question. In particular, Artemov proved Theorem 5 in this stronger formulation. All the realization results in this paper presuppose this stronger formulation, unless stated otherwise.

There are three main methods of proving realization theorems:
- syntactically by induction on a cut-free sequent-style derivation, see [Art01], [Bre00], and [Fit09] for sequents, [AKS99] for hypersequents, and [GK12] for nested sequents;
- semantically using the so-called Model Existence Property [Fit05], [Rub06], [Fit13];
- by embedding a modal logic into another logic with a known realization theorem and bringing the obtained realization to the requisite form by justification transformations [GK12], [BKS14].

The semantic method is slightly less preferable because it does not ordinarily provide a constructive realization procedure. Goetschi and Kuznets in [GK12] have proved the realization for the whole modal cube by combining the syntactic and embedding methods. Our goal in this paper is to achieve the same result by the syntactic method only and in a modular manner.

It should come as no surprise that a justification counterpart JL of a given modal logic ML is often built by justification axioms similar to the modal axioms of ML. Indeed, $\mathsf{S4} = \mathsf{K} + \mathsf{t} + \mathsf{4}$ and $\mathsf{LP} = \mathsf{J} + \mathsf{jt} + \mathsf{j4}$. This can, however, lead to a situation where different axiomatizations of the same modal logic correspond to axiomatizations of different justification logics: there are 24 justification logics corresponding to only 15 modal logics of the modal cube. Thus, while the forgetful projection of a justification logic is unique, a modal logic may have more than one justification counterpart. In the modal cube, the logic with the most axiomatizations and, hence, the most justification counterparts is $\mathsf{S5} = \mathsf{S4} + \mathsf{5}$: its justification counterparts include $\mathsf{JT45}$, $\mathsf{JT5}$, $\mathsf{JTB5}$, $\mathsf{JTB45}$, $\mathsf{JDB5}$, $\mathsf{JDB45}$, $\mathsf{JDB4}$, and $\mathsf{JTB4}$ (see [GK12]).

4 Modular Nested Sequent Calculi

To achieve a fully modular realization theorem, we are using slightly modified (see Remark 12) modular nested sequents by Marin and Straßburger from [MS14]. In this section, we give all the necessary definitions for and modifications of their formalism.

Definition 7 (Nested sequents). *A* nested sequent *is a sequence of formulas and brackets defined by the following grammar:*

$$\Gamma ::= \varepsilon \mid \Gamma, A \mid \Gamma, [\Gamma] \enspace ,$$

where ε is the empty sequence, and A is a modal formula.

The comma denotes sequence concatenation and plays the role of structural disjunction, whereas the brackets $[\cdot]$ are called *structural box*. From now on, by

$$\frac{}{\Gamma\{p,\overline{p}\}}\ \mathsf{id} \qquad \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}\ \vee \qquad \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}}\ \wedge$$

$$\frac{\Gamma\{A,A\}}{\Gamma\{A\}}\ \mathsf{ctr} \qquad \frac{\Gamma\{\Delta,\Sigma\}}{\Gamma\{\Sigma,\Delta\}}\ \mathsf{exch} \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}\ \mathsf{k}$$

Fig. 2. Nested sequent calculus NK for the modal logic K

sequent we mean a nested sequent. Sequents are denoted by uppercase Greek letters. Nested sequent calculi are an internal formalism, meaning that every sequent has a *formula interpretation*.

Definition 8 (Formula interpretation). *The* corresponding formula *of a sequent Γ, denoted by $\underline{\Gamma}$ is defined as follows:*

$$\underline{\varepsilon} := \bot; \quad \underline{\Gamma, A} := \begin{cases} \underline{\Gamma} \vee A & \textit{if } \Gamma \neq \varepsilon, \\ A & \textit{otherwise;} \end{cases} \quad \underline{\Gamma, [\Delta]} := \begin{cases} \underline{\Gamma} \vee \Box\underline{\Delta} & \textit{if } \Gamma \neq \varepsilon, \\ \Box\underline{\Delta} & \textit{otherwise.} \end{cases}$$

To describe the application of nested rules deeply inside a nested structure, the concept of *context* is used.

Definition 9 (Context). *A* context *is a sequent with the symbol* hole $\{\,\}$ *in place of one of the formulas. Formally,*

$$\Gamma\{\,\} ::= \Delta, \{\,\} \mid [\Gamma\{\,\}] \mid \Gamma\{\,\}, \Delta \ ,$$

where Δ is a sequent. A sequent Π can be inserted into a context $\Gamma\{\}$ by replacing the hole $\{\,\}$ in $\Gamma\{\,\}$ with Π. The result of such an insertion is denoted $\Gamma\{\Pi\}$.

Example 10. Let $\Gamma\{\,\} = \big[\{\,\}, [D]\big]$ and $\Pi = [F], B$. Then $\Gamma\{\Pi\} = \big[[F], B, [D]\big]$.

Definition 11 (Nested sequent calculi for the modal cube, [MS14]). *The rules of* Marin–Straßburger's modular nested sequent calculi *are divided into three groups. The rules of the calculus NK for the basic modal logic K can be found in Fig. 2. Additional rules used to obtain calculi for the remaining 14 logics of the modal cube are divided into logical rules in Fig. 3 and structural rules in Fig. 4.*

Remark 12. In [MS14], the calculus NK and its extensions are based on multisets. However, since sequence-based nested sequents are necessary to use the realization method from [GK12], we modify the system in the same way as Goetschi and Kuznets did in [GK12] with Brünnler's nested sequent calculi from [Brü09]: namely, we add the exchange rule exch. The only other modification compared to [MS14] is the use of the rules $\mathsf{5a}^{\Diamond}$, $\mathsf{5b}^{\Diamond}$, $\mathsf{5c}^{\Diamond}$, $\mathsf{5a}^{\Box}$, $\mathsf{5b}^{\Box}$, and $\mathsf{5c}^{\Box}$ instead of two more compact but non-local two-hole rules. The possibility to replace such a two-hole rule with three single-hole rules was first observed by Brünnler [Brü09].

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d}^{\Diamond} \qquad \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t}^{\Diamond} \qquad \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathsf{b}^{\Diamond} \qquad \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{4}^{\Diamond}$$

$$\frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathsf{5a}^{\Diamond} \qquad \frac{\Gamma\{[\Delta], [\Pi, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Pi]\}}\ \mathsf{5b}^{\Diamond} \qquad \frac{\Gamma\{[\Delta, [\Pi, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Pi]]\}}\ \mathsf{5c}^{\Diamond}$$

Fig. 3. Additional logical rules corresponding to the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$

$$\frac{\Gamma\{[\varepsilon]\}}{\Gamma\{\varepsilon\}}\ \mathsf{d}^{\Box} \qquad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\ \mathsf{t}^{\Box} \qquad \frac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{[\Sigma], \Delta\}}\ \mathsf{b}^{\Box} \qquad \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}}\ \mathsf{4}^{\Box}$$

$$\frac{\Gamma\{[\Pi, [\Delta]]\}}{\Gamma\{[\Pi], [\Delta]\}}\ \mathsf{5a}^{\Box} \qquad \frac{\Gamma\{[\Pi, [\Delta]], [\Sigma]\}}{\Gamma\{[\Pi], [[\Delta], \Sigma]\}}\ \mathsf{5b}^{\Box} \qquad \frac{\Gamma\{[\Pi, [\Delta], [\Sigma]]\}}{\Gamma\{[\Pi, [[\Delta], \Sigma]]\}}\ \mathsf{5c}^{\Box}$$

Fig. 4. Additional structural rules corresponding to the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$

Marin and Straßburger [MS14] showed that these calculi are complete with respect to the corresponding modal logics in a modular way:

Theorem 13 (Modular completeness of nested calculi). *For a set of axioms $X \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we denote by $X^{\Diamond}$ the set of corresponding nested rules: $X^{\Diamond} := \{\mathsf{r}^{\Diamond} \mid \mathsf{r} \in X\}$, where $\mathsf{5}^{\Diamond}$ abbreviates the set of three rules $\mathsf{5a}^{\Diamond}$, $\mathsf{5b}^{\Diamond}$, and $\mathsf{5c}^{\Diamond}$. The definition of $X^{\Box}$ is analogous. For any modal formula A,*

$$\mathsf{K} + X \vdash A \qquad \Longleftrightarrow \qquad \mathsf{NK} + X^{\Diamond} + X^{\Box} \vdash A \ .$$

5 Auxiliary Definitions and Lemmas

Unlike the realization method applied by Artemov to a sequent calculus for $\mathsf{S4}$, the method developed for nested sequents in [GK12] requires complex manipulations with the realizing terms, which necessitates careful bookkeeping and, hence, additional notation. In particular, all modalities, structural or otherwise, are *annotated* with integers, so that it becomes possible to refer to particular occurrences of modality and to record the realizing term for each occurrence.

5.1 Annotations

Definition 14 (Annotation, proper annotation). Annotated modal formulas *are defined in the same way as modal formulas, except that each occurrence of $\Box$ ($\Diamond$) must be annotated with an odd (even) natural number.*

$p^r := p$	$(\overline{p})^r := \neg p$	$(\Box_{2k-1} A)^r := r(2k-1) : A^r$
$\bot^r := \bot$	$\top^r := \top$	$(\Diamond_{2l} A)^r := \neg r(2l) : \neg A^r$
$(A \vee B)^r := A^r \vee B^r$	$(A \wedge B)^r := A^r \wedge B^r$	

Fig. 5. Realization of modal formulas

Annotated sequents (contexts) *are defined in the same way as sequents (contexts), except that annotated modal formulas are used instead of modal formulas and that each occurrence of the structural box must be annotated by an odd natural number. The* corresponding formula *of an annotated sequent is an annotated formula defined as in Definition 8, except for the last case, which now reads:*

$$\underline{\Sigma, [\Delta]_k} := \begin{cases} \underline{\Sigma} \vee \Box_k \underline{\Delta} & \textit{if } \Sigma \neq \varepsilon, \\ \Box_k \underline{\Delta} & \textit{otherwise.} \end{cases}$$

A formula, sequent, or context is called properly annotated *if no index occurs twice in it.*

If all indices are erased in an annotated formula A (sequent Δ, context $\Gamma\{\,\}$), the result A' (Δ', $\Gamma'\{\,\}$) is called its unannotated version *and, vice versa, we call A (Δ, $\Gamma\{\,\}$) an* annotated version *of A' (Δ', $\Gamma'\{\,\}$).*

The translation from modal to justification formulas is defined by means of *realization functions* that assign realizations to each occurrence of modalities. Proper realizations have to respect the skolemized structure of modal formulas.

Definition 15 (Pre-realization and realization functions). *A* pre-realization function *r is a partial function from natural numbers to justification terms. A pre-realization function r is called a* realization function *if $r(2l) = x_l$ whenever $r(2l)$ is defined. If r is defined on all indices occurring in a given annotated formula A, then r is called a* (pre-)realization function on *A.*

Definition 16. *The translation of an annotated formula A under a given pre-realization function r on A is defined by induction on the construction of A, as shown in Fig. 5.*

In our realization proof, we will use some additional notation:

Definition 17. *Let A be an annotated formula and r be a pre-realization function on A. We define*

$$\mathsf{vars}_\Diamond(A) := \{x_k \mid \Diamond_{2k} \textit{ occurs in } A\} \quad \textit{and} \quad r \restriction A := r \restriction \{i \mid i \textit{ occurs in } A\} \ .$$

Here $f \restriction S$ denotes the restriction of f to the set $S \cap \mathsf{dom}(f)$.

5.2 Substitutions

It is easy to see that, due to the schematic nature of their axioms, justification logics enjoy the Substitution Property: if $\Gamma(x,p) \vdash_{\mathsf{JL}} B(x,p)$ for some justification variable x and propositional variable p, then for any term t and any formula F we have $\Gamma(x/t,p/F) \vdash_{\mathsf{JL}} B(x/t,p/F)$. The realization method, however, requires a more precise notation for substitutions of terms for justification variables and uses some additional standard definitions from term rewriting.

Definition 18 (Substitution). *A* term substitution*, or simply a* substitution*, is a total mapping from justification variables to justification terms. It is extended to all terms in the standard way. For a justification formula A, we denote by $A\sigma$ the result of simultaneous replacement of each term t in A with $t\sigma$. The* domain $\mathsf{dom}(\sigma)$ *and the* variable range $\mathsf{vrange}(\sigma)$ *of σ are defined by*

$$\mathsf{dom}(\sigma) := \{x \mid \sigma(x) \neq x\} \ ,$$
$$\mathsf{vrange}(\sigma) := \big\{y \mid \big(\exists x \in \mathsf{dom}(\sigma)\big)\big(y \textit{ occurs in } \sigma(x)\big)\big\} \ .$$

Definition 19 (Compositions). *A substitution σ can be composed with another substitution σ' or with a pre-realization function r:*

$$(\sigma' \circ \sigma)(x) := \sigma(x)\sigma' \qquad \textit{and} \qquad (\sigma \circ r)(n) := r(n)\sigma \ .$$

Definition 20 (Substitution Residence). *A substitution σ is said to*
- live on *an annotated modal formula A if $\mathsf{dom}(\sigma) \subseteq \mathsf{vars}_{\Diamond}(A)$,*
- live away from *A if $\mathsf{dom}(\sigma) \cap \mathsf{vars}_{\Diamond}(A) = \varnothing$.*

The following lemma is an easy corollary of the given definitions (see also [GK12]):

Lemma 21. *If r is a realization function on an annotated formula A and if a substitution σ lives away from A, then $\sigma \circ (r \restriction A)$ is a realization function on A.*

5.3 Internalization

Since in justification logics the modal necessitation rule is replaced with the zero-premise axiom necessitation rule, which can be treated as an axiom, justification logics clearly enjoy the Deduction Theorem. One of the fundamental properties peculiar to justification logics is their ability to internalize their own proofs. Various aspects of this property are referred to as the *Lifting Lemma*, the *Constructive Necessitation*, or the *Internalization Property*. They are easily proved by induction on the derivation. We use the following form of this property.

Lemma 22 (Internalization). *If $\mathsf{JL} \vdash A_1 \supset \cdots \supset A_n \supset B$, then there is a term $t(x_1,\dots,x_n)$ such that*

$$\mathsf{JL} \vdash s_1 : A_1 \supset \cdots \supset s_n : A_n \supset t(s_1,\dots,s_n) : B$$

for any terms $s_1,\dots,s_n$. In particular, for $n = 0$, if $\mathsf{JL} \vdash B$, then there exists a ground[3] term t such that $\mathsf{JL} \vdash t : B$.

[3] Containing no justification variables.

5.4 Realizable Rules

We now lay the foundation for the realization method: we define what it means to realize one rule in a given cut-free nested derivation. The complexity of this definition is due mainly to the necessity to reconcile realizations of the premises of two-premise rules. The reconciliation mechanism is based on Fitting's merging technique from [Fit09]. However, most of the details are described in [GK12] and will not be repeated here.

Since realization relies on indices, we first need to define what it means to annotate nested sequent rules. In defining this, we exploit the fact that all the rules from Figs. 2–4 are *context-preserving*, i.e., all changes happen within the hole while the context, which can be arbitrary, remains unchanged.

Definition 23 (Annotated rules). *Consider an instance of a context-preserving nested rule with common context $\Gamma'\{\ \}$:*

$$\frac{\Gamma'\{\Lambda'_1\} \ \ldots\ \Gamma'\{\Lambda'_n\}}{\Gamma'\{\Lambda'\}} \ . \tag{1}$$

Its annotated version *has the form*

$$\frac{\Gamma\{\Lambda_1\} \ \ldots\ \Gamma\{\Lambda_n\}}{\Gamma\{\Lambda\}} \ , \tag{2}$$

where

- *the sequents $\Gamma\{\Lambda_1\}, \ldots, \Gamma\{\Lambda_n\}$, and $\Gamma\{\Lambda\}$ are properly annotated;*
- *$\Gamma\{\ \}, \Lambda_1, \ldots, \Lambda_n$, and Λ are annotated versions of $\Gamma'\{\ \}, \Lambda'_1, \ldots, \Lambda'_n$, and Λ' respectively; and*
- *no index occurs in both Λ_i and Λ_j for arbitrary $1 \leq i < j \leq n$.*

Realization functions on annotated formulas were defined in Definition 16. A realization function on an annotated nested sequent, as well as the properties of living on/away from an annotated nested sequent, are understood with respect to its corresponding formula from Definition 14. We are now ready to define what it means to realize one rule instance in a nested sequent derivation.

Definition 24 (Realizable rule). *An instance* (1) *of a context-preserving rule with common context $\Gamma'\{\ \}$ is called* realizable *in a justification logic* JL *if there exists such an annotated version* (2) *of it that, for arbitrary realization functions $r_1, \ldots, r_n$ on the premises $\Gamma\{\Lambda_1\}$, ..., $\Gamma\{\Lambda_n\}$ respectively, there exists a realization function r on the conclusion $\Gamma\{\Lambda\}$ and a substitution σ that lives on $\Gamma\{\Lambda_i\}$ for each $i = 1, \ldots, n$, such that*

$$\mathsf{JL} \vdash \Gamma\{\Lambda_1\}^{r_1}\sigma \supset \ldots \supset \Gamma\{\Lambda_n\}^{r_n}\sigma \supset \Gamma\{\Lambda\}^r \ .$$

In particular, for $n = 0$ it is sufficient that there be a realization function r on $\Gamma\{\Lambda\}$ such that $\mathsf{JL} \vdash \Gamma\{\Lambda\}^r$.

A rule is called realizable *in a justification logic* JL *if all its instances are realizable in* JL.

Goetschi and Kuznets in [GK12] showed that, in order to prove the realizability of a rule, it is sufficient to show the realizability of all its shallow instances.

Definition 25 (Shallow rule instance). *The* shallow version *of an instance* (1) *of a context-preserving rule is obtained by making the context empty:*

$$\frac{\Lambda'_1 \ \ldots \ \Lambda'_n}{\Lambda} \ .$$

The method of constructive realization based on nested sequents hinges on the following theorem:

Theorem 26 (Realization method, [GK12]). *If a modal logic* ML *is described by a cut-free nested sequent calculus* NML*, such that all rules of* NML *are context-preserving and all shallow instances of these rules are realizable in a justification logic* JL*, then there is a constructive realization of* ML *into* JL*.*

5.5 Auxiliary Lemmas

We use the following lemmas to shorten the proofs of realizability for several shallow rules.

Lemma 27 (Internalized Positive Introspection, [GK12]). *There exist justification terms* $t_!(x)$ *and* $\mathsf{pint}(x)$ *such that for any term* s *and any justification formula* A*:* $\quad \mathsf{J5} \vdash \mathsf{pint}(s) : (s : A \supset t_!(s) : s : A)$.

Lemma 28. *There is a term* $\mathsf{qm}(x)$ *such that for any justification formula* A *and any term* s*:* $\quad \mathsf{JB} \vdash \neg A \supset \mathsf{qm}(s) : \neg s : A$.

Proof. Consider the following derivation in JB:

0.	$p \quad \supset \quad \neg\neg p$	propositional tautology
1.	$x : p \quad \supset \quad t_1(x) : \neg\neg p$	from 0. by Lemma 22
2.	$\neg t_1(x) : \neg\neg p \quad \supset \quad \neg x : p$	from 1. by prop. reasoning
3.	$\bar{?}\, t_1(x) : \neg t_1(x) : \neg\neg p \quad \supset \quad t_2(\bar{?}\, t_1(x)) : \neg x : p$	from 2. by Lemma 22
4.	$\neg p \quad \supset \quad \bar{?}\, t_1(x) : \neg t_1(x) : \neg\neg p$	instance of jb
5.	$\neg p \quad \supset \quad t_2(\bar{?}\, t_1(x)) : \neg x : p$	from 3. and 4. by prop. reasoning

Let $\mathsf{qm}(x) := t_2(\bar{?}\, t_1(x))$. Note that $\mathsf{qm}(x)$ depends neither on s nor on A. By substitution, it follows that $\mathsf{JB} \vdash \neg A \supset \mathsf{qm}(s) : \neg s : A$. □

6 Modular Realization Theorem for the Modal Cube

In order to use the realization method from Theorem 26, we need to show that all shallow instances of all rules from Figs. 2–4 are realizable. For rules in Figs. 2–3, this has been proved in [GK12]. Thus, the main contribution of this paper, which makes the modular realization theorem possible is the proof of realizability for the rules in Fig. 4. Due to space limitations, we only provide the proofs for select representative cases.

Lemma 29 (Main lemma).

1. *Each shallow instance of* $\rho \in \{\mathsf{id}, \vee, \wedge, \mathsf{ctr}, \mathsf{exch}, \Box, \mathsf{k}\}$ *is realizable in* J.
2. *For each* $\rho \in \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$ *each shallow instance of* $\rho^{\Diamond}$ *and of* $\rho^{[]}$ *is realizable in* JP, *where* $\mathsf{J5A} = \mathsf{J5B} = \mathsf{J5C} := \mathsf{J5}$.

Proof. For each rule ρ, we consider its arbitrary shallow instance. Statement 2 for $\rho^{\Diamond}$ and Statement 1 have been proved in [GK12]. Statement 2 for $\rho^{[]}$ with $\rho \in \{\mathsf{d}, \mathsf{4}, \mathsf{5a}, \mathsf{5b}\}$ is left for the reader. We give proofs for the remaining 3 rules:

Case $\rho = \mathsf{t}^{[]}$: Let $\dfrac{[\Delta']}{\Delta'}$ be an arbitrary shallow instance of $\mathsf{t}^{[]}$. Let $[\Delta]_k$ be a properly annotated version of $[\Delta']$. Then Δ properly annotates Δ' and $\dfrac{[\Delta]_k}{\Delta}$ is an annotated version of this instance. Let r_1 be a realization function on $[\Delta]_k$. For $r := r_1$ and the identity substitution σ, we have

$$([\Delta]_k)^{r_1}\sigma \supset \Delta^r \quad = \quad r_1(k) : \Delta^{r_1} \supset \Delta^{r_1} \ ,$$

which is derivable in JT as an instance of jt.

Case $\rho = \mathsf{b}^{[]}$: Let $\dfrac{[\Sigma', [\Delta']]}{[\Sigma'], \Delta'}$ be an arbitrary shallow instance of $\mathsf{b}^{[]}$. Let $[\Sigma, [\Delta]_i]_k$ and $[\Sigma]_l, \Delta$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Sigma, [\Delta]_i]_k}{[\Sigma]_l, \Delta}$ is an annotated version of this instance. Let r_1 be a realization function on $[\Sigma, [\Delta]_i]_k$. The following is a derivation in JB:

0. $\neg\Delta^{r_1} \quad \supset \quad \mathsf{qm}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$ — by Lemma 28
1. $\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1} \quad \supset \quad \neg r_1(i) : \Delta^{r_1} \quad \supset \quad \Sigma^{r_1}$ — propositional tautology
2. $r_1(k) : \big(\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}\big) \quad \supset \quad \mathsf{qm}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \quad \supset$
   $t\big(r_1(k), \mathsf{qm}(r_1(i))\big) : \Sigma^{r_1}$ — from 1. by Lemma 22
3. $r_1(k) : \big(\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}\big) \quad \supset$ — from 2. by prop. reasoning
   $t\big(r_1(k), \mathsf{qm}(r_1(i))\big) : \Sigma^{r_1} \quad \vee \quad \neg\mathsf{qm}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$
4. $\neg\mathsf{qm}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \quad \supset \quad \Delta^{r_1}$ — from 0. by prop reasoning
5. $r_1(k) : \big(\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}\big) \quad \supset$ — from 3. and 4. by prop. reasoning
   $t\big(r_1(k), \mathsf{qm}(r_1(i))\big) : \Sigma^{r_1} \quad \vee \quad \Delta^{r_1}$

Thus, for $s := t\big(r_1(k), \mathsf{qm}(r_1(i))\big)$,

$$\mathsf{JB} \vdash \qquad r_1(k) : \big(\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}\big) \quad \supset \quad s : \Sigma^{r_1} \vee \Delta^{r_1} \ . \qquad (3)$$

The index l occurs neither in Σ nor in Δ because $[\Sigma]_l, \Delta$ is properly annotated. Hence, $r := (r_1 \restriction \Sigma, \Delta) \cup \{l \mapsto s\}$ is a realization on $[\Sigma]_l, \Delta$. For the identity substitution σ and this r, it follows from (3) that

$$\mathsf{JB} \vdash \qquad \big([\Sigma, [\Delta]_i]_k\big)^{r_1}\sigma \quad \supset \quad \big([\Sigma]_l, \Delta\big)^r \ .$$

Case ρ = $\mathsf{5c}^{[]}$: Let $\dfrac{[\Pi', [\Delta'], [\Sigma']]}{[\Pi', [[\Delta'], \Sigma']]}$ be an arbitrary shallow instance of $\mathsf{5c}^{[]}$. Let $\big[\Pi, [\Delta]_i, [\Sigma]_j\big]_h$ and $\big[\Pi, [[\Delta]_i, \Sigma]_k\big]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{\big[\Pi, [\Delta]_i, [\Sigma]_j\big]_h}{\big[\Pi, [[\Delta]_i, \Sigma]_k\big]_l}$ is an annotated version of this instance. Let r_1 be a realization function on $\big[\Pi, [\Delta]_i, [\Sigma]_j\big]_h$. The following is a derivation in $\mathsf{J5}$:

0. $\Sigma^{r_1} \quad\supset\quad r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — propositional tautology
1. $r_1(j) : \Sigma^{r_1} \quad\supset\quad t_1(r_1(j)) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)$ — from 0. by Lemma 22
2. $\mathsf{pint}(r_1(i)) : \Big(r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1}\Big)$ — by Lemma 27
3. $r_1(i) : \Delta^{r_1} \quad\supset\quad r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — propositional tautology
4. $t_!(r_1(i)) : r_1(i) : \Delta^{r_1} \quad\supset$ $t_4(t_!(r_1(i))) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)$ — from 3. by Lemma 22
5. $\Big(r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1}\Big) \quad\supset$ $r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)$ — from 4. by prop. reasoning
6. $\mathsf{pint}(r_1(i)) : \Big(r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1}\Big) \quad\supset$ $t_5(\mathsf{pint}(r_1(i))) : \Big(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big)$ — from 5. by Lem. 22
7. $t_5(\mathsf{pint}(r_1(i))) : \Big(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big)$ — from 2. and 6. by MP
8. $\Big(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big) \quad\supset$ $\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \quad\supset\quad \Pi^{r_1} \vee s : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)$, where $s := t_4(t_!(r_1(i))) + t_1(r_1(j))$ — from 1. by prop. reasoning and sum
9. $t_5(\mathsf{pint}(r_1(i))) : \Big(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})\Big) \quad\supset$ $t_6(t_5(\mathsf{pint}(r_1(i)))) : \Big(\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset \Pi^{r_1} \vee s : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big)$ — from 8. by Lemma 22
10. $t_6(t_5(\mathsf{pint}(r_1(i)))) : \Big(\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset \Pi^{r_1} \vee s : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big)$ — from 7. and 9. by MP
11. $r_1(h) : \Big(\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1}\Big) \quad\supset\quad t : \Big(\Pi^{r_1} \vee s : \big(r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}\big)\Big)$, where $t := t_6\big(t_5(\mathsf{pint}(r_1(i)))\big) \cdot r_1(h)$ — from 10. by app and MP

The indices k and l do not occur in any of Π, $[\Delta]_i$, or Σ because $\big[\Pi, [[\Delta]_i, \Sigma]_k\big]_l$ is properly annotated. Thus, $r := (r_1 \restriction \Pi, [\Delta]_i, \Sigma) \cup \{k \mapsto s, l \mapsto t\}$ is a realization on $\big[\Pi, [[\Delta]_i, \Sigma]_k\big]_l$. For the identity substitution σ and this r, from 11. it follows

$$\mathsf{J5} \vdash \qquad \Big(\big[\Pi, [\Delta]_i, [\Sigma]_j\big]_h\Big)^{r_1}\sigma \quad\supset\quad \Big(\big[\Pi, [[\Delta]_i, \Sigma]_k\big]_l\Big)^{r} \ .$$

This concludes the proof that all shallow instances of each rule used for a modular nested sequent calculus for the logics of the modal cube are realizable into the justification logic containing the justification axiom corresponding to this rule. ◻

It now follows from Theorem 13, Theorem 26, and Lemma 29 that

Theorem 30 (Modular realization theorem). *For each possible axiomatization* $\mathsf{K}+X$ *of a modal logic* ML *from the modal cube, there is a constructive realization of* ML *into* $\mathsf{J}+\mathsf{j}X$ *using the nested sequent calculus* $\mathsf{NK}\cup X^{\Diamond}\cup X^{\Box}$ *for* ML*, where* $\mathsf{j}X := \{\mathsf{j}\rho \mid \rho \in X\}$*.*

Proof. Let $\mathsf{K}+X$ be an axiomatization of ML. By Theorem 13, $\mathsf{NK}\cup X^{\Diamond}\cup X^{\Box} \vdash A$ for each theorem A of ML. It was shown in Lemma 29 that all shallow instances of each rule from $\mathsf{NK}\cup X^{\Diamond}\cup X^{\Box}$ are realizable in a sublogic of $\mathsf{J}+\mathsf{j}X$ and, thus, also in $\mathsf{J}+\mathsf{j}X$ itself. By Theorem 26, there is a constructive realization of ML into $\mathsf{J}+\mathsf{j}X$. ◻

7 Conclusion

This paper completes the project of finding a uniform, modular, and constructive realization method for a wide range of modal logics. In this paper, we applied it to all the logics of the modal cube and all the justification counterparts based on their various axiomatizations. We are now confident that this method can be easily extended to other classical modal logics captured by nested sequent calculi. The natural challenge is to extend this method to the nested sequent calculi for intuitionistic modal logics from [MS14] and for constructive modal logics from [ADS15]. The size of LP-terms constructed for realizing $\mathsf{S4}$ by using sequent calculi was analyzed in [BK06]. It would be interesting to compare the size of terms produced by using nested sequent calculi.

Acknowledgments. The authors would like to thank the anonymous reviewers for the valuable comments and suggestions on clarifying issues of potential interest to the readers. The authors are indebted to Agata Ciabattoni for making this research possible.

References

[ADS15] Ryuta Arisaka, Anupam Das, and Lutz Straßburger. On nested sequents for constructive modal logics. E-print 1505.06896, arXiv, May 2015.

[AF12] Sergei [N.] Artemov and Melvin Fitting. Justification logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Fall 2012 edition, 2012.

[AKS99] Sergei N. Artemov, E. L. Kazakov, and D. Shapiro. Logic of knowledge with justifications. Technical Report CFIS 99–12, Cornell University, 1999.

[Art01] Sergei N. Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, March 2001.

[BK06] Vladimir N. Brezhnev and Roman Kuznets. Making knowledge explicit: How hard it is. *Theoretical Computer Science*, 357(1–3):23–34, July 2006.

[BKS14] Samuel Bucheli, Roman Kuznets, and Thomas Studer. Realizing public announcements by justifications. *Journal of Computer and System Sciences*, 80(6):1046–1066, September 2014.

[Bre00] Vladimir N. Brezhnev. On explicit counterparts of modal logics. Technical Report CFIS 2000–05, Cornell University, 2000.

[Brü09] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, July 2009.

[Fit05] Melvin Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, February 2005.

[Fit09] Melvin Fitting. Realizations and LP. *Annals of Pure and Applied Logic*, 161(3):368–387, December 2009.

[Fit13] Melvin Fitting. Realization using the model existence theorem. *Journal of Logic and Computation*, Advance Access, July 2013.

[Fit14] Melvin Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014.

[FK15] Melvin Fitting and Roman Kuznets. Modal interpolation via nested sequents. *Annals of Pure and Applied Logic*, 166(3):274–305, March 2015.

[FM98] Melvin Fitting and Richard L. Mendelsohn. *First-Order Modal Logic*, volume 277 of *Synthese Library*. Kluwer Academic Publishers, 1998.

[Gar14] James Garson. Modal logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Summer 2014 edition, 2014.

[GK12] Remo Goetschi and Roman Kuznets. Realization for justification logics via nested sequents: Modularity through embedding. *Annals of Pure and Applied Logic*, 163(9):1271–1298, September 2012.

[HN12] Raul Hakli and Sara Negri. Does the deduction theorem fail for modal logic? *Synthese*, 187(3):849–867, August 2012.

[Kuz10] Roman Kuznets. Self-referential justifications in epistemic logic. *Theory of Computing Systems*, 46(4):636–661, May 2010.

[Min92] Grigorii E. Mints. Lewis' systems and system T (1965–1973). In *Selected Papers in Proof Theory*, volume 3 of *Studies in Proof Theory*, pages 221–294. Bibliopolis and North-Holland, 1992.

[MS14] Sonia Marin and Lutz Straßburger. Label-free modular systems for classical and intuitionistic modal logics. In *Advances in Modal Logic, Volume 10*, pages 387–406. College Publications, 2014.

[Rub06] Natalia [M.] Rubtsova. On realization of S5-modality by evidence terms. *Journal of Logic and Computation*, 16(5):671–684, October 2006.

[Yu14] Junhua Yu. Self-referentiality of Brouwer–Heyting–Kolmogorov semantics. *Annals of Pure and Applied Logic*, 165(1):371–388, January 2014.

A Remaining Proofs Lemma 29

Proof. **Case** $\rho = \mathsf{d}^{[]}$: Let $\dfrac{[\varepsilon]}{\varepsilon}$ be an arbitrary shallow instance of $\mathsf{d}^{[]}$. Let $[\varepsilon]_k$ be a properly annotated version of the premise. Then $\dfrac{[\varepsilon]_k}{\varepsilon}$ is an annotated version of this instance. Let r_1 be a realization function on $[\varepsilon]_k$. Note that $\varepsilon^{r^*} = \bot$ for any realization r^*. For $r := r_1$ and the identity substitution σ, we have $\mathsf{JD} \vdash ([\varepsilon]_k)^{r_1}\sigma \supset (\varepsilon)^r$ as an instance of jd.

Case $\rho = 4^{[]}$: Let $\dfrac{[\Delta'], [\Sigma']}{[[\Delta'], \Sigma']}$ be an arbitrary shallow instance of $4^{[]}$. Let $[\Delta]_i, [\Sigma]_k$ and $[[\Delta]_i, \Sigma]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Delta]_i, [\Sigma]_k}{[[\Delta]_i, \Sigma]_l}$ is an annotated version of this instance. Let r_1 be a realization function on $[\Delta]_i, [\Sigma]_k$. The following is a derivation in $\mathsf{J4}$:

0. $r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — Propositional tautology
1. $!\,r_1(i) : r_1(i) : \Delta^{r_1} \supset t_1(!\,r_1(i)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 0. by Lemma 22
2. $r_1(i) : \Delta^{r_1} \supset !\,r_1(i) : r_1(i) : \Delta^{r_1}$ — Instance of $\mathsf{j4}$
3. $r_1(i) : \Delta^{r_1} \supset t_1(!\,r_1(i)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 1. and 2. by prop. reason.
4. $\Sigma^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — Propositional tautology
5. $r_1(k) : \Sigma^{r_1} \supset t_2(r_1(k)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 4. by Lemma 22
6. $r_1(i) : \Delta^{r_1} \vee r_1(k) : \Sigma^{r_1} \supset (t_1(!\,r_1(i)) + t_2(r_1(k))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 3. and 5., using sum and prop. reasoning

Thus, for $t := (t_1(!\,r_1(i)) + t_2(r_1(k)))$,

$$\mathsf{J4} \vdash r_1(i) : \Delta^{r_1} \vee r_1(k) : \Sigma^{r_1} \supset t : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}).$$

The indices i and l do not occur in either $[\Delta]_i$ or Σ since $[[\Delta]_i, \Sigma]_l$ is properly annotated. Hence: $r := (r_1 \upharpoonright [\Delta]_i, \Sigma) \cup \{l \mapsto t\}$ is a realization on the conclusion $[[\Delta]_i, \Sigma]_l$. For the identity substitution σ and r, we have

$$\mathsf{J4} \vdash ([\Delta]_i, [\Sigma]_k)^{r_1}\sigma \supset ([[\Delta]_i, \Sigma]_l)^r.$$

Case $\rho = 5\mathsf{a}^{[]}$: Let $\dfrac{[\Pi', [\Delta']]}{[\Pi'], [\Delta']}$ be an arbitrary shallow instance of $5\mathsf{a}^{[]}$, let $[\Pi, [\Delta]_i]_k$ and $[\Pi]_l, [\Delta]_i$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Pi, [\Delta]_i]_k}{[\Pi]_l, [\Delta]_i}$ is an annotated version of this instance. Let r_1 be a realization function on $[\Pi, [\Delta]_i]_k$. The following is a derivation in $\mathsf{J5}$:

0. $(\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (\neg r_1(i) : \Delta^{r_1} \supset \Pi^{r_1})$ — Propositional tautology
1. $r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset ?\,r_1(i) : \neg r_1(i) : \Delta^{r_1} \supset t(r_1(k), ?\,r_1(i) : \Pi^{r_1}$ — From 0. by Lemma 22
2. $r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (\neg(?\,r_1(i) : \neg r_1(i) : \Delta^{r_1}) \vee t(r_1(k), ?\,r_1(i)) : \Pi^{r_1})$ — From 1. by prop. reasoning
3. $\neg r_1(i) : \Delta^{r_1} \supset ?\,r_1(i) : \neg r_1(i) : \Delta^{r_1}$ — Instance of $\mathsf{j5}$

4. $\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset r_1(i):\Delta^{r_1}$ From 3. by prop. reasoning
5. $(\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \vee t(r_1(k),?\,r_1(i)):\Pi^{r_1}) \supset$
   $(r_1(i):\Delta^{r_1} \vee t(r_1(k),?\,r_1(i)):\Pi^{r_1})$ From 4. by prop. reasoning
6. $r_1(k):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset (r_1(i):\Delta^{r_1} \vee t(r_1(k),?\,r_1(i)):\Pi^{r_1})$
   From 2. and 5. by prop. reasoning

For $s := t(r_1(k),?\,r_1(i))$,

$$\mathsf{J5} \vdash r_1(k):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset (r_1(i):\Delta^{r_1} \vee s:\Pi^{r_1}).$$

The index l does not occur in either $[\Delta]_i$ or Π, since $[\Pi]_l,[\Delta]_i$ is properly annotated. Thus, $r := (r_1 \restriction \Pi,[\Delta]_i) \cup \{l \mapsto s\}$ is a realization on the conclusion $[\Pi]_l,[\Delta]_i$. For the identity substitution σ and r, we have

$$\mathsf{J5} \vdash ([\Pi,[\Delta]_i]_k)^{r_1}\sigma \supset ([\Pi]_l,[\Delta]_i)^r.$$

Case $\rho = \mathsf{5b}^{[]}$: Let $\dfrac{[\Pi',[\Delta']],[\Sigma']}{[\Pi'],[[\Delta'],\Sigma]'}$ be an arbitrary shallow instance of $\mathsf{5b}^{[]}$, let $[\Pi,[\Delta]_i]_j,[\Sigma]_h$ and $[\Pi]_k,[[\Delta]_i,\Sigma]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Pi,[\Delta]_i]_j,[\Sigma]_h}{[\Pi]_k,[[\Delta]_i,\Sigma]_l}$ is an annotated version of this instance. Let r_1 be a realization function on $[\Pi,[\Delta]_i]_j,[\Sigma]_h$. The following is a derivation in $\mathsf{J5}$:

0. $\neg r_1(i):\Delta^{r_1} \supset ?\,r_1(i):\neg r_1(i):\Delta^{r_1}$ Instance of $\mathsf{j5}$
1. $\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset \neg\neg r_1(i):\Delta^{r_1}$ From 0. by prop. reasoning
2. $\neg\neg r_1(i):\Delta^{r_1} \supset r_1(i):\Delta^{r_1}$ Propositional tautology
3. $\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset r_1(i):\Delta^{r_1}$ From 1. and 2. by prop. reasoning
4. $p:(\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset r_1(i):\Delta^{r_1})$ From 3. by Lemma 22
5. $\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset ?\,?\,r_1(i):\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1}$ Instance of $\mathsf{j5}$
6. $?\,?\,r_1(i):\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset p\cdot ?\,?\,r_1(i):r_1(i):\Delta^{r_1}$
   From 4. by app and MP
7. $\neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset p\cdot ?\,?\,r_1(i):r_1(i):\Delta^{r_1}$
   From 5. and 6. by prop. reasoning
8. $\neg r_1(i):\Delta^{r_1} \supset \Pi^{r_1} \vee r_1(i):\Delta^{r_1} \supset \Pi^{r_1}$ Propositional tautology
9. $?\,r_1(i):\neg r_1(i):\Delta^{r_1} \supset r_1(j):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset t_3(?\,r_1(i),r_1(j)):\Pi^{r_1}$
   From 8. by Lemma 22
10. $r_1(j):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset \neg\,?\,r_1(i):\neg r_1(i):\Delta^{r_1} \vee t_3(?\,r_1(i),r_1(j)):\Pi^{r_1}$
   From 9. by prop. reasoning

Let $s' := p\cdot ?\,?\,r_1(i)$ and $s := t_3(?\,r_1(i),r_1(j))$, then:

11. $r_1(j):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset s':r_1(i):\Delta^{r_1} \vee s:\Pi^{r_1}$
   From 7. and 10. by prop. reasoning
12. $r_1(i):\Delta^{r_1} \supset (r_1(i):\Delta^{r_1} \vee \Sigma^{r_1})$ Propositional tautology
13. $s':r_1(i):\Delta^{r_1} \supset t_1(s'):(r_1(i):\Delta^{r_1} \vee \Sigma^{r_1})$ From 12. by Lemma 22
14. $r_1(j):(\Pi^{r_1} \vee r_1(i):\Delta^{r_1}) \supset t_1(s'):(r_1(i):\Delta^{r_1} \vee \Sigma^{r_1}) \vee s:\Pi^{r_1}$
   From 11. and 13. by prop. reasoning

15. $\Sigma^{r_1} \supset (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ Propositional tautology

16. $r_1(h) : \Sigma^{r_1} \supset t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ From 15. by Lemma 22

17. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee$
$t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ From 14. and 16. by prop. reasoning

18. $t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \supset$
$(t_1(s') + t_2(r_1(h))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ By sum and prop. reasoning

19. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee$
$(t_1(s') + t_2(r_1(h))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$
From 17. and 18. by prop. reasoning

For $t := (t_1(s') + t_2(r_1(h)))$, it follows from 19. that:

$$\mathsf{J5} \vdash r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee t : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}).$$

The indices k and l do not occur in Π, $[\Delta]_i$ or Σ, since $[\Pi]_k, [[\Delta]_i, \Sigma]_l$ is properly annotated. Thus, $r := (r_1 \restriction \Pi, [\Delta]_i, \Sigma) \cup \{k \mapsto s, l \mapsto t\}$ is a realization on the conclusion $[\Pi]_k, [[\Delta]_i, \Sigma]_l$. For the identity substitution σ and r we have

$$\mathsf{J5} \vdash ([\Pi, [\Delta]_i]_j, [\Sigma]_h)^{r_1}\sigma \supset ([\Pi]_k, [[\Delta]_i, \Sigma]_l)^r.$$

This proves the remaining cases of Lemma 29. □